

On 4-chromatic Schrijver graphs: their structure,
non-3-colorability, and critical edges

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Abstract

We give an elementary proof for the non-3-colorability of 4-chromatic Schrijver graphs thus providing such a proof also for 4-chromatic Kneser graphs. To this end we use a complete description of the structure of 4-chromatic Schrijver graphs that was already given by Braun in [3, 4] and even earlier in an unpublished manuscript by Li [15]. We also address connections to surface quadrangulations. In particular, we show that a spanning subgraph of 4-chromatic Schrijver graphs quadrangulates the Klein bottle, while another spanning subgraph quadrangulates the projective plane. The latter is a special case of a result by Kaiser and Stehlík [11]. We characterize the color-critical edges of 4-chromatic Schrijver graphs and also present preliminary results toward the characterization of color-critical edges in Schrijver graphs of higher chromatic number. Finally, we show that (apart from two cases of small parameters) the subgraphs we present that quadrangulate the Klein bottle are edge-color-critical. The analogous result for the subgraphs quadrangulating the projective plane is an immediate consequence of earlier results by Gimbel and Thomassen and was already noted by Kaiser and Stehlík.

1 Introduction

Lovász’s celebrated proof of Kneser’s conjecture [17] is based on the Borsuk-Ulam theorem from algebraic topology and its appearance triggered a lot of activities in applying algebraic topological tools, in particular, the Borsuk-Ulam theorem in combinatorics. For several examples of this the reader is referred to Jiří Matoušek’s excellent book [19]. Naturally, it was always a question, whether the topological tools are really necessary to prove Kneser’s conjecture, or a purely combinatorial argument could substitute it. Matoušek himself gave such a proof in [20] that Ziegler [25] generalized later to make it apply to generalized versions of the Lovász-Kneser theorem. Among others, Ziegler also gave a combinatorial proof of Schrijver’s theorem [23]. These proofs, as Ziegler puts it, “are combinatorial (‘elementary’) in the sense that they do not rely on topological concepts (such as continuous maps, simplicial approximation, homology) or results.” Admittedly, however, all these proofs are inspired by topology and (quoting Ziegler again) “it is neither desirable nor practical to eliminate this background intuition.”

One can still ask, however, whether the topological background is also necessary for such special cases of these theorems, where the parameters involved are small. Kneser graphs depend on two positive integer parameters, n and k (satisfying $n \geq 2k$; see Definition 1 below) and the Lovász-Kneser theorem states that their chromatic number is $n - 2k + 2$. The difficult part of the proof is the lower bound part, that is proving that this many colors are needed for a proper coloring. This is completely trivial, however, when this number is 2, and is also quite easy when it is 3, as it only needs the demonstration of an odd cycle in the graph. (In case of Schrijver graphs, their 3-chromatic versions are just the odd cycles themselves.) The 4-chromatic case, however, is already non-trivial, and this is what we focus on in this paper.

To be able to say more let us give the necessary definitions. We start with standard notation. For $n \geq 1$, we write $[n]$ to denote the set of positive integers up to n . For $k \geq 0$ and a set S we write $\binom{S}{k}$ for the set of size k subsets of S . In this paper we deal with simple graphs only. All the graphs mentioned are finite except for the Borsuk graphs used in Section 4.1. If G is a graph we write $V(G)$ for its set of vertices and $E(G)$ for the set of edges. An edge connecting the vertices a and b is denoted by ab (or equivalently by ba). A map f from the vertex set of a graph is called a *coloring* of the graph and the value $f(v)$ is called the *color* of the vertex v . We call an edge *monochromatic* if it connects two vertices of the same color and we call a coloring *proper* if no edge is monochromatic. We write $\chi(G)$ for the *chromatic number* of the graph G , that is, the smallest number of colors in a proper coloring of G . In order to be able to name the vertices of paths and cycles we use P_m and C_m for path or cycle specifically on the vertex set $[m]$. Namely for $m \geq 2$ integer we write P_m for the path with vertex set $[m]$ and with edges connecting consecutive numbers. For $m \geq 3$ we write C_m for the cycle obtained from P_m by adding the edge $m1$. For a graph G and a subset S of $V(G)$ we write $G[S]$ for the subgraph induced by S and $G \setminus S = G[V(G) \setminus S]$. Similarly, $G \setminus S$ for $S \subseteq E(G)$ means the subgraph obtained from G by deleting the edges in S . For graphs G and H a map

$f : V(G) \rightarrow V(H)$ is called a *homomorphism* if for every edge $ab \in E(G)$ $f(a)f(b)$ is an edge in H . Clearly, if a homomorphism from G to H exists, then we have $\chi(G) \leq \chi(H)$. We call a map $f : V(G) \rightarrow V(H)$ a *cover* if for any vertex v of G f maps the neighbors of v in G bijectively to the neighbors of $f(v)$ in H . We call such a cover a *double cover* if every vertex in H is the image of exactly two vertices of G .

Definition 1. *Let $n \geq 2k$ be positive integers. The Kneser graph $\text{KG}(n, k)$ is defined as follows.*

$$V(\text{KG}(n, k)) = \binom{[n]}{k},$$

$$E(\text{KG}(n, k)) = \{ab : a, b \in \binom{[n]}{k}, a \cap b = \emptyset\}.$$

The Schrijver graph $\text{SG}(n, k)$ is defined as the subgraph of $\text{KG}(n, k)$ induced by the vertices of the Kneser graph that are independent sets in the cycle C_n .

Kneser [14] in 1955 observed (using different terminology) that the chromatic number $\chi(\text{KG}(n, k))$ is at most $n - 2k + 2$ and conjectured that this inequality is sharp. This was proved by Lovász [17] in 1978.

Theorem 1. *(Lovász-Kneser theorem)*

$$\chi(\text{KG}(n, k)) = n - 2k + 2.$$

Soon afterwards, Schrijver generalized (actually Bárány's simplified version [2] of) the proof and showed that the induced subgraphs now named after him have the same chromatic number. He also observed that these graphs are already vertex-color-critical.

Theorem 2. *(Schrijver [23])*

$$\chi(\text{SG}(n, k)) = n - 2k + 2.$$

Furthermore, for every $a \in V(\text{SG}(n, k))$ we have $\chi(\text{SG}(n, k) \setminus \{a\}) = n - 2k + 1$.

In this paper we give an elementary proof of the statement that $\chi(\text{SG}(2k + 2, k)) \geq 4$. This proof is combinatorial in the sense that it can be followed without any knowledge in topology. Nevertheless, an expert can feel the topological connection even in this argument: we will use a parameter called winding number that, though completely elementary and can be understood without any topological background, undeniably bears some flavour of topology. Our proof is very closely related to Mohar and Seymour's proof in [21] of non-3-colorability of certain surface quadrangulations, where the winding number also plays a crucial role. In fact, 4-chromatic Schrijver graphs have a close connection to such quadrangulations that we will discuss in Section 5.

Our proof builds on an understanding of how 4-chromatic Schrijver graphs look like. Such a structural description of $\text{SG}(2k + 2, k)$ is already implicit in Braun's proof of his result in [3] about the automorphism group of Schrijver graphs and is made more explicit in [4]. It also appears in the even earlier manuscript by Li [15]. Nevertheless, in Section 2

we also present such a description (in somewhat different and more general terms) for the sake of completeness. Section 3 contains the actual proof of non-3-colorability. Using the structural description from Section 2 again, we show in Section 5 that a spanning subgraph of every 4-chromatic Schrijver graph quadrangulates the Klein bottle. This observation itself also provides a proof of non-3-colorability if we use a theorem independently proved by Archdeacon, Hutchinson, Nakamoto, Negami, Ota [1] and Mohar and Seymour [21]. With slight modification of the previous (and somewhat more natural) quadrangulation we will also see how another spanning subgraph of $SG(2k + 2, k)$ quadrangulates the projective plane. This gives a special case of the result of Kaiser and Stehlík in [11] and also implies non-3-colorability by a result of Youngs [24], which is generalized in [1] and [21], and also in [10] in a different direction.

As we have seen Schrijver proved that Schrijver graphs are vertex-color-critical. They are not edge-color-critical, however. In Section 4 we characterize the color-critical edges of 4-chromatic Schrijver graphs and also present preliminary results toward the characterization of color-critical edges in Schrijver graphs of higher chromatic number. In Section 5.1 we show that (except for very small values of the parameters) the special subgraphs of $SG(2k + 2, k)$ quadrangulating the Klein bottle are edge-color-critical. (The analogous result for the subgraphs quadrangulating the projective plane is a straightforward consequence of a theorem in [8] as noted in [11].)

As we wrote this paper we found that some ingredients of our argument (in particular, the structure of graphs $SG(2k + 2, k)$ and the usefulness of the winding number in similar investigations) are already known, the relevant references are given in the foregoing. Nevertheless, these ingredients seem not to have ever appeared together the way we present them and therefore they are given here in full detail. Apart from summarizing these facts, the main novelties in this paper are the way we present the structure of 4-chromatic Schrijver graphs via the more general family of graphs we call reduced drums (see Definition 4 in Section 2), their direct use in the proof of non-3-colorability (Section 3), our results on critical edges of Schrijver graphs (Section 4), and the edge-color-criticality of certain subgraphs of four-chromatic Schrijver graphs quadrangulating the Klein bottle (Section 5). In the final short section we formulate a conjecture that would characterize critical edges in Schrijver graphs that are sufficiently large for their chromatic number.

2 The structure of 4-chromatic Schrijver graphs

To describe the structure of the graphs $SG(2k + 2, k)$, we will use the Cartesian product of graphs and also identification of vertices.

Definition 2. *Let F and G be two graphs. Their Cartesian product $F \square G$ has vertex set $V(F \square G) = V(F) \times V(G)$, while the edge set is defined by*

$$E(F \square G) = \{(a, c)(b, d) : (a = b \in V(F) \text{ and } cd \in E(G)) \text{ or } (ab \in E(F) \text{ and } c = d \in V(G))\}.$$

The subgraphs of $F \square G$ induced by the vertex sets $\{a\} \times V(G)$ for $a \in V(F)$ are called *layers*. Note that all layers are isomorphic to G . Symmetrically, the subgraphs induced by $V(F) \times \{c\}$ for $c \in V(G)$ are isomorphic to F , but we will not consider this type of subgraphs.

Definition 3. Let F be a graph and $a, b \in V(F)$. By identifying vertices a and b in F we obtain the graph with vertex set $(V(F) \setminus \{a, b\}) \cup \{v_{a,b}\}$ in which two vertices are connected if neither is $v_{a,b}$ and they are connected in F or one of them is $v_{a,b}$ and the other is a neighbor of either a or b in F . Clearly, we can also identify multiple pairs of vertices in a graph.

The following structure will give a kind of skeleton of the graphs $\text{SG}(2k+2, k)$, therefore we give them a name.

Definition 4. The drum of height h and perimeter n , denoted $D_{h,n}$, is defined for integers $h \geq 2$ and even numbers $n \geq 4$ as follows.

$$V(D_{h,n}) = V(P_h \square C_n),$$

$$E(D_{h,n}) = E(P_h \square C_n) \cup \{(i, j), (i, k)\} : i \in \{1, h\}, j - k \text{ is odd}\}.$$

That is, $D_{h,n}$ is obtained from $P_h \square C_n$ by extending the even cycles on the bottom and top layers (induced by $\{1\} \times V(C_n)$ and $\{h\} \times V(C_n)$) into complete bipartite graphs $K_{n/2, n/2}$.

For $i \in [h]$ and $j \in [n/2]$, we call the vertices (i, j) and $(h+1-i, j+n/2)$ of $D_{h,n}$ a pair of opposite vertices. The reduced drum $D'_{h,n}$ is obtained from the drum $D_{h,n}$ by identifying all opposite pairs of vertices.

Observe that every vertex of the drum $D_{h,n}$ has a unique opposite vertex and the involution γ switching each opposite pair is an automorphism of $D_{h,n}$.

As opposite vertices are not adjacent, the map ι that brings a vertex v of $D_{h,n}$ to the vertex of $D'_{h,n}$ representing the pair of opposite vertices containing v is a graph homomorphism. For $h > 2$ and $n > 4$ opposite vertices do not have a common neighbor. In this case, as the switching γ of opposite pairs is an automorphism of $D_{h,n}$, the homomorphism ι is, in fact, a double cover. We note that opposite vertices have no common neighbors in either $D_{2,n}$ if 4 divides n or in $D_{h,4}$ if h is even. So the map ι is a double cover in these cases, too.

Here we constructed the reduced drum $D'_{h,n}$ as a factor of the drum $D_{h,n}$. It may be instructive to also construct it from “half of the drum $D_{h,n}$ ”, that is from $P_{\lceil h/2 \rceil} \square C_n$, where the bottom layer is extended to a complete bipartite graph. In order to do so we only have to adjust opposite vertices in the top layer. For even h we connect them, for odd h we identify them as follows.

Lemma 3. For even integers $h \geq 2$ and $n \geq 4$ the reduced drum $D'_{h,n}$ can be obtained from $P_{h/2} \square C_n$ by adding the edges $\{(1, i)(1, j) : i - j \text{ is odd}\} \cup \{(h/2, i)(h/2, j) : |i - j| = n/2\}$.

For odd $h \geq 3$ and even $n \geq 4$ the reduced drum $D'_{h,n}$ can be obtained from $P_{(h+1)/2} \square C_n$ by adding the edges $\{(1, i)(1, j) : i - j \text{ is odd}\}$ and then identifying the vertex $((h+1)/2, i)$ with the vertex $((h+1)/2, i + n/2)$ for all $1 \leq i \leq n/2$.

Proof. We map the vertex (i, j) with $i \leq h/2$ of the graph constructed in the lemma to the vertex in $D'_{h,n}$ obtained from identifying the vertex (i, j) of $D_{h,n}$ with its opposite vertex. In case h is odd we further have the vertices w_i obtained by identifying $((h+1)/2, i)$ with $((h+1)/2, i + n/2)$ for $1 \leq i \leq n/2$. As $((h+1)/2, i)$ and $((h+1)/2, i + n/2)$ are opposite in $D_{h,n}$ they are also identified in $D'_{h,n}$ and the resulting vertex is where we map w_i . It is straightforward to check that the mapping so defined is an isomorphism. \square

Looking at the reduced drum $D'_{h,n}$ as constructed in Lemma 3 we see $\lceil h/2 \rceil$ layers of the product $P_{\lceil h/2 \rceil} \square C_n$. The bottom and top layers (corresponding to the vertex 1 and $\lceil h/2 \rceil$ of $P_{\lceil h/2 \rceil}$, respectively) are modified, the rest of the layers are copies of C_n . The bottom layer is a complete bipartite graph $K_{n/2, n/2}$. For odd h the top layer is the cycle $C_{n/2}$ (or K_2 in case $n = 4$), while for even h it is the cycle C_n with the opposite vertices connected. Using notation from [9], we call this last graph a *Möbius ladder* and denote it by M_n . Only neighboring layers are connected and they are connected by a perfect matching connecting corresponding vertices except in case of the top layer for odd h , where each vertex is connected to two opposite vertices of the previous layer. The case $h = 2$ is degenerate because the whole graph is a single layer that is both the bottom and the top layer. It is the complete bipartite graph $K_{n/2, n/2}$ if n is not divisible by 4, and in case n is divisible by 4 one has to add a perfect matching on both sides of $K_{n/2, n/2}$ to obtain $D'_{2,n}$.

Now we are ready to describe the structure of 4-chromatic Schrijver graphs. As mentioned in the Introduction, this description is already implicit in the proof of the main result of Braun's paper [3]. It is also given more explicitly (though in somewhat different terms) in [4] as well as in an earlier unpublished manuscript by Li [15].

Theorem 4. (cf. [3, 4, 15]) For $k \geq 1$ the Schrijver graph $\text{SG}(2k+2, k)$ is isomorphic to the reduced drum $D'_{k+1, 2k+2}$.

Proof. In case $k = 1$ the theorem claims that $\text{SG}(4, 1)$ is isomorphic to $D'_{2,4}$. This holds as both of these graphs are complete graphs on four vertices. For the rest of the proof, we assume $k \geq 2$ and set $C = C_{2k+2}$, $\text{SG} = \text{SG}(2k+2, k)$ and $D = D_{k+1, 2k+2}$. All our calculations will be modulo $2k+2$. We denote the edge of C connecting a and $a+1$ by e_a .

Recall that the vertices of SG are the size k independent sets in C . As C is a cycle, each of the k elements of a set $v \in V(\text{SG})$ is incident to two edges of C and since v is an independent set these $2k$ edges must all be distinct. Therefore, there are exactly two edges in $C \setminus v$. Let us define $e(v) = E(C \setminus v)$, so $e(v) = \{e_a, e_b\}$ for two distinct edges e_a and e_b of C . It is easy to see that for indices a and b of distinct parity there is exactly one independent set of size k among the vertices of C outside the endpoints of e_a and e_b ,

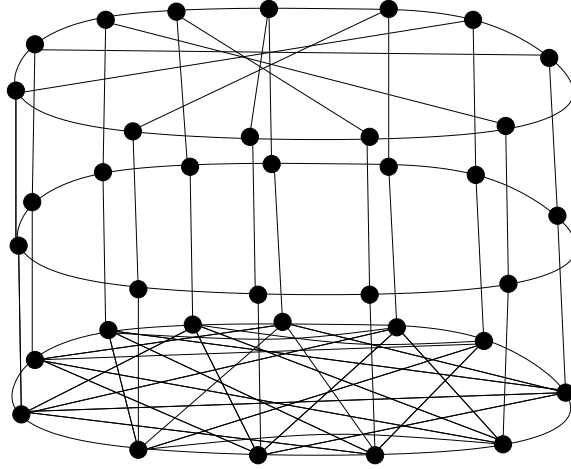


Figure 1: This figure shows the graph $\text{SG}(12, 5) \cong D'_{6,12}$ the way we think about it: the “bottom cycle” induces a $K_{6,6}$, while the “top cycle” induces the Möbius ladder M_{12} . See also Figures 2 and 3 for more symmetric pictures of the graphs $\text{SG}(6, 2)$ and $\text{SG}(8, 3)$, respectively.

while for distinct indices a and b of the same parity, there are none. Thus, $e(v)$ determines $v \in V(\text{SG})$ and $e(v)$ must be of the form $\{e_a, e_b\}$ with $a - b$ odd.

Consider the map $f : V(D) \rightarrow V(\text{SG})$ defined by $e(f((i, j))) = \{e_{i+j-1}, e_{j-i}\}$. The function is well defined as $i + j - 1$ and $j - i$ are of different parity.

We claim that f is a cover. Consider an arbitrary vertex $(i, j) \in V(D)$. The neighbors of the vertex $v = f((i, j))$ in SG are the independent sets of size k in $C \setminus v$. Here $C \setminus v$ has $k + 2$ vertices and its edge set is $e(v) = \{e_{i+j-1}, e_{j-i}\}$. Notice that these are adjacent edges if $i = 1$ or $i = k + 1$, but they are independent otherwise. Let us first assume $1 < i < k + 1$. Then v has exactly 4 neighbors, each obtained from $X = V(C) \setminus v$ by removing one endpoint of both edges in $e(v)$. Notice that these are exactly the vertices $f(w)$, where w runs over the four neighbors of (i, j) . We have:

$$\begin{aligned} f((i - 1, j)) &= X \setminus \{i + j - 1, j - i + 1\}, \\ f((i + 1, j)) &= X \setminus \{i + j, j - i\}, \\ f((i, j - 1)) &= X \setminus \{i + j - 1, j - i\}, \\ f((i, j + 1)) &= X \setminus \{i + j, j - i + 1\}. \end{aligned}$$

It remains to deal with the vertices (i, j) of D with $i \in \{1, k + 1\}$. We may assume by symmetry that $i = 1$. Now the vertex $v = f((1, j))$ has $e(v) = \{e_{j-1}, e_j\}$. Here j is the common vertex of the two edges. Therefore v has a single neighbor containing j : it is obtained by removing the two neighbors of j in C from $X = V(C) \setminus v$ and is the image of the only neighbor of $(1, j)$ outside its layer:

$$f((2, j)) = X \setminus \{j - 1, j + 1\}.$$

The rest of the neighbors of v are k -subsets of $S = X \setminus \{j\}$. Note that we have $S = \{u \in [2k+2] : u - j \text{ is odd}\}$. Any k -subset of S is of the form $w = S \setminus \{j'\}$ for some $j' \in S$, therefore we have $e(w) = \{e_{j'-1}, e_{j'}\}$, which means $w = f((1, j'))$ for the neighbor $(1, j')$ of $(1, j)$. This finishes the proof of the claim that f is a covering.

It remains to find the inverse image of a vertex of v in SG. Let $e(v) = \{e_a, e_b\}$. We have $f((i, j)) = v$ if and only if $i + j - 1 = a$ and $j - i = b$ or the other way around, $i + j - 1 = b$ and $j - i = a$. Recall that we still do calculations modulo $2k+2$, so these are systems of congruences. Both of these systems have two solutions modulo $2k+2$. Namely, if i_0, j_0 represent a solution, then the other three solutions will be $(k+2 - i_0, j_0 + k+1)$, $(i_0 + k+1, j_0 + k+1)$ and $(2k+3 - i_0, j_0)$. A pair (i, j) represent a vertex of D only if $i \in [k+1]$. Thus, always two of the four solutions represent vertices in D , either the first two or the last two. Notice that they represent a pair of opposite vertices in D . This means that f is a double cover that collapses the opposite pairs of vertices. To finish the proof of the theorem notice that the image of a cover f is always isomorphic to the graph obtained by identifying the sets of vertices in the domain of f that are mapped to the same vertex. We obtained $D'_{k+1, 2k+2}$ from D exactly like this: we identified the pairs of opposite vertices. \square

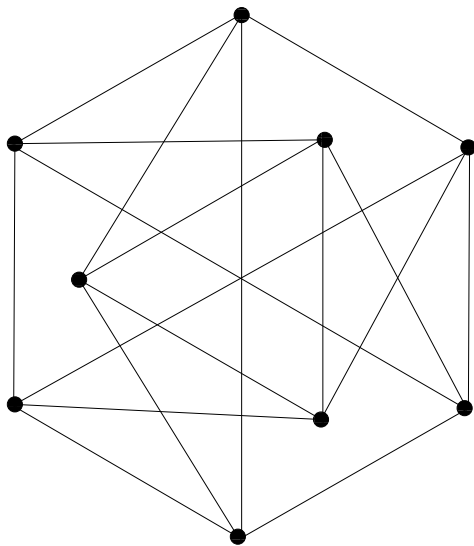


Figure 2: The graph SG(6, 2). This is the first “interesting” Schrijver graph, that is, the smallest one which is not just a complete graph or an odd cycle. The “inner cycle” here is the “top cycle” C_3 obtained after identifying opposite points of a C_6 , while the “outer cycle” is the C_6 extended to a complete bipartite graph $K_{3,3}$.

3 The chromatic number of reduced drums

Our goal in this chapter is to prove the following statement in an elementary way.

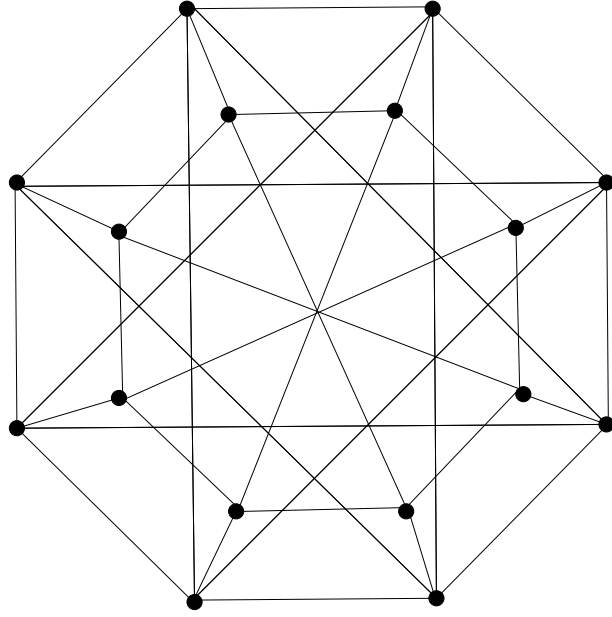


Figure 3: The graph $SG(8, 3)$. We have drawn the complete bipartite graph $K_{4,4}$ as the “outer cycle”, while the “inner cycle” induces the Möbius ladder M_8 .

Theorem 5. *For $h \geq 2$ integer and $n \geq 4$ even, we have $\chi(D'_{h,n}) = 2$ if $h + n/2$ is odd and $\chi(D'_{h,n}) = 4$ if $h + n/2$ is even.*

Note that by Theorem 4 this implies that $SG(2k + 2, k) = 4$ for all $k \geq 1$.

The main part of Theorem 5 is the non-three-colorability in case $h + n/2$ is even. We state that separately in Lemma 6 and give the simple proof for the rest of Theorem 5 right after stating this lemma.

Lemma 6. *There is no proper 3-coloring of the reduced drum $D'_{h,n}$ if $h \geq 2$, $n \geq 4$ even and $h + n/2$ is even.*

Proof of Theorem 5 using Lemma 6. $D_{h,n}$ is always bipartite, with the color of the vertex (i, j) in the proper 2-coloring given by the parity of $i + j$. This coloring gives identical colors to opposite vertices (i, j) and $(h + 1 - i, j + n/2)$ whenever $h + n/2$ is odd. So in this case the factor $D'_{h,n}$ inherits the same 2-coloring.

If $h + n/2$ is even, then Lemma 6 gives $\chi(D'_{h,n}) \geq 4$, so it is enough to find a proper 4-coloring. We consider $D'_{h,n}$ as constructed in Lemma 3. We start with an arbitrary proper coloring of the top layer with colors from $\{1, 2, 3, 4\}$. The top layer is a cycle $C_{n/2}$ or a Möbius ladder M_n , so even three colors suffice in all cases except for $M_4 \simeq K_4$. For vertices (i, j) in all other layers, that is for $i < \lceil h/2 \rceil$ we use a color of the same parity as $i + j$. For the vertices $(\lceil h/2 \rceil - 1, j)$ we choose from the two colors allowed by this parity rule to avoid the color of the only neighbor of this vertex in the top layer. For all other vertices we can choose arbitrarily and we obtain a proper 4-coloring. \square

Our key tool in the proof of Lemma 6 is the following notion.

Definition 5. When speaking of a 3-coloring of a graph we will always assume that the colors are 0, 1 and 2. Let $f : V(G) \rightarrow \{0, 1, 2\}$ be a proper 3-coloring of a graph G . Consider an arbitrary edge $e = ab$ of G . We define $W_f(\vec{e}) \in \{-1, 1\}$ for an orientation \vec{e} of e from a to b such that $W_f(\vec{e}) \equiv f(b) - f(a)$ modulo 3. Note that this is possible as f is a proper coloring. Let C be a cycle in G and let \vec{C} be a cyclic orientation of C . We set

$$W_f(\vec{C}) := \frac{1}{3} \sum_{\vec{e} \in E(\vec{C})} W_f(\vec{e})$$

and

$$W_f(C) := |W_f(\vec{C})|.$$

We call $W_f(C)$ the winding number of f on C .

Note that $\sum_{\vec{e} \in E(\vec{C})} W_f(\vec{e}) \equiv 0$ modulo 3 by the definition, so the winding number is an integer. Informally, it is the number of times we wind around the triangle K_3 on vertices 0, 1, 2 as we follow how the homomorphism f maps C to K_3 . The opposite orientation \overleftarrow{C} of C yields $W_f(\overleftarrow{C}) = -W_f(\vec{C})$, so the winding number is well defined.

The proof will be obtained as a consequence of the following lemmata.

Lemma 7. If f is a proper 3-coloring of a graph G and C is an m -cycle in G , then $W_f(C) \equiv m$ modulo 2. We have $W_f(C) = 0$ if C is a 4-cycle.

Proof. The first statement immediately follows by observing that $3W_f(\vec{C})$ is the sum of m values $W_f(\vec{e})$ and each of these values is -1 or 1 . The winding number is at most $m/3$, so in case $m = 4$ it has to be even and less than 2, so it is 0. \square

Lemma 8. Let f be a proper 3-coloring of the graph $P_2 \square C_m$ and let $C^{(1)}$ and $C^{(2)}$ denote its two layers. We have

$$W_f(C^{(1)}) = W_f(C^{(2)}).$$

Proof. For $j \in [2]$, consider the orientation $\vec{C}^{(j)}$ of $C^{(j)}$ going as $(j, 1) \rightarrow (j, 2) \rightarrow (j, 3) \rightarrow \dots \rightarrow (j, m) \rightarrow (j, 1)$. For $i \in [m]$, consider the oriented 4-cycle $\vec{D}^{(i)}$: $(1, i) \rightarrow (1, i+1) \rightarrow (2, i+1) \rightarrow (2, i) \rightarrow (1, i)$. Here $m+1$ is meant to be 1. We have

$$W_f(\vec{C}^{(1)}) - W_f(\vec{C}^{(2)}) = \sum_{i=1}^m W_f(\vec{D}^{(i)}).$$

Indeed, when applying the definition of the winding number to the right hand side, the terms $W_f(\vec{e})$ for the “vertical” edges $e = (1, i)(2, i)$ cancel because both orientations of these edges come up once. The terms corresponding to “horizontal” edges remain but the same terms show up on both sides of the equation.

From Lemma 7 the right hand side of the above equation is zero, so we have $W_f(\vec{C}^{(1)}) = W_f(\vec{C}^{(2)})$ and also $W_f(C^{(1)}) = W_f(C^{(2)})$. \square

Corollary 9. *Let f be a proper 3-coloring of the graph $P_h \square C_n$ and denote by $C^{(j)}$ the layer induced by the vertices $\{(j, i) : 1 \leq i \leq n\}$. Then*

$$W_f(C^{(1)}) = W_f(C^{(h)}).$$

Proof. Lemma 8 implies $W_f(C^{(j)}) = W_f(C^{(j+1)})$ for $j = 1, \dots, h-1$, thus we have the equalities $W_f(C^{(1)}) = W_f(C^{(2)}) = \dots = W_f(C^{(h)})$. \square

Lemma 10. *If f is a proper 3-coloring of a complete bipartite graph $K_{m,m}$, and C is any cycle of this complete bipartite graph, then $W_f(C) = 0$.*

Proof. The statement immediately follows from the fact, that one full side of the complete bipartite graph, say H , must be colored with one color if only 3 colors are used. Let us cut the cycle C into 2-edge paths at the vertices in H . At any cyclic orientation \vec{C} of the cycle C the contributions of the edges in one of these paths to $W_f(\vec{C})$ cancel out, so we must have $W_f(\vec{C}) = 0$ and therefore $W_f(C)$ must also be zero. \square

Lemma 11. *Let $n \geq 4$ be even and f a proper 3-coloring of the cycle C_n that also puts different colors on vertices that are opposite to each other. In other words, f remains a proper coloring even if we extend the cycle into a Möbius ladder M_n by adding the edges connecting the $n/2$ pairs of opposite vertices. Then $W_f(C_n) \equiv n + 2$ modulo 4.*

Proof. As usual, we assume $V(C_n) = [n]$. When referring to the vertex $n+1$ we mean the vertex 1. For $i \in [n]$, let \vec{e}_i stand for the oriented edge from i to $i+1$. Let \vec{C}_n stand for the orientation of C_n so given.

Consider the sums $m_j := \sum_{i=j}^{j+n/2-1} W_f(\vec{e}_i)$ for $j \in [n/2+1]$. Clearly, $m_j \equiv f(j+n/2) - f(j)$ modulo 3. As j and $j+n/2$ are opposite vertices in C_n we have $f(j+n/2) \neq f(j)$ so m_j is not divisible by 3. Also, as m_j is the sum of $n/2$ terms, each -1 or 1 , we have $m_j \equiv n/2$ modulo 2. So if $n/2$ is even, then m_j is congruent with 2 or 4 modulo 6, while if $n/2$ is odd, then m_j is congruent to -1 or 1 modulo 6. We can write $m_j = 6s_j + t_j$, where s_j is an integer and $t_j \in \{2, 4\}$ for all j if $n/2$ is even and $t_j \in \{-1, 1\}$ for all j if $n/2$ is odd.

For $j \in [n/2]$ we have $m_{j+1} - m_j = W_f(\vec{e}_{j+n/2}) - W_f(\vec{e}_j)$ and therefore $|m_{j+1} - m_j| \leq 2$. This implies $s_{j+1} = s_j$. As this holds for all $j \in [n/2]$ we also have $s_1 = s_{n/2+1}$.

By the definition of the winding number we have $3W_f(\vec{C}_n) = m_1 + m_{n/2+1}$. Here $m_1 + m_{n/2+1} = (6s_1 + t_1) + (6s_{n/2+1} + t_{n/2+1}) = 12s_1 + t_1 + t_{n/2+1}$. We see that $t_1 + t_{n/2+1}$ must be divisible by 3. In case $n/2$ is even we have $t_1, t_{n/2+1} \in \{2, 4\}$, so $t_1 + t_{n/2+1} = 6$ must hold. This yields $3W_f(\vec{C}_n) = 12s_1 + 6$, so $W_f(\vec{C}_n) = 4s_1 + 2$ as claimed. If $n/2$ is odd, we have $t_1, t_{n/2+1} \in \{-1, 1\}$, so we must have $t_1 + t_{n/2+1} = 0$ and $W_f(\vec{C}_n) = 4s_1$. \square

Proof of Lemma 6. Assume that for some integer $h \geq 2$ and even $n \geq 4$ the reduced drum $D'_{h,n}$ admits a proper 3-coloring f . We need to prove that $h + n/2$ is odd.

Let us first assume that h is even. By Lemma 3 we can consider $D'_{h,n}$ as the graph $P_{h/2} \square C_n$ with a few additional edges. Let C^{bottom} stand for the bottom layer of $P_{h/2} \square C_n$ and C^{top} for its top layer. The additional edges make the subgraph of $D'_{h,n}$ induced by C^{bottom} a complete bipartite graph, so by Lemma 10 we must have $W_f(C^{\text{bottom}}) = 0$. By Corollary 9 we have $W_f(C^{\text{top}}) = W_f(C^{\text{bottom}})$. The additional edges make the subgraph of $D'_{h,n}$ induced by C^{top} a Möbius ladder M_n , so by Lemma 11 we have $0 = W_f(C^{\text{top}}) \equiv n + 2$ modulo 4. This means $n \equiv 2$ modulo 4, so $h + n/2$ is odd.

Now we assume that h is odd. By Lemma 3 we can consider $D'_{h,n}$ as constructed from $P_{(h+1)/2} \square C_n$ in two steps. We refer to the bottom and top layers of $P_{(h+1)/2} \square C_n$ as C^{bottom} and C^{top} . In the first step we construct \hat{D} from $P_{(h+1)/2} \square C_n$ by adding edges to C^{bottom} to make it into a complete bipartite graph. Then we identify the opposite vertices in C^{top} to obtain $D'_{h,n}$. Clearly, the 3-coloring f can be extended to a proper 3-coloring \hat{f} of \hat{D} by keeping the color of the vertices outside C^{top} and assigning both opposite vertices in C^{top} the color that f assigns to the new vertex resulting from their identification. As in the previous case we must have $W_{\hat{f}}(C^{\text{top}}) = W_{\hat{f}}(C^{\text{bottom}}) = 0$ by Corollary 9 and Lemma 10. Let C^* stand for the cycle of length $n/2$ obtained from the cycle C^{top} by identifying the opposite vertices. In case $n = 4$ the graph C^* is not a cycle but then $h + n/2$ is odd anyway. We clearly have $W_{\hat{f}}(C^{\text{top}}) = 2W_f(C^*)$ as each contribution $W_f(\vec{e})$ to the winding number of an orientation of C^* appears twice (on two opposite edges) as the contribution to the winding number of the corresponding orientation of C^{top} . So we must have $W_f(C^*) = 0$, and by Lemma 7 this means that $n/2$ is even. This makes $h + n/2$ odd again and completes the proof. \square

Finally we state a similar result that is not directly related to Schrijver graphs but follows easily from the lemmata above. For $h \geq 2$ integer and even $n \geq 4$ let us construct the graph $X_{h,n}$ from $P_h \square C_n$ by adding the edges $(1, i)(1, i + n/2)$ for $i \in [n/2]$ and identifying the vertices (h, i) and $(h, i + n/2)$ for all $i \in [n/2]$. Note that $X_{h,n}$ is a 4-regular graph and in case $n/2$ is odd, it is a subgraph of $D'_{2h-1,n}$.

Theorem 12. *For all $h \geq 2$ and even $n \geq 4$ we have $\chi(X_{h,n}) = 4$.*

Proof. We write C^{bottom} and C^{top} for the bottom and top layers of $P_h \square C_n$ and, in case $n \geq 6$, let C^* be the $n/2$ -cycle of $X_{h,n}$ obtained by identifying the opposite vertices in C^{top} . Let \hat{X} be $P_h \square C_n$ plus the additional edges making C^{bottom} into the Möbius ladder M_n .

Assume for a contradiction that f is a proper 3-coloring of $X_{h,n}$. We can extend f to \hat{f} , a proper 3-coloring of \hat{X} that assigns the same color to opposite vertices of C^{top} as f assigned to the vertex obtained by identifying them. We have $W_{\hat{f}}(C^{\text{bottom}}) \equiv n + 2$ modulo 4 by Lemma 11. We have $W_{\hat{f}}(C^{\text{top}}) \equiv n$ modulo 4 if $n \geq 6$ because $W_{\hat{f}}(C^{\text{top}}) = 2W_f(C^*)$ and $W_f(C^*) \equiv n/2$ modulo 2 by Lemma 7. In case $n = 4$ Lemma 7 gives $W_{\hat{f}}(C^{\text{top}}) = 0$, so the same congruence $W_{\hat{f}}(C^{\text{top}}) \equiv n$ modulo 4 also holds. Therefore

we have $W_{\hat{f}}(C^{\text{top}}) \neq W_{\hat{f}}(C^{\text{bottom}})$ contradicting Corollary 9. The contradiction proves $\chi(X_{h,n}) \geq 4$.

As we noted, $X_{h,n}$ is a subgraph of $D'_{2h-1,n}$ if $n/2$ is odd. In this case $\chi(X_{h,n}) \leq \chi(D'_{2h-1,n}) = 4$ by Theorem 5. It remains to prove $\chi(X_{h,n}) \leq 4$ if $n/2$ is even. In that case ignoring the edges added to C^{bottom} to make it a Möbius ladder the graph is bipartite. These extra edges form a matching, so another bipartite graph. Therefore $X_{h,n}$, as the union of two bipartite graphs, has chromatic number at most 4. \square

4 Critical edges

In this section we study when will the deletion of a single edge of a Schrijver graph reduce its chromatic number. Here we consider arbitrary Schrijver graphs, not only 4-chromatic ones, but point out how our results apply to 4-chromatic Schrijver graphs. (To emphasize that the vertices of Schrijver graphs are subsets of $[n]$, in what follows they will be denoted by capital letters.)

Definition 6. *We call the edge e of a graph G color-critical or simply critical if the removal of that edge decreases the chromatic number, that is $\chi(G \setminus \{e\}) < \chi(G)$.*

We call an edge $XY \in E(\text{SG}(n, k))$ interlacing if the elements of X and Y alternate on the cycle C_n .

We will see that distinguishing interlacing and non-interlacing edges of the Schrijver graphs is *not* enough to characterize critical edges, but it nevertheless seems to be an important distinction. In Section 4.1 we will show that non-interlacing edges are not critical, in Section 4.2 we show that several categories of interlacing edges are critical. In fact, we will see in Section 4.3 that our results cover all edges in the 4-chromatic Schrijver graphs that are the main focus of this paper. In Section 4.4 we characterize the critical edges in the Schrijver graphs $\text{SG}(n, 2)$ and will see that for $n \geq 8$ some of the interlacing edges are not critical.

4.1 Critical edges are interlacing

As a preparation for our proof that all critical edges are interlacing, we recall a version of Bárány's proof [2] of the Lovász-Kneser theorem and also the proof of Schrijver's result [23]. We start with a definition of the following infinite graphs first defined by Erdős and Hajnal [5].

Definition 7. *Let S^d stand for the unit sphere around the origin in \mathbb{R}^{d+1} . We consider \mathbb{R}^{d+1} (hence also S^d) equipped with the Euclidean distance, so the diameter of S^d is 2. The open hemisphere around a point $x \in S^d$ is the set $\{y \in S^d \mid x \cdot y > 0\} = \{y \in S^d \mid |x - y| < \sqrt{2}\}$. (Here $x \cdot y$ is meant to be the scalar product of the vectors from the origin to x and y , respectively.) The Borsuk graph $B(d, \varepsilon)$ is the infinite graph whose vertices*

are the points of S^d and two vertices are connected by an edge if $|x + y| < \varepsilon$ (that is y is ε -close to $-x$). Here ε is a positive real.

The following proposition is an equivalent form of the Borsuk-Ulam theorem (cf. Lovász [18]):

Proposition 13. *For any $d \geq 0$ integer and $\varepsilon > 0$ real we have $\chi(B(d, \varepsilon)) \geq d + 2$.*

Bárány's proof uses the following lemma of Gale [6]:

Lemma 14. *For any integers $d, k > 0$ there is a set of $d + 2k$ points on S^d such that each open hemisphere contains at least k of these points.*

Although Bárány's proof of the Lovász-Kneser theorem used another form of the Borsuk-Ulam theorem and did not refer to the Borsuk graph, it can be reformulated as the next proposition. The statement $\text{KG}(n, k) \geq n - 2k + 2$ then follows through Proposition 13.

Proposition 15. *For positive integers $n \geq 2k$ one can find $\varepsilon > 0$ and a graph homomorphism from $B(n - 2k, \varepsilon)$ to $\text{KG}(n, k)$.*

Proof. Using Lemma 14 we find points $x_1, \dots, x_n \in S^d$ for $d = n - 2k$ such that each open hemisphere contains at least k of these points. For $x \in S^d$ let $f(x)$ stand for the k 'th smallest distance $|x - x_i|$ for $1 \leq i \leq n$. By the choice of the points x_i we have $f(x) < \sqrt{2}$ for all $x \in S^d$. As f is a continuous function on a compact domain it takes a maximum value $a < \sqrt{2}$. Now choose $\varepsilon = \sqrt{2} - a > 0$. We map a vertex x of $B(d, \varepsilon)$ to any vertex V of $\text{KG}(n, k)$ satisfying that for all elements $i \in V$ we have $|x - x_i| \leq a$. Such a vertex V exists since $f(x) \leq a$. This map is a graph homomorphism from $B(d, \varepsilon)$ to $\text{KG}(n, k)$ because if xy is an edge of $B(d, \varepsilon)$ and we map x to V and y to W , then x_i for $i \in V$ is contained in the hemisphere around x , while x_j for $j \in W$ is contained in the hemisphere around $-x$. These two hemispheres are disjoint so V and W must also be disjoint making VW an edge in the Kneser graph. \square

Schrijver [23] found the following stronger version of Gale's lemma:

Lemma 16. *For integers $d, k > 0$ there are points $x_1, \dots, x_{d+2k} \in S^d$ such that one can find a vertex V in $\text{SG}(d + 2k, k)$ for each open hemisphere H such that $x_i \in H$ for all $i \in V$.*

Using this stronger lemma one can make the statement of Proposition 15 stronger as follows. Note that this implies Schrijver's result through Proposition 13. Although Schrijver did not refer to Borsuk graphs in his proof, this is the essence of his proof.

Proposition 17. *For positive integers $n \geq 2k$ one can find $\varepsilon > 0$ and a graph homomorphism from $B(n - 2k, \varepsilon)$ to $\text{SG}(n, k)$.*

Proof. This proof is essentially identical to the proof of Proposition 15, the only difference is that we use Lemma 16 in place of Lemma 14. We choose $x_1, \dots, x_n \in S^d$ for $d = n - 2k$ to satisfy the statement of Lemma 16. For $x \in S^d$ we define $f(x) = \min_{V \in V(\text{SG}(n,k))} (\max_{i \in V} |x - x_i|)$. As before $f(x) < \sqrt{2}$ by the choice of the points x_i and since f is continuous on a compact domain we have $f(x) \leq a$ for some fixed $a < \sqrt{2}$ and all $x \in S^d$. We choose $\varepsilon = \sqrt{2} - a > 0$ as before. We map a vertex x of $B(d, \varepsilon)$ to any vertex V of $\text{SG}(n, k)$ with $|x - x_i| \leq a$ for all $i \in V$. Just as in the proof of Proposition 15 such a vertex V exists for all x and the resulting map is a graph homomorphism. \square

We can state an even stronger form of Gale's lemma as follows. Gale's lemma is stated as Lemma 3.5.1 in Matoušek's book [19]. The proof there (attributed to an observation by Ziegler) explicitly shows this stronger form.

Lemma 18. *For integers $d, k > 0$ there are points $x_1, \dots, x_{d+2k} \in S^d$ such that for every $x \in S^d$ one can find an interlacing edge VW in $\text{SG}(d+2k, k)$ such that x_i is contained in the open hemisphere around x for every $i \in V$ while x_j is contained in the open hemisphere around $-x$ for each $j \in W$.*

At first glance one would think that with this strong form of Gale's lemma we can prove that the subgraph of a Schrijver graph formed by the interlacing edges has the same chromatic number as the entire Schrijver graph. Unfortunately, the lemma only ensures that in a homomorphism of a Borsuk graph to the Schrijver graph we can make the image of *any one edge* to be an interlacing edge, but we cannot ensure that the image of *all edges* are interlacing. Furthermore, the strong statement alluded to above is false in general. Removing *all* the non-interlacing edges typically drastically reduces the chromatic number. The subgraph of $\text{SG}(n, k)$ formed by the interlacing edges is called the *interlacing graph* $\text{IG}_{n,k}$. Litjens, Polak, Sevenster and Vena [16] show that its chromatic number is $\lceil n/k \rceil$.

Nevertheless, Lemma 18 is enough to prove that none of the non-interlacing edges is critical in a Schrijver graph. The proof is through the following proposition.

Proposition 19. *For positive integers $n \geq 2k$ and a vertex V_0 of $\text{SG}(n, k)$ one can find $\varepsilon > 0$ and a graph homomorphism from $B(n - 2k, \varepsilon)$ to the graph obtained from $\text{SG}(n, k)$ by removing all non-interlacing edges incident to V_0 .*

Proof. Once again, we choose $x_1, \dots, x_n \in S^d$ for $d = n - 2k$, this time to satisfy the statement of Lemma 18. For $x \in S^d$ we define

$$f(x) = \min_{VW} (\max(\max_{i \in V} |x - x_i|, \max_{j \in W} |-x - x_j|)),$$

where the minimum is for interlacing edges VW in $\text{SG}(n, k)$. As before $f(x) < \sqrt{2}$ by the choice of the points x_i and since f is continuous on a compact domain we have $f(x) \leq a$ for some fixed $a < \sqrt{2}$ and all $x \in S^d$.

We choose $\varepsilon = (\sqrt{2} - a)/2 > 0$ and set $b = a + \varepsilon = \sqrt{2} - \varepsilon$. We map a vertex x of $B(d, \varepsilon)$ to a vertex V of $\text{SG}(n, k)$ such that $|x - x_i| \leq b$ for all $i \in V$. Such a vertex exists by the choice of the points x_i . This time we do not choose an arbitrary such vertex V but follow this order of priorities: We map x to V_0 only if no other choice is available and we map x to a non-interlacing neighbor of V_0 only if all other available choices are also non-interlacing neighbors of V_0 or possibly V_0 itself.

To see that this defines a graph homomorphism, we need to show that if x is mapped to V and y is mapped to W for an edge xy of $B(d, \varepsilon)$, then VW is an edge of $\text{SG}(n, k)$, and further if $V = V_0$ or $W = V_0$, then VW is interlacing. As before, x_i for $i \in V$ is contained in the hemisphere around x , while x_j for $j \in W$ is contained in the hemisphere around $-x$, so V and W must be disjoint and thus an edge of $\text{SG}(n, k)$. Assume now that $V = V_0$ (the case $W = V_0$ is symmetric). We need to prove that VW is an interlacing edge. By $f(x) \leq a$ we have an interlacing edge $V'W'$ in $\text{SG}(n, k)$ with $|x - x_i| \leq a$ for $i \in V'$ and $|-x - x_j| \leq a$ for $j \in W'$. This makes V' available as the image of x and W' available as the image of y . (Note that we may have y at distance ε away from $-x$, but even then the furthest point of W' from y is still within distance b .) We map x to V_0 only as a last resort, therefore we must have $V' = V_0$. But then W' is an interlacing neighbor of V_0 where y could be mapped, therefore y cannot be mapped to a non-interlacing neighbor of V_0 . This makes VW an interlacing edge. \square

Theorem 20. *No non-interlacing edge of a Schrijver graph is critical, further if we remove all non-interlacing edges incident to a fixed vertex of $\text{SG}(n, k)$ the chromatic number of the remaining graph is still $n - 2k + 2$.*

Proof. This follows directly from Propositions 13 and 19. \square

Let us remark that with a more involved order of priorities among the vertices of the Schrijver graph one can extend the size of the set of non-interlacing edges that can be removed without effecting the chromatic number. But finding an edge-critical subgraph this way does not seem to be within reach.

4.2 Many interlacing edges are critical

Here we show several classes of interlacing edges of Schrijver graphs that are critical. To show that an edge VW of the Schrijver graph $\text{SG}(n, k)$ is critical one has to give a proper $(n - 2k + 1)$ -coloring of the graph $\text{SG}(n, k) \setminus \{VW\}$, or equivalently an $(n - 2k + 1)$ -coloring of the vertices of $\text{SG}(n, k)$ in which only the edge VW is monochromatic. For this we will start with the following coloring.

Definition 8. *We call a (non-proper) $(n - 2k + 1)$ -coloring of the Schrijver graph $\text{SG}(n, k)$ a basic coloring for the edge VW if $n - 2k$ of the colors used are identified with the $n - 2k$ elements of $[n] \setminus (V \cup W)$ and the color of a vertex U is selected from $U \setminus (V \cup W)$ whenever this set is not empty and the color of U is the $(n - 2k + 1)$ th color, called special color if $U \setminus (V \cup W)$ is empty.*

Proposition 21. *In a basic coloring of a Schrijver graph for an edge VW we have that:*

1. *Both endpoints of all the monochromatic edges are colored the special color and these edges form an induced matching.*
2. *If $C_n[V \cup W]$ is connected or has two components, both on an odd number of vertices, then VW is the only monochromatic edge in the coloring making it a critical edge in $SG(n, k)$.*

Proof. Two vertices in the same non-special color class x both contain x , so they are intersecting. Two vertices in the special color class form an edge if they partition $V \cup W$, proving part 1.

The vertices of $SG(n, k)$ are independent k -sets in C_n . If $C_n[V \cup W]$ has a single component (a path or the entire cycle C_n in the trivial case $n = 2k$), then it has a single partition into independent sets, while if it has two components (necessarily paths), then it has two such partitions. If both of those two components contain an odd number of vertices, then the parts in one of these partitions are unequal, so only one of the partitions correspond to a monochromatic edge of $SG(n, k)$, so VW is the only monochromatic edge. This makes VW critical and proves part 2 as the basic coloring uses $n - 2k + 1$ colors. \square

Definition 9. *We call a vertex V of the Schrijver graph $SG(n, k)$ regular if V contains at least one of any three consecutive vertices on the cycle C_n .*

Note that for $n > 3k$ there are clearly no regular vertices in $SG(n, k)$, but for a fixed chromatic number $n - 2k + 2 = d + 2$ and large k , the Schrijver graph $SG(2k + d, k)$ has $\Theta(k^d)$ vertices and all but only $\Theta(k^{d-1})$ of them are regular.

Proposition 22. *If V is a regular vertex of the Schrijver graph $SG(2k + d, k)$, then the degree of V is 2^d and all the 2^d edges incident to V are interlacing.*

Proof. Let $n = 2k + d$. Any vertex V of $SG(n, k)$ is an independent set of size k in C_n , so $2k$ of the n edges of C_n is incident to an element of V leaving $n - 2k = d$ edges in $C_n \setminus V$. Note that we have already made this observation in the 4-chromatic case in the proof of Theorem 4. A neighbor of V in $SG(n, k)$ is an independent set of size k in C_n that is disjoint from V , that is, it is obtained by removing d elements from the set $[n] \setminus V$ in such a way that we remove at least one endpoint of all the d edges in $C_n \setminus V$.

Now assume that V is regular. In this case the d edges in $C_n \setminus V$ form a matching. To obtain a neighbor of V we then must remove exactly one endpoint of these d edges from $[n] \setminus V$ in an arbitrary way, hence the 2^d possibilities. Clearly, all these neighbors form interlacing edges with V . \square

Theorem 23. *Any edge connecting two regular vertices in a Schrijver graph is critical.*

Proof. Let VW be an edge in the Schrijver graph $SG(n, k)$ connecting two regular vertices. By symmetry among the nodes of C_n we may assume $1 \in V$. Let $D = [n] \setminus (V \cup W)$. D consist of $n - 2k$ nodes of C_n . Among any three consecutive nodes on C_n at least one is

contained in V and another one in W (to make these vertices regular), so at most one of the three nodes is in D . This means that two elements of D are separated by at least two nodes not in D along the cycle C_n .

We start with a basic coloring f_0 of $\text{SG}(n, k)$ for the edge VW in which we choose the highest possible index whenever we have a choice. That is, we have $f_0(U) = \max(U \cap D)$ for vertices U of $\text{SG}(n, k)$ intersecting D and $f_0(U)$ is the special color if U and D are disjoint.

We will obtain the coloring f of the vertices of $\text{SG}(n, k)$ by modifying f_0 at one end of each monochromatic edge except for VW to make those other edges non-monochromatic. We will also make sure not to create new monochromatic edges in the process. By Proposition 21, all monochromatic edges form an induced matching, so the modifications in the coloring do not interact: it is enough if we find a non-special color that is not represented among the f_0 -colors of the neighbors of one of the endvertices of the monochromatic edges considered.

Let therefore XY be a monochromatic edge of f_0 other than VW . The sets X and Y partition $V \cup W$, so one of them, say X , contains 1. As XY is different from VW , neither of the sets $X \setminus V$ or $Y \setminus W$ is empty. Let $a = \min(X \setminus V)$ and $b = \min(Y \setminus W)$ and note that a and b are distinct and at least 2. Let us consider the case $b < a$ first. In this case we recolor X by setting $f(X) = b - 1$. We need to show first that $b - 1 \in D$, so this is an available non-special color. The independent set Y contains b , so it cannot contain $b - 1$. We have $b \notin W$, so by $X \cup Y = V \cup W$ we have $b \in V$ and therefore $b - 1 \notin V$. This makes $b - 1 \notin X$, as otherwise we had $b - 1 \in X \setminus V$, making $a < b$, a contradiction. So $b - 1 \notin X \cup Y = V \cup W$ and therefore $b - 1 \in D$ as claimed.

It remains to show that X does not have a neighbor Z with $f_0(Z) = b - 1$. Let us assume that such a neighbor Z does exist. Let $D' = [n] \setminus (X \cup Z)$.

We have $b - 1 = f_0(Z) = \max(Z \cap D)$, so Z avoids all nodes $d \in D$ with $d > b - 1$. These nodes are then contained in D' as X is disjoint from D . Let us consider now $d \in D$ with $d < b - 1$. We have $d \notin W$, so as W is regular we must have a neighbor (in C_n) $d' \in W$ with $d' = d - 1$ or $d' = d + 1$. This makes $d, d' \notin V$ and as $a > b$ we must also have $d, d' \notin X$. As Z is an independent set in C_n , it cannot contain both d and d' , hence either d or d' shows up in D' . Finally consider $b - 1 \in D$. As $f_0(Z) = b - 1$, Z must contain $b - 1$, so it contains neither of its neighbors in C_n . One of these neighbors is b and $b \in Y$, implying $b \notin X$ and $b \in D'$. We have $b, b - 1 \notin W$ and W is regular so it must contain $b - 2$, the other neighbor of $b - 1$ (note that $b - 1 \in D$, therefore $b - 1 > 1$). From $b - 2 \in W$ we conclude $b - 2 \notin V$ and since $a > b$ we also have $b - 2 \notin X$ and thus $b - 2 \in D'$.

In summary, every element $d \in D$ is either contained in D' or a neighbor of d is contained in D' , and for element $b - 1$ of D both of its neighbors are contained in D' . We saw that elements of D are separated by at least two other nodes along C_n , so there is no double counting, we must have $|D'| > |D|$. This is a contradiction, as we clearly have $|D'| = |D| = n - 2k$. The contradiction proves that the recoloring of X does not create new monochromatic edges.

We can deal with the case $a < b$ similarly: we recolor Y by setting $f(Y) = a - 1$. We can show that $a - 1 \in D$ and that this does not create a monochromatic edge the same way as we did in the case $a > b$. After dealing with each monochromatic edge of f_0 other than VW (and defining $f(U) = f_0(U)$ for all other the vertices U) we obtain the coloring f of the vertices of $\text{SG}(n, k)$ with $n - 2k + 1$ colors, where VW is the only monochromatic edge showing that VW is critical. \square

4.3 Critical edges of $\text{SG}(2k + 2, k)$

In this short section we show that in case of the 4-chromatic Schrijver graphs (that are the focus of this paper) the results in Sections 4.1 and 4.2 are enough to characterize all edges with respect to color-criticality.

Theorem 24. *An edge of the Schrijver graph $\text{SG}(2k + 2, k)$ is color-critical if and only if it is interlacing.*

Proof. The only if part of the theorem is true for all Schrijver graphs as stated in Theorem 20.

For the if part consider an interlacing edge VW in $\text{SG}(2k + 2, k)$. The set $D = [n] \setminus (V \cup W)$ has two elements. Therefore $C_n[V \cup W] = C_n \setminus D$ is either a path or the disjoint union of two paths. In the former case or if there are two paths, both on an odd number of vertices, then VW is critical by Proposition 21. If, however, there are two paths and both have at least 2 vertices, then V and W are regular (otherwise they cannot be interlacing), so VW is critical by Theorem 23. We have covered all the possible cases as $C_n[V \cup W]$ has $2k$ vertices, so if it consists of two components, then the parity of their sizes agree. \square

Recall that the Schrijver graph $\text{SG}(2k + 2, k)$ is isomorphic to the reduced drum $D'_{k+1, 2k+2}$ by Theorem 4 and the latter can be obtained from $P_{\lfloor k/2 \rfloor + 1} \square C_{2k+2}$ by slightly modifying the top and bottom layers as described in Lemma 3. This gives another (more geometric) view of the 4-chromatic Schrijver graphs and it is instructive to identify the critical edges in this view of them.

Proposition 25. *In the view of the reduced drum $D'_{k+1, 2k+2}$ given in Lemma 3, the regular vertices of the isomorphic graph $\text{SG}(2k + 2, k)$ correspond to exactly the vertices outside the bottom layer.*

The non-critical edges of the reduced drum $D'_{k+1, 2k+2}$ are exactly the edges we added to the bottom layer to make it a complete bipartite graph. That is, all edges not contained in the bottom layer are critical and so are also the edges of the copy of the cycle C_{2k+2} that is the bottom layer of the graph $P_{\lfloor k/2 \rfloor + 1} \square C_{2k+2}$. All remaining edges are non-critical.

Proof. Recall from the proof of Theorem 4 that the double cover mapping the drum $D_{k+1, 2k+2}$ to the Schrijver graph $\text{SG}(2k + 2, k)$ maps the vertex (i, j) of the drum to the vertex V of the Schrijver graph satisfying that the two edges in $C_n \setminus V$ are the edge connecting $i + j - 1$ to $i + j$ and the one connecting $j - i$ to $j - i + 1$ (all calculations here and

in the rest of this proof are modulo $2k+2$). Here V is regular if and only if these edges are non-adjacent, so it is non-regular exactly if $i+j = j-i$ or $i+j-1 = j-i+1$. The former equality happens exactly if $i = k+1$, while the latter happens exactly if $i = 1$. Thus, the vertices in the top and bottom layer are mapped to non-regular vertices, while the rest of the layers are mapped to regular vertices. The isomorphism between the reduced drum and the Schrijver graph is obtained as the factor of the above map, therefore the vertices in the bottom layer of the reduced drum (corresponding to identification of a vertex from the bottom layer of the drum and its opposite vertex in the top layer) correspond to non-regular vertices in $\text{SG}(2k+2, k)$, while the rest of the vertices in the reduced drum correspond to regular vertices as claimed.

Theorem 24 characterizes the critical edges in the Schrijver graph $\text{SG}(2k+2, k)$ as the interlacing edges. By Proposition 22, all edges incident to a regular vertex are interlacing, so all edges of the reduced drum $D'_{k+1, 2k+2}$ are critical except perhaps some edges in the bottom layer. Using the argument from the previous paragraph we can see that the vertex $(1, j)$ of the reduced drum corresponds to vertex $V_j = \{a \in [2k+2] \setminus \{j\} \mid a-j \text{ even}\}$ of the Schrijver graph. A neighbor of $(1, j)$ in the bottom layer of the reduced drum is $(1, i)$ with $i-j$ odd. If $j-1, j+1 \in V_i$, then these nodes of C_n are not separated by the set V_j , so the edge $V_i V_j$ is not interlacing making this edge (and the edge $(1, i)(1, j)$ of the reduced drum) non-critical. We have $j-1, j+1 \in V_i$ unless $i = j-1$ or $i = j+1$. In these two cases the edge $V_i V_j$ is indeed interlacing, making it (and the edge $(1, i)(1, j)$ of the reduced drum) critical. \square

4.4 Critical edges of $\text{SG}(n, 2)$

Definition 10. *The length $l(\{i, j\})$ of a vertex $\{i, j\}$ in the Schrijver graph $\text{SG}(n, 2)$ is the distance of i from j in the graph C_n , that is $l(\{i, j\}) = \min(|i-j|, n-|i-j|)$. We have $2 \leq l(\{i, j\}) \leq \lfloor n/2 \rfloor$.*

The following theorem characterizes the critical edges in $\text{SG}(n, 2)$. It shows that all interlacing edges are critical for $n \leq 7$ but this is not true for $n \geq 8$, further most interlacing edges are non-critical as n grows. We remark that this is consistent with a new result of Kaiser and Stehlík [12] which proves the edge-color-criticality of a spanning subgraph of $\text{SG}(n, 2)$ they obtain by deleting several non-interlacing edges.

Theorem 26. *An edge XY of the Schrijver graph $\text{SG}(n, 2)$ is critical if and only if it is interlacing and either $l(X) \leq 3$ or $l(Y) \leq 3$.*

Proof. For brevity we write SG for $\text{SG}(n, 2)$.

For the if part consider an edge XY of SG satisfying the conditions in the theorem. If both X and Y has length 2, then $C_n[X \cup Y]$ is a path on four vertices, while if one of them has length 2, while the other is longer, then $C_n[X \cup Y]$ consists of a path on three vertices and an isolated vertex. Proposition 21 claims that the edge XY is critical in SG in both cases.

In the remaining case either X or Y has length 3. By symmetry we may assume that $X = \{1, 4\}$ and $Y = \{2, i\}$ for some $i > 4$. We define a coloring f of the vertices of the graph SG as follows. If a vertex U has an element $j > 4$, $j \neq i$, then we set $f(U)$ to such a value j . This uses $n - 5$ colors and creates no monochromatic edges. All vertices not colored yet are subsets of $\{1, 2, 3, 4, i\}$. We use two special colors for these vertices, namely we set $f(U) = 0$ if $U = \{1, 3\}$, $U = \{1, i\}$ or $U = \{3, i\}$ and set $f(U) = -1$ if $U = X$, $U = Y$, $U = \{2, 4\}$ or $U = \{4, i\}$. Note that this covers all remaining vertices and $\{1, i\}$ and $\{4, i\}$ may not even be vertices of the Schrijver graph depending on the value of i .

Clearly, the color class 0 forms an independent set in the Schrijver graph while the color class -1 induces the single monochromatic edge XY . In total we use $n - 3$ colors, so the coloring shows that the edge XY is critical in SG as claimed.

For the only if part of the theorem let us consider a critical edge XY of SG . By Theorem 20 XY is interlacing, so we may assume $X = \{i, k\}$, $Y = \{j, l\}$ with $1 \leq i < j < k < l \leq n$. Our goal is to prove that either X or Y has length at most 3.

As XY is critical we have a proper $(n - 3)$ -coloring f of $SG \setminus \{XY\}$. We identify vertices of SG with the corresponding edge of the complement \overline{C}_n of C_n . This makes f an edge-coloring of \overline{C}_n where non-adjacent edges receive distinct colors except in the case of X and Y . As f cannot be a proper coloring of SG , X and Y must indeed receive the same color. We call the color of X and Y red. Any further red edge must be adjacent to both X and Y , so beyond X and Y only ij , jk , kl and li may be red (some of these may not even be edges in \overline{C}_n). Further, ij and kl are not adjacent, so at most one of them is red and similarly at most one of jk and li is red. Thus, we have at most 4 red edges.

Any other color class contains pairwise adjacent edges, so these color classes form either triangles or stars. Note that no color class can be a star centered at i or k , because then we could modify f by recoloring X to this color to obtain a proper $(n - 3)$ -coloring of SG . Similarly no color class can be a star centered at j or l . Also, no two distinct color classes can be stars with the same center as then we could join these classes and recolor X to a new color to obtain a proper $(n - 3)$ -coloring of SG .

Let s be the number of triangular color classes. We must have $n - 4 - s$ color classes forming a star then. (Note that the class of the red edges is neither triangular nor forms a star.) We remove the center of these stars from $[n]$ to obtain a size $s + 4$ subset Q . In case a color class consists of a single edge we designate one end vertex as the center and remove only that. Note that by the observation in the previous paragraph $i, j, k, l \in Q$. Clearly, all edges of $\overline{C}_n[Q]$ belong to a triangular color class or are red. Out of the $\binom{s+4}{2}$ edges of the complete graph on Q at most $s + 4$ belong to the cycle C_n , the rest must be colored by the s triangular color classes (at most 3 edges per class) or must be red (at most 4 edges), so we have

$$\binom{s+4}{2} - (s+4) \leq 3s+4.$$

The inequality shows $s \leq 2$. We use a case analysis to conclude that either X or Y

has length at most 3. We spell out this case analysis here for completeness, but the reader is probably better off proving this statement oneself than reading the rest of this proof.

We start with the case $s = 0$. There is no triangular color class, so all edges in $\overline{C}_n[\{i, j, k, l\}]$ must be red. But as ij and kl cannot be both red, one of them must belong to C_n . Similarly, one of jk and li must also belong to C_n and this makes the length of X or Y two.

In case $s = 1$ we have $|Q| = 5$. We have at most seven edges in two colors in $\overline{C}_n[Q]$. This means that $C_n[Q]$ has at least 3 edges. If it has at least four edges, then at most one edge of $\overline{C}_n[Q]$ has length larger than 3, so one of X and Y has length at most 3 and we are done. So we assume that $C_n[Q]$ has exactly 3 edges. This means that $\overline{C}_n[Q]$ has 7 edges, and among these a triangle is one color class and 4 red edges form another. Now consider the node $m \in Q \setminus \{i, j, k, l\}$. The edges incident to m are not red, therefore 2 of the 4 edges connecting m to i, j, k and l must come from the triangular color class, the other two from C_n . Say $\{i, j, k, l\} = \{a, b, c, d\}$ with ma and mb from the triangular color class and mc and md from C_n . Now ab must also come from the triangular color class, therefore the third edge of $C_n[Q]$ must connect one of a or b to one of c or d . We may assume by symmetry that it is bc . As $bc, cm,$ and md are all from C_n the length of the edge cd is 2, while the length of bd is 3. One of these two edges is either X or Y , so we are done again.

In case $s = 2$ we have $|Q| = 6$. Let $Q = \{i, j, k, l, m, m'\}$. Out of the edges connecting m to the other 5 elements of Q at most two belong to C_n , none are red, so at least three belong to the two triangular color classes. This implies that both the triangular color classes must include edges incident to m . Similar claim can be said about m' . Thus, in particular, the pair mm' should belong to both triangular color classes that is clearly impossible. This finishes the proof of the theorem. \square

5 Relation to quadrangulations of nonorientable surfaces

A graph is said to quadrangulate a surface if it can be embedded in the surface in such a way that all the faces of the embedding are quadrangles. The Schrijver graph $SG(6, 2)$ shown on Figure 2 quadrangulates the Klein bottle as shown on Figure 4.

Similarly, we can draw $X_{h,n}$ (recall its definition from Section 3 given before stating Theorem 12) as a quadrangulation of the Klein bottle by simply drawing a centrally symmetric grid of $2h - 1$ rows and $n/2$ columns in the rectangle representing the Klein bottle. The vertices of the top and bottom rows induce the Möbius ladder M_n , the vertices of the middle row induce $C_{n/2}$, while symmetric pairs of other rows induce a copy of C_n . Recall that for even k the graph $X_{k/2+1, 2k+2}$ is a spanning subgraph of $D'_{k+1, 2k+2} = SG(2k + 2, k)$. See Figure 5 for the drawing of $X_{4,14}$ which is a subgraph of $D'_{7,14} = SG(14, 6)$ as a quadrangulation of the Klein bottle.

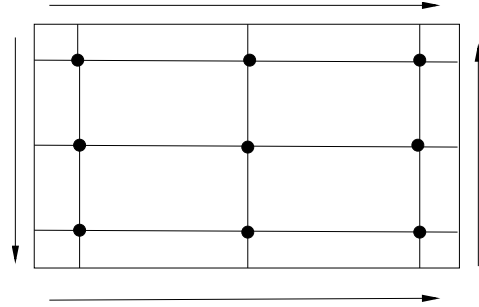


Figure 4: The graph $Y_2 = \text{SG}(6, 2)$ drawn as a quadrangulation of the Klein bottle.

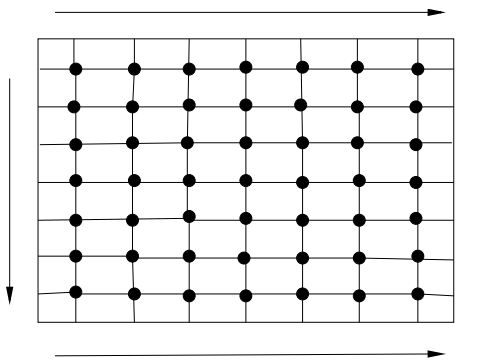


Figure 5: The subgraph $Y_6 = X_{4,14}$ of $\text{SG}(14, 6)$ drawn as a quadrangulation of the Klein bottle.

A spanning subgraph of $SG(2k + 2, k)$ quadrangulates the Klein bottle also in case of odd $k > 1$, but its drawing is a little more tricky. The smallest case, $SG(8, 3)$ can be seen on Figure 6.

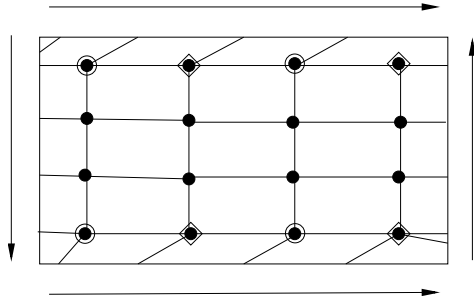


Figure 6: The subgraph Y_3 of $SG(8, 3)$ drawn as a quadrangulation of the Klein bottle. The circled and squared vertices belong to the “outer cycle” vertices of Figure 3 that form a complete bipartite graph there. The circles and squares refer to their different color classes in that complete bipartite graph. Here only 12 edges run among them out of the original 16 ones, four edges of $SG(8, 3)$ are deleted to form this quadrangulation.

Again, it is not hard to see how to obtain a similar subgraph of $SG(2k + 2, k)$ quadrangulating the Klein bottle for any odd $k \geq 3$. One has to draw the same grid-like picture with $k + 1$ horizontal and vertical lines, where the top and bottom segment of the vertical lines is made skew, so that the corresponding edge connects the top vertex of a vertical line with the bottom vertex of the next vertical line to the right. The top vertex of the rightmost line is not connected this way to any vertex (it has degree 3), while there is an extra edge connecting the bottom left and bottom right vertices as in Figure 6. We will denote this graph by Y_k . In case k is even, we write Y_k to denote $X_{k/2+1, 2k+2}$.

Note that Y_k is typically not the only subgraphs of $SG(2k + 2, k)$ quadrangulating the Klein bottle. We chose it for its relative simplicity.

Concerning the relevance of quadrangulations of the Klein bottle we remark that a closely related graph G was demonstrated by Kaiser, Stehlík, and Škrekovski [13] to provide a counterexample to a conjecture about square-free monomial ideals in polynomial rings. Graph G , originally defined by Gallai [7] as remarked in [13], is simply the graph we obtain by putting one more vertical line into the picture of $SG(6, 2)$ on Figure 4. For some other relations between algebraic questions and surface quadrangulations cf. [22].

We can obtain a quadrangulation of the projective plane similarly. If we place the $k + 1$ by $k + 1$ grid symmetrically in the rectangle representing the projective plane the opposite corner vertices are connected by two parallel edges, a horizontal one and a vertical one. Removing one edge from each of these pairs of parallel edges (say the vertical one) we obtain the quadrangulation Z_k of the projective plane. Just as Y_k is a subgraph of $SG(2k + 2, k)$, Z_k is also a subgraph of $SG(2k + 2, k)$ (with the vertices located similarly in Y_k and Z_k in the two pictures representing the same vertex of $SG(2k + 2, k)$). Only

the edges connecting the top and bottom row change. Note that the horizontal edges on these vertices form a cycle C_{2k+2} in both cases. If k is even the vertical edges connect opposite vertices of this cycle in Y_k . In case k is odd the not-quite-vertical extra edges connect almost opposite vertices in the cycle. In both cases the subgraph formed by the cycle and these connecting edges is bipartite. In the case of Z_k the vertical edges give diagonals extending the cycle C_{2k+2} to a graph isomorphic to $P_{k+1} \square P_2$, which is always bipartite.

Let us add a remark here on the appearance that Z_k is a subgraph of Y_k at least in the case of even k . Indeed, one can delete two edges from the (picture of) Y_k if k is even to obtain the (picture of) Z_k . But this view is misleading. The two pictures may be equal but they are drawn in rectangles representing different surfaces and the change in the surface results in similarly placed segments joining different pairs of vertices. A similar relationship holds between Y_k and Z_k even if k is odd if we draw the edges of Z_k crossing the top edge of the bounding rectangle slanted as in Figure 9. This change of drawing does not effect the graph drawn (it is still Z_k quadrangulating the projective plane), but now the picture is contained in the usual picture representing Y_k (although in this case we cannot delete a few edges in Y_k to obtain the exact same picture). To emphasize that the view of considering Z_k as part of Y_k is wrong we note that though the graph Z_2 happens to be a subgraph of $Y_2 = \text{SG}(6, 2)$, but the similar relation is not true for Z_3 and Y_3 (shown on Figures 9 and 6) nor for Z_4 and Y_4 or beyond (for example for Z_6 and Y_6 shown on Figures 8 and 5).

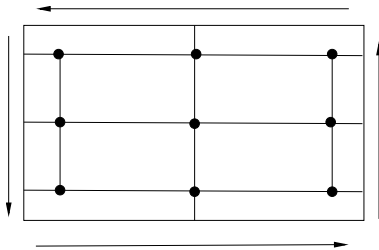


Figure 7: The subgraph Z_2 of $\text{SG}(6, 2)$ drawn as a quadrangulation of the projective plane.

Chromatic properties of surface quadrangulations are widely investigated, see for example [24, 1, 21, 22]. Higher dimensional quadrangulations of projective spaces are also investigated in [10, 11]. Youngs [24] proved the surprising result that any quadrangulation of the projective plane is either bipartite or 4-chromatic, that is, it simply cannot have chromatic number 3. Archdeacon, Hutchinson, Nakamoto, Negami, Ota [1] and independently Mohar and Seymour [21] generalized this result to what are called odd quadrangulations of nonorientable surfaces. They proved that if a graph G quadrangulates a nonorientable surface in such a way that cutting the surface along an odd cycle of G results in an orientable surface, then the chromatic number of G is at least 4. Note that these results,

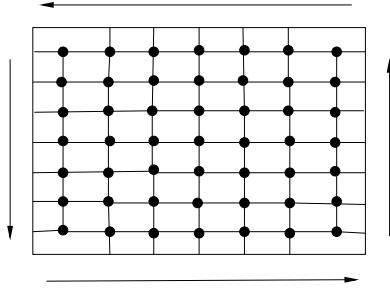


Figure 8: The subgraph Z_6 of $SG(14,6)$ drawn as a quadrangulation of the projective plane.

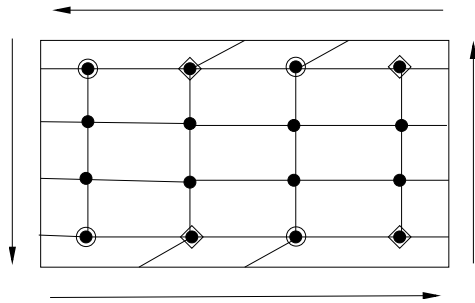


Figure 9: The subgraph Z_3 of $SG(8,3)$ drawn as a quadrangulation of the projective plane.

along with the observation above that a subgraph of every 4-chromatic Schrijver graph quadrangulates the Klein bottle, or the one that another subgraph quadrangulates the projective plane, already imply that these graphs, and thus also the graphs $\text{SG}(2k+2, k)$ are not 3-colorable. Youngs' theorem and the projective quadrangulations give this as Z_k is not bipartite, in case of quadrangulating the Klein bottle one also has to verify the property that cutting along the edges of some odd cycle the surface would become orientable. It is not hard to find such a cycle though in our pictures. (In Figures 4 and 5 any odd cycle formed by some "vertical line" would do, while only a little modification is needed in case of odd k .) We also remark that the proof given in [1] also makes use of the observation we stated in Lemma 7.

Kaiser and Stehlík [10] define quadrangulations of topological spaces in any dimension as an extension of the quadrangulation of surfaces and generalize Youngs' theorem for quadrangulations of the projective space of higher dimensions. In [11] they prove that a spanning subgraph of every Schrijver graph $\text{SG}(n, k)$ quadrangulates the projective space of dimension $n - 2k$. (With their generalization of Youngs' theorem this gives the right value for the chromatic number of $\text{SG}(n, k)$.) Their construction is quite involved and even in the case $n = 2k + 2$ which is relevant for us, it is not easy to compare it to our construction. (In fact, Figures 9 and 11 demonstrate that the spanning subgraph of $\text{SG}(2k + 2, k)$ quadrangulating the projective plane need not be unique.) They also conjecture that their spanning subgraph (for every n and k) is edge-color-critical and note that in case of $n = 2k + 2$ it follows from a result of Gimbel and Thomassen [8]. Indeed, Gimbel and Thomassen proved that if G is a graph in the projective plane such that all contractible cycles have length at least four, then G is 3-colorable if and only if G does not contain a nonbipartite quadrangulation of the projective plane. This implies that a nonbipartite quadrangulation of the projective plane is edge-color-critical if all of its contractible four-cycles are faces.

5.1 Critical edges in the Klein bottle quadrangulating subgraphs

As we have already mentioned at the end of the last section, a non-bipartite quadrangulation of the projective plane is edge-color-critical if all contractible 4-cycles are facial by a result of Gimbel and Thomassen [8]. This applies to the quadrangulations Z_k . Here we consider the subgraph Y_k of $\text{SG}(2k + 2, k)$ that quadrangulates the Klein bottle. We saw that Z_2 happens to be a proper subgraph of Y_2 but no similar relation holds between Z_k and Y_k for $k \geq 3$. This implies that Y_2 is not edge-color-critical, but does not say anything about Y_k for $k \geq 3$. We will see that the graph Y_3 (see on Figure 6) is *not* edge-color-critical either but Y_k is edge-color-critical for $k \geq 4$.

To see the structure of Y_k clearly we introduce the following graphs.

Definition 11. *For even values of k let L_k be obtained from the cycle C_{2k+2} by adding the extra edges connecting opposite vertices in this cycle. So L_k is the Möbius ladder M_{2k+2}*

in this case.

For odd values of k let us obtain L_k from the cycle C_{2k+2} by adding the extra edges $i(i+k)$ for $i \in [k+1]$.

Note that the graph L_k is bipartite for all values of k .

Considering $\text{SG}(2k+2, k)$ as the reduced drum $D'_{k+1, 2k+2}$ as described by Lemma 3, we can see (cf. the figures in the previous section) that Y_k is isomorphic to the graph we obtain from the reduced drum $D'_{k+1, 2k+2}$ if we change its bottom layer (a complete bipartite graph) to its subgraph isomorphic to L_k in the natural way (i.e., such that the vertex i of L_k corresponds to the vertex $(1, i)$ of $D'_{k+1, 2k+2}$). This means deleting some edges of $D'_{k+1, 2k+2}$ from its bottom layer. We identify L_k with this subgraph of Y_k so we can refer to extra edges in Y_k : these are the extra edges of L_k in the bottom layer.

Note that Y_2 is isomorphic to $\text{SG}(6, 2)$, therefore its critical edges are described by Theorem 24 (or, rather by Proposition 25). It has 3 non-critical edges. The following theorem deals with all larger values of k .

Theorem 27. *For $k \geq 4$ the subgraph Y_k of $\text{SG}(2k+2, k)$ quadrangulating the Klein bottle is edge-color-critical. In Y_3 all edges but two are critical.*

Proof. Set again $\text{SG} = \text{SG}(2k+2, k)$ for the sake of brevity of notation. Having the characterization of critical edges of SG by Proposition 25, we only have to prove that the non-critical edges of SG that still appear in Y_k for $k \geq 4$ become critical there (while exactly two of them become critical for $k = 3$). The edges of Y_k non-critical in SG are exactly the extra edges of L_k .

Let us denote by U_k the following graph.

$$V(U_k) := V(P_2 \square C_{2k+2})$$

and for even $k \geq 4$

$$E(U_k) := E(P_2 \square C_{2k+2}) \cup \{(1, i)(1, j) : \{i, j\} \in E(L_k)\},$$

while for odd $k \geq 3$

$$E(U_k) := E(P_2 \square C_{2k+2}) \cup \{(1, i)(1, j) : ij \in E(L_k)\} \cup \{(2, i)(2, i+k+1) : i \in [k+1]\}.$$

Note that the bottom layer of U_k induces L_k , while the top layer induces the cycle C_{2k+2} or the Möbius ladder M_{2k+2} depending on the parity of k . When speaking of an extra edge of U_k we mean an extra edge of L_k in the bottom layer of U_k .

We observe that it will be enough to prove that deleting any extra edge of U_k for odd $k \geq 5$, the remaining graph has a proper 3-coloring, while deleting any extra edge of U_k for even $k \geq 4$, the remaining graph has a proper 3-coloring in such a way that the opposite vertices of the induced C_{2k+2} of its second layer get identical colors. (The case of $k = 3$ will be dealt with separately, but in fact we have $Y_3 \cong U_3$.) This is because for odd $k \geq 5$ Y_k differs from U_k only by having some “middle layers” inducing cycles C_{2k+2}

between the two layers of U_k . However, once a 3-coloring of $U_k \setminus \{e\}$ is found for some extra edge e of U_k , then it can be extended to Y_k in a very simple way. Indeed, copy such a 3-coloring to the two bottommost layers of Y_k and once the j th layer is colored, color the $(j + 1)$ th layer by the rule $c(j + 1, i) := c(j, i + 1)$. This will give a proper 3-coloring of Y_k when k is odd. The even case is similar, the only difference is that the topmost layer of Y_k is in fact a cycle C_{k+1} of half length. But it is clear that its coloring is equivalent to coloring a double length cycle C_{2k+2} in such a way that the opposite vertices get identical colors. (Identifying these opposite vertices will give the required coloring of the half-length cycle.)

We consider first the case when $k \geq 4$ is even. We have to prove that if any extra edge is deleted from U_k , then the remaining graph has a proper 3-coloring c for which $c(2, i) = c(2, i + k + 1)$ for every $i \in [k + 1]$. We may assume by symmetry that we remove the edge $(1, 1)(1, k + 2)$, since an automorphism of U_k can bring this edge to any other extra edge of U_k while permuting the second layer within itself. We give the coloring c explicitly. Reading the colors of the vertices in the first layer in order we obtain

$$012(01)^{k/2-1}020(12)^{k/2-1},$$

while in the second layer we similarly obtain

$$201(20)^{k/2-1}201(20)^{k/2-1}.$$

Here the exponential notation $(ab)^i$ means that the colors a and b alternately repeat i times.

Note that this coloring gives the same color (namely 0) to both endpoints of the deleted edge $(1, 1)(1, k + 2)$, but creates no other monochromatic edge in U_k and gives identical colors to opposite vertices of the cycle in the second layer. See Figure 10 for an illustration of this 3-coloring.

Now assume k is odd. We should give a proper 3-coloring of the vertices of U_k after any one extra edge is deleted. Now U_k has only one non-trivial automorphism, so it is not enough to consider the removal of a single extra edge.

Let us first consider the removal of the extra edge $e = (1, i)(1, i + k)$ from U_k for some $2 \leq i \leq k$. Note that e is also an extra edge in U_{k-1} , so we have a proper 3-coloring c of $U_{k-1} \setminus \{e\}$ that gives the same color to opposite vertices in the second layer. We extend c to a proper 3-coloring of $U_k \setminus \{e\}$ as follows. We set $c(1, 2k + 1) := c(1, 1)$ and $c(2, 2k + 1) := c(2, 1)$ leaving the vertices $(1, 2k + 2)$ and $(2, 2k + 2)$ uncolored for now. We claim we have created no monochromatic edge yet. Indeed, the edges of U_k among the vertices $(1, j)$ for $j \in [2k]$ are also edges of U_{k-1} , so (other than the deleted edge e) none is monochromatic. All neighbors of the vertex $(1, 2k + 1)$ in the first layer of U_k are also neighbors of $(1, 1)$ in U_{k-1} , so these edges did not become monochromatic either. Clearly, no “vertical” edge $(1, j)(2, j)$ became monochromatic either for $j \in [2k + 1]$, nor did any of the edges $(2, j)(2, j + 1)$ for $j \in [2k]$. Finally, we have to check the

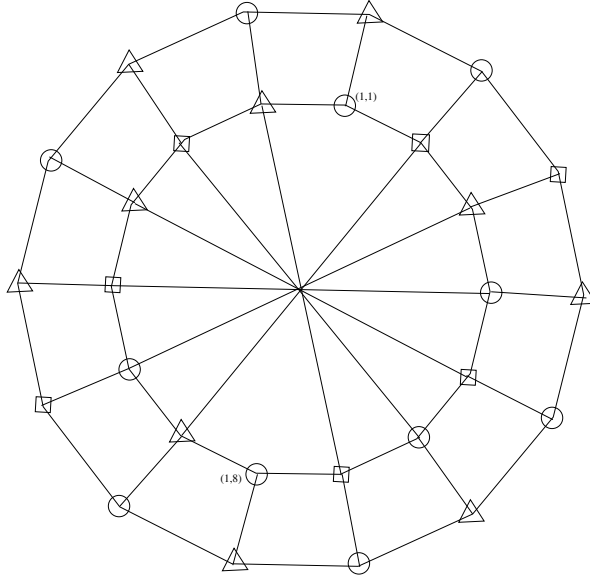


Figure 10: Our 3-coloring of $U_6 \setminus \{e\}$, $e = (1, 1)(1, 8)$ where opposite vertices of the second layer get identical colors. Colors 0, 1, and 2 are indicated by circle, square, and triangle, respectively.

“diagonal” edges $(2, j)(2, j + k + 1)$ for $j \in [k]$. these are not monochromatic, because $c(2, j) = c(2, j + k) \neq c(2, j + k + 1)$ by the properties of our coloring of $U_{k-1} \setminus \{e\}$.

It remains to further extend c to the two remaining vertices of $Y_k \setminus \{e\}$. We start with the vertex $(2, 2k + 2)$. It has three neighbors already colored, two of which are $(2, 1)$ and $(2, 2k + 1)$ with identical colors, so we can find a color for the vertex $(2, 2k + 2)$ not represented in its neighborhood and thus keeping our coloring proper. Finally we consider the last remaining vertex $(1, 2k + 2)$. It has three neighbors in U_k out of which $(1, 1)$ and $(1, 2k + 1)$ have the same color, so we can extend our proper 3-coloring to this vertex too. This proves that $\chi(U_k \setminus \{e\}) \leq 3$ as needed.

We still have to consider the case when the deleted edge e is one of $(1, 1)(1, k + 1)$ and $(1, k + 1)(1, 2k + 1)$. These two cases are symmetric (an automorphism of U_k takes one to the other), so we may assume that the deleted edge is the former one. We show an explicit proper 3-coloring c of $U_k \setminus \{e\}$ in this case. With the same notation as before, reading the colors of the vertices in the first layer in order we obtain

$$0(21)^{(k-1)/2}0(10)^{(k-1)/2}21,$$

while in the second layer we obtain

$$1(02)^{(k-1)/2}1(01)^{(k-1)/2}02.$$

This coloring gives color 0 to both endpoints of e , but makes no other edge of U_k monochromatic. This makes c a proper 3-coloring of $U_k \setminus \{e\}$ and finishes the proof that Y_k is edge-color critical for $k \geq 4$.

Finally, we consider the case $k = 3$, namely the graph $Y_3 = U_3$. The 3-coloring given explicitly in the previous paragraph works in the $k = 3$ case, too, that is, it gives a proper 3-coloring of $Y_3 \setminus \{(1, 1)(1, 4)\}$ and thus it shows that the edge $(1, 1)(1, 4)$ is critical in Y_3 . By symmetry, the edge $(1, 4)(1, 7)$ is also critical in Y_3 . However, neither of the two remaining extra edges, namely $(1, 2)(1, 5)$ and $(1, 3)(1, 6)$ are critical. Even if we delete both of these edges, the remaining graph Q is 4-chromatic, albeit not isomorphic to Z_3 . One way to see that Q has no proper 3-coloring is to notice that the remaining two extra edges, that is, $(1, 1)(1, 4)$ and $(1, 4)(1, 7)$ divide the cycle C_8 in the bottom layer into three quadrangles and then applying the argument of Lemmata 7, 8, and 11 on the two layers of Q . Alternatively, one can show that Q quadrangulates the projective plane (as Z_3 does also) and (since it is not bipartite) we have $\chi(Q) = 4$ by Youngs' theorem. This completes the proof of the theorem. \square

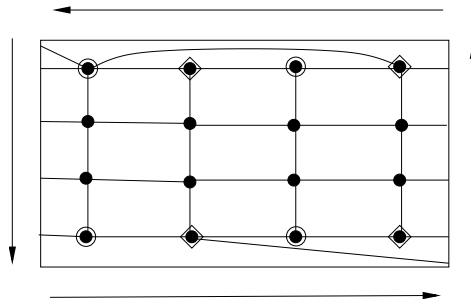


Figure 11: A subgraph of $SG(8, 3)$ quadrangulating the projective plane different from the one on Figure 9. (Note that unlike the graph on Figure 9 this one has a vertex of degree 5.) This is the quadrangulation mentioned at the end of the proof of Theorem 27.

6 Open problems

In this final section we come back to the question of critical edges of Schrijver graphs of higher chromatic numbers. Recall that our results characterized them for 4-chromatic Schrijver graphs as being the interlacing edges (Theorem 24), but this characterization does not hold in general (see Theorem 26). We cannot even come up with a conjecture for the full characterization, nevertheless, we formulate an open problem (that, if true, would generalize Theorem 23) and a conjecture that would characterize the critical edges for large Schrijver graphs with small chromatic number.

Problem 1. *Are all edges incident to a regular vertex in a Schrijver graph critical?*

Conjecture 1. *For any d and large enough k (that is $k > k_d$ for some threshold k_d depending on d) all the interlacing edges of the Schrijver graph $SG(2k + d, k)$ are critical.*

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