Homework Set 7.

1. Suppose a complex function is of the form $f = \frac{g}{h}$ with g and h holomorphic on an open disc around $z_0 \in \mathbf{C}$. Suppose f has an isolated singularity at z_0 and $g(z_0) = 0$ there.

   a.) Put extra conditions "of your choice" on h, and g in order to obtain a formula for $Res_{z_0}(f)$ in terms of h and g. Prove your formula.

   (So there is a variation on the kind of statements one can get, but of course you do not have entire freedom on the conditions on h and g.)

   b.) Illustrate your formula on an example.

2. a.) Determine singularities and residues of $f(z) = \frac{\text{cosec}z}{z^2}$, where $\text{cosec}z = \frac{1}{\sin z}$, as usual.

   b.) For $n = 1, 2, ...$ let γ_n be the square of side-length $(2n+1)\pi$; with sides parallel to the real and imaginary axes; going through $\pm(n + \frac{1}{2})\pi$ (same as the one we used in class).

   Show that $\text{cosec}z = \frac{1}{\sin z}$ is bounded on γ_n for every n.

   c.) Use the integral of $f(z) = \frac{\text{cosec}z}{z^2}$ over γ_n and parts a.) and b.) to calculate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

3. Use any method you like to find

   a.)

$$\int_{|z+1|=2} \frac{e^z}{(z+1)^{34}}\, dz$$

   b.)

$$\int_{|z-1|=1} \left(\frac{z}{z-1}\right)^n dz$$

   where $n \geq 0$ integer;

   c.)

$$\int_{|z|=10} \frac{2z+5}{z^2+5z+4}\, dz$$