THE LANGUAGE OF SCHEMES
Homework 4

1. This exercise assembles some facts needed for the solution of the next one.

a) Let $\mathcal{F}$ be a quasi-coherent sheaf on an integral scheme X. Show that there is a unique subsheaf $\mathcal{F}_{\text{tors}} \subset \mathcal{F}$ such that for all affine open subsets $U = \text{Spec}(A) \subset X$

$$\mathcal{F}_{\text{tors}}(U) = \{s \in \mathcal{F}(U) \,:\, as = 0 \,\text{for some}\; a \in A\}.$$

b) Show that if moreover X is a smooth curve over a field and $\mathcal{F}$ is coherent, then $\mathcal{F}/\mathcal{F}_{\text{tors}}$ is locally free. [*Hint:* You will need the algebraic fact that a finitely generated torsion-free module over a principal ideal ring is free.]

2. Let X be a smooth projective curve over a field k and $\mathcal{F}$ a locally free sheaf of rank r on X.

a) Assume $\mathcal{F}$ has a global section $s \in \mathcal{F}(X)$. Set

$$\mathcal{G} := \mathcal{F}/(\mathcal{O}_X s), \quad \mathcal{F}'' := \mathcal{G}/\mathcal{G}_{\text{tors}}, \quad \mathcal{F}' := \ker(\mathcal{F} \to \mathcal{F}'').$$

Show that

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of locally free sheaves of ranks $1, r, r-1$, respectively.

b) Show that an exact sequence with the above properties exists for any locally free sheaf $\mathcal{F}$ of rank r on X. [*Hint:* Tensor with $\mathcal{O}_X(n)$ for large n.]

3. If $\mathcal{F}$ is a locally free sheaf of rank r on a scheme X, define an invertible sheaf $\det(\mathcal{F})$ by choosing an affine open covering $\{U_i \,:\, i \in I\}$ of X such that $\mathcal{F}|_{U_i}$ is free for all i and setting $\det(\mathcal{F})|_{U_i} := (\wedge^r(\mathcal{F}(U_i)))^{\sim}$.

[It is easy to see that this definition does not depend on the choice of the U_i.]

Verify that if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of locally free sheaves, then

$$\det(\mathcal{F}) \cong \det(\mathcal{F}') \otimes_{\mathcal{O}_X} \det(\mathcal{F}'').$$

b) Given a locally free sheaf $\mathcal{F}$ of rank r on a smooth projective curve X, prove by induction on r the *generalized Riemann-Roch formula:*

$$\chi(\mathcal{F}) - r\chi(\mathcal{O}_X) = \deg(\det(\mathcal{F})).$$

4. Let X be a smooth projective curve of genus g over an algebraically closed field. Show that there is a finite morphism $X \to \mathbf{P}^1$ of degree at most $g+1$ (i.e. the induced extension $K(X)|k(t)$ has degree $\leq g+1$).

5. Let X be a smooth projective curve of genus 3 over an algebraically closed field k of characteristic $\neq 2$.

a) Check that either X is isomorphic to a smooth plane curve of degree 4 (i.e. defined by a homogeneous polynomial of degree 4), or X is hyperelliptic (i.e. it has a finite morphism of degree 2 onto $\mathbf{P}^1$).

b) In the latter case show that $K(X) \cong k(t)[y]$, where $y^2 = f$ for some polynomial $f \in k[t]$. Find the possible degrees of f.