Practice Set

1. For each of the following functions $u = u(x, y)$ and $v = v(x, y)$, check if the Cauchy-Riemann equations hold. If they do hold, find a function $f(z) = u + iv$. (Here $z = x + iy$ as usual.)
   a.) $u = y,\ v = -x$
   b.) $u = 5x - 3y,\ v = 3x + 5y$
   c.) $u = \cos(x)\sin(y)$ and $v = \sin(x)\cos(y)$
   d.) $u = x^3 - 3xy^2 + 1$ and $v = 3x^2y - y^3$

2. Verify the Cauchy-Riemann equations for $f(z) = z^3, e^z$.

3. Show the real part of $z^{\frac{1}{2}}$ is always positive.

4. Suppose that $f : \mathbf{C} \to \mathbf{C}$ is a function which satisfies $f(z) = f(\frac{z}{2})$ for all $z \in \mathbf{C}$ and which is continuous at $z = 0$. Show that f is a constant function.

5. Calculate principal values for $i^i \cdot (i-1)^i$ and $[i(i-1)]^i$. Deduce that $(ab)^c \neq a^c b^c$, in general. (ref: hw2)

6. Consider the series $e^z = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$ and $e^w = \sum\limits_{n=0}^{\infty} \frac{w^n}{n!}$. Use the multiplication thm of power series (as mentioned in class), to verify that $e^z \cdot e^w = e^{z+w}$.

7. If you have not yet done so, verify $e^{iz} = \cos z + i \sin z$ for all z.

8. Show that in polar coordinates, the Cauchy-Riemann equations take the form
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$
   Use these equations to show that the logarithm function defined by
$$\log z = \log r + i\theta$$
   where $z = re^{i\theta}$ with $-\pi < \theta < \pi$ is holomorphic on the region $r > 0$ and $-\pi < \theta < \pi$.
   Will clarify this in class.

9. Suppose that f is holomorphic in an open set Ω. Prove that $Im(f)$ =constant implies that f is constant.

10. Suppose that f is holomorphic in an open set Ω. Prove that $|f|$ =constant implies that f is constant.

11. *On the relationship between a holomorphic function $f(z) : \mathbf{C} \to \mathbf{C}$ and the corresponding $F : \mathbf{R}^2 \to \mathbf{R}^2$ map.*
    a.) Show that for the Jacobian J of F we have, $Jh = f'(z)h$. What is the meaning of this equation???
    b.) Show $\det J = |f'|^2$.

12. Investigate differentiability of $f(z) = \frac{z}{\overline{z}}$.

13. Prove that $f(z) = \overline{z}(|z|^2 - 2)$ is differentiable only on $|z| = 1$. Show $f'(z) = \overline{z}^2$ for these z.

14. Suppose we know that $f : \mathbf{C} \to \mathbf{C}$ is holomorphic and that for $f = u + iv$ we have $u(x, y) = x^3 - 3xy^2$. What is $f'(z)$?

15. If $f : \mathbf{C} \to \mathbf{C}$ is complex differentiable at $a \in \mathbf{C}$ then $g(z) = \overline{f(\overline{z})}$ is complex differentiable at $\overline{a} \in \mathbf{C}$ and we have $g'(\overline{a}) = \overline{f'(a)}$.

16. Sometimes complex numbers can be used to prove statements that do not appear to be about complex numbers at all, such as the extra credit problem in hw1 (closed formula for $1 + \cos\phi + ... + \cos n\phi$). Here are two more examples.
a.) Prove that

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$

b.) Compute the value of

$$\sum_{k=0}^{1004} (-1)^k \binom{2010}{2k+1} = \binom{2010}{1} - \binom{2010}{3} + \binom{2010}{5} - ... + \binom{2010}{2009}$$

Hint: find $(1 + i)^{2010}$.