PRACTICE SET 3.

1. One of the first facts one learns about differentiation of functions of a real variable is that $f(x) = |x|$ is continuous but not differentiable at 0. Explain why there cannot be a similar complex function (i.e. one that is continuous, but not holomorphic at the origin, while it is holomorphic in a punctured neighbourhood of zero.

2. a.) Find (using any method you like) $\int\limits_{|z|=10} \frac{1}{(z-i)(z-2i)(z-3i)}\, dz$

   b.) Use an appropriate estimate to find $\int\limits_{|z|=R} \frac{1}{z^2-i}\, dz$ for large $R$.

   c.) Explain why in fact for any polynomial $p(z)$ of degree at least 2, there is an $R_0$ such that if $R > R_0$, then $f(z) = \frac{1}{p(z)}$ is holomorphic on $\{|z| > R\}$ and if $\gamma$ is a circle of radius $R$ then you should get the same answer for $\int\limits_{|z|=R} \frac{1}{p(z)}\, dz$ as in part a.) and b.)

   c.) What about degree 1 polynomials?

3. Show that if $g$ is holomorphic at $z_0$ and $f$ has a simple pole there, then
$$Res_{z_0} gf = g(z_0)\, Res_{z_0} f.$$

4. Find $z$ such that $|f(z)|$ is maximal for $|z| \le 1$ where $f(z) = z^2 - 3iz$.

5. a.) Give an example of a domain $\Omega$ and a holomorphic function $f : \Omega \to \mathbf{C}$ such that $|f(z)|$ has a minimum in $\Omega$, but $f(z)$ isn't constant. (i.e. there is no "minimum modulus theorem")
   b.) Suppose that $\Omega$ is a domain and $f : \Omega \to \mathbf{C}$ is a never zero holomorphic function on $\Omega$ that has a minimum in $\Omega$. Show that in that case $f$ is constant.

6. Let $\gamma$ be any closed piecewise smooth path, and let $f$ be holomorphic on $\gamma$ such that $|f(w)| < M$ for $w \in \gamma$. Prove that if $|z| > M$ then
$$\int_\gamma \frac{f'(w)}{f(w) - z}\, dw = 0.$$
   *Hint:* What does this integral represent in terms of winding numbers?.

7. Evaluate $\int\limits_\gamma \frac{e^z - e^{-z}}{z}\, dz$ on each of the curves $\gamma$ given in a jpg file that you can download separately.

8. Let $P(z) = z^7 - 2z^5 + 6z^3 - z + 1$. How many of its roots (counted with multiplicity) are in the open disc $B(0, 1)$? Explain.

9. Let $\gamma : [0, 2] \to \mathbf{C}$ be defined by
$$\gamma(t) = \begin{cases} e^{2\pi i t} & \text{if} \quad t \in [0, 1] \\ -2 + 3 \cdot e^{2\pi i t} & \text{if} \quad t \in [1, 2] \end{cases}$$
   a.) Sketch $\gamma$, indicating orientation.

   b.) Use the definition of winding numbers to determine **(i)** $n(\gamma, 0)$, **(ii)** $n(\gamma, -4)$, **(iii)** $n(\gamma, 2)$.

10. Prove that the smooth paths $\gamma_1(t) = 2 + 3e^{it}$ and $\gamma_2(t) = 3 + 2e^{it}$, $t \in [0, 2\pi]$ are not homotopic in $\mathbf{C} \setminus \{0\}$

11. Assume that $f : \mathbf{C} \to \mathbf{C}$ is entire and $f(z) = f(z + 1) = f(z + i)$ for all $z$. Show that $f(2009\pi + \frac{i}{\sqrt{2}}) = f(0)$.

12. Suppose that $f : \Omega \to \mathbf{C}$ is holomorphic and $z_0 \in \Omega$ is a zero of order $n$ of the equation $f(z) - w_0 = 0$ for some $w_0 \in \mathbf{C}$.

    Show that if $f$ is not constant then there is an open disc $D_2$ centered at $w_0$ in $\mathbf{C}$ where each $w \in D_2$ is in the image of $f$ and each $w$ has $n$ preimages (counted with multiplicity).