- poles
- essential singularities.

The first type is harmless since a function can actually be extended to be holomorphic at its removable singularities (hence the name). Near the third type, the function oscillates and may grow faster than any power, and a complete understanding of its behavior is not easy. For the second type the analysis is more straight forward and is connected with the calculus of residues, which arises as follows.

Recall that by Cauchy's theorem a holomorphic function f in an open set which contains a closed curve γ and its interior satisfies

$$\int_\gamma f(z)\,dz = 0.$$

The question that occurs is: what happens if f has a pole in the interior of the curve? To try to answer this question consider the example $f(z) = 1/z$, and recall that if C is a (positively oriented) circle centered at 0, then

$$\int_C \frac{dz}{z} = 2\pi i.$$

This turns out to be the key ingredient in the calculus of residues.

A new aspect appears when we consider indefinite integrals of holomorphic functions that have singularities. As the basic example $f(z) = 1/z$ shows, the resulting "function" (in this case the logarithm) may not be single-valued, and understanding this phenomenon is of importance for a number of subjects. Exploiting this multi-valuedness leads in effect to the "argument principle." We can use this principle to count the number of zeros of a holomorphic function inside a suitable curve. As a simple consequence of this result, we obtain a significant geometric property of holomorphic functions: they are open mappings. From this, the maximum principle, another important feature of holomorphic functions, is an easy step.

In order to turn to the logarithm itself, and come to grips with the precise nature of its multi-valuedness, we introduce the notions of homotopy of curves and simply connected domains. It is on the latter type of open sets that single-valued branches of the logarithm can be defined.

1 Zeros and poles

By definition, a **point singularity** of a function f is a complex number z_0 such that f is defined in a neighborhood of z_0 but not at the point z_0 itself. We shall also call such points **isolated singularities**. For example, if the function f is defined only on the punctured plane by $f(z) = z$, then the origin is a point singularity. Of course, in that case, the function f can actually be defined at 0 by setting $f(0) = 0$, so that the resulting extension is continuous and in fact entire. (Such points are then called removable singularities.) More interesting is the case of the function $g(z) = 1/z$ defined in the punctured plane. It is clear now that g cannot be defined as a continuous function, much less as a holomorphic function, at the point 0. In fact, $g(z)$ grows to infinity as z approaches 0, and we shall say that the origin is a pole singularity. Finally, the case of the function $h(z) = e^{1/z}$ on the punctured plane shows that removable singularities and poles do not tell the whole story. Indeed, the function $h(z)$ grows indefinitely as z approaches 0 on the positive real line, while h approaches 0 as z goes to 0 on the negative real axis. Finally h oscillates rapidly, yet remains bounded, as z approaches the origin on the imaginary axis.

Since singularities often appear because the denominator of a fraction vanishes, we begin with a local study of the zeros of a holomorphic function.

A complex number z_0 is a **zero** for the holomorphic function f if $f(z_0) = 0$. In particular, analytic continuation shows that the zeros of a non-trivial holomorphic function are isolated. In other words, if f is holomorphic in Ω and $f(z_0) = 0$ for some $z_0 \in \Omega$, then there exists an open neighborhood U of z_0 such that $f(z) \neq 0$ for all $z \in U - \{z_0\}$, unless f is identically zero. We start with a local description of a holomorphic function near a zero.

Theorem 1.1 *Suppose that f is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then there exists a neighborhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z) \quad \textit{for all } z \in U.$$

Proof. Since Ω is connected and f is not identically zero, we conclude that f is not identically zero in a neighborhood of z_0. In a small disc centered at z_0 the function f has a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Since f is not identically zero near z_0, there exists a smallest integer n

such that $a_n \neq 0$. Then, we can write

$$f(z) = (z - z_0)^n [a_n + a_{n+1}(z - z_0) + \cdots] = (z - z_0)^n g(z),$$

where g is defined by the series in brackets, and hence is holomorphic, and is nowhere vanishing for all z close to z_0 (since $a_n \neq 0$). To prove the uniqueness of the integer n, suppose that we can also write

$$f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z)$$

where $h(z_0) \neq 0$. If $m > n$, then we may divide by $(z - z_0)^n$ to see that

$$g(z) = (z - z_0)^{m-n} h(z)$$

and letting $z \to z_0$ yields $g(z_0) = 0$, a contradiction. If $m < n$ a similar argument gives $h(z_0) = 0$, which is also a contradiction. We conclude that $m = n$, thus $h = g$, and the theorem is proved.

In the case of the above theorem, we say that f has a **zero of order** n (or **multiplicity** n) at z_0. If a zero is of order 1, we say that it is **simple**. We observe that, quantitatively, the order describes the rate at which the function vanishes.

The importance of the previous theorem comes from the fact that we can now describe precisely the type of singularity possessed by the function $1/f$ at z_0.

For this purpose, it is now convenient to define a **deleted neighborhood** of z_0 to be an open disc centered at z_0, minus the point z_0, that is, the set

$$\{z : 0 < |z - z_0| < r\}$$

for some $r > 0$. Then, we say that a function f defined in a deleted neighborhood of z_0 has a **pole** at z_0, if the function $1/f$, defined to be zero at z_0, is holomorphic in a full neighborhood of z_0.

Theorem 1.2 *If f has a pole at $z_0 \in \Omega$, then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that*

$$f(z) = (z - z_0)^{-n} h(z).$$

Proof. By the previous theorem we have $1/f(z) = (z - z_0)^n g(z)$, where g is holomorphic and non-vanishing in a neighborhood of z_0, so the result follows with $h(z) = 1/g(z)$.

The integer n is called the **order** (or **multiplicity**) of the pole, and describes the rate at which the function grows near z_0. If the pole is of order 1, we say that it is **simple**.

The next theorem should be reminiscent of power series expansion, except that now we allow terms of negative order, to account for the presence of a pole.

Theorem 1.3 *If f has a pole of order n at z_0, then*

$$(1) \qquad f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + G(z),$$

where G is a holomorphic function in a neighborhood of z_0.

Proof. The proof follows from the multiplicative statement in the previous theorem. Indeed, the function h has a power series expansion

$$h(z) = A_0 + A_1(z - z_0) + \cdots$$

so that

$$\begin{aligned} f(z) &= (z - z_0)^{-n}(A_0 + A_1(z - z_0) + \cdots) \\ &= \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + G(z). \end{aligned}$$

The sum

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)}$$

is called the **principal part** of f at the pole z_0, and the coefficient a_{-1} is the **residue** of f at that pole. We write $\text{res}_{z_0} f = a_{-1}$. The importance of the residue comes from the fact that all the other terms in the principal part, that is, those of order strictly greater than 1, have primitives in a deleted neighborhood of z_0. Therefore, if $P(z)$ denotes the principal part above and C is any circle centered at z_0, we get

$$\frac{1}{2\pi i} \int_C P(z)\, dz = a_{-1}.$$

We shall return to this important point in the section on the residue formula.

As we shall see, in many cases, the evaluation of integrals reduces to the calculation of residues. In the case when f has a simple pole at z_0, it is clear that

$$\text{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$