LECTURES ON LINEAR ALGEBRAIC GROUPS

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This is a somewhat expanded version of the notes from a course given at the Eötvös and Technical Universities of Budapest in the spring of 2006. Its aim was to provide a quick introduction to the main structural results for affine algebraic groups over algebraically closed fields with full proofs but assuming only a very modest background. The necessary techniques from algebraic geometry are developed from scratch along the way. In fact, some readers may regard the text as a good example of applying the basic theory of quasi-projective varieties in a nontrivial way; the low-dimensional varieties usually discussed in introductory courses do not always provide a suitable testing ground.

The experts should be warned at once: there is no theorem here that cannot be found in the standard textbooks of Borel, Humphreys or Springer. There are some differences, however, in the exposition. We do not leave the category of quasi-projective varieties for a single moment, and carry a considerably lighter baggage of algebraic geometry than the above authors. Lie algebra techniques are not used either. Finally, for two of the main theorems we have chosen proofs that have been somewhat out of the spotlight for no apparent reason. Thus for Borel's fixed point theorem we present Steinberg's beautiful correspondence argument which reduces the statement to a slightly enhanced version of the Lie-Kolchin theorem, and for the conjugacy of maximal tori we give Grothendieck's proof from Séminaire Chevalley that replaces a lot of technicalities by a clever use of basic facts about group extensions.

We work over an algebraically closed base field throughout. We feel that the case of a general base field should be treated either within the more general framework of affine group schemes, or together with methods of Galois cohomology. Both are outside the modest scope of these notes.

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Chapter 1. Basic Notions

The concept of a linear algebraic group may be introduced in two equivalent ways. One is to define it as a subgroup of some general linear group GL_n which is closed for the Zariski topology. The other, more intrinsic approach is to say that a linear algebraic group is a group object in the category of affine varieties. In this chapter we develop the foundational material necessary for making the two definitions precise, and prove their equivalence. Some basic examples are also discussed.

1. Affine varieties

Throughout these notes we shall work over an *algebraically closed* field k. We identify points of *affine n-space* $\mathbf{A}^n_k$ with

$$\{(a_1, \dots, a_n) : a_i \in k\}.$$

Given an ideal $I \subset k[x_1, \dots, x_n]$, set

$$V(I) := \{P = (a_1, \dots, a_n) \in \mathbf{A}^n : f(P) = 0 \quad \text{for all } f \in I\}.$$

Definition 1.1. *$X \subset \mathbf{A}^n$ is an* affine variety *if there exists an ideal $I \subset k[x_1, \dots, x_n]$ for which $X = V(I)$.*

The above definition is not standard; a lot of textbooks assume that I is in fact a prime ideal.

According to the Hilbert Basis Theorem there exist finitely many polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ with $I = (f_1, \dots, f_m)$. Therefore

$$V(I) = \{P = (a_1, \dots, a_n) \in \mathbf{A}^n : f_i(P) = 0 \quad i = 1, \dots, m\}.$$

The following lemma is easy.

Lemma 1.2. *Let $I_1, I_2, I_\lambda\ (\lambda \in \Lambda)$ be ideals in $k[x_1, \dots, x_n]$. Then*

- $I_1 \subseteq I_2 \Rightarrow V(I_1) \supseteq V(I_2)$;
- $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$;
- $V(\langle I_\lambda : \lambda \in \Lambda \rangle) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

The last two properties imply that the affine varieties may be used to define the closed subsets in a topology on X (note that $\mathbf{A}^n = V(0)$, $\emptyset = V(1)$). This topology is called the *Zariski topology* on $\mathbf{A}^n$, and affine varieties are equipped with the induced topology.

Next another easy lemma.

Lemma 1.3. *The open subsets of the shape*

$$D(f) := \{P \in \mathbf{A}^n : f(P) \neq 0\},$$

where $f \in k[x_1, \dots, x_n]$ is a fixed polynomial, form a basis of the Zariski topology on $\mathbf{A}^n$.

Now given an affine variety $X \subset \mathbf{A}^n$, set

$$I(X) := \{f \in k[x_1, \ldots, x_n] : f(P) = 0 \text{ for all } P \in X\}.$$

Now arises the natural question: which ideals $I \subset k[x_1, \ldots, x_n]$ satisfy $I(V(I)) = I$? An obvious necessary condition is that $f^m \in I$ should imply $f \in I$ for all $m > 0$, i.e. I should equal its *radical* $\sqrt{I}$. Such ideals are called *radical ideals.*

The condition is in fact sufficient:

Theorem 1.4. (Hilbert Nullstellensatz) $I = \sqrt{I} \Leftrightarrow I(V(I)) = I$.

According to the Noether-Lasker theorem, given an ideal I with $I = \sqrt{I}$, there exist *prime ideals* $P_1, \ldots, P_r \subset k[x_1, \ldots, x_n]$ satisfying $I = P_1 \cap \cdots \cap P_r$. (Recall that an ideal I is a prime ideal if $ab \in I$ implies $a \in I$ or $b \in I$.) Since in this case $V(I) = \cup V(P_i)$, it is enough to prove the theorem in the case when I is a prime ideal.

This we shall do under the following additional condition:

$(*)$ There exists a subfield $F \subset k$ with tr.deg.$(k|F) = \infty$.

(Recall that this last condition means that there exist infinite systems of elements in k that are algebraically independent over F, i.e. there is no polynomial relation with F-coefficients among them.)

The condition $(*)$ is satisfied, for instance, for $k = \mathbf{C}$. See Remark 1.6 below on how to get rid of it.

Lemma 1.5. *Let $I \subset k[x_1, \ldots, x_n]$ be a prime ideal, and $F \subset k$ a subfield satisfying $(*)$. Then there exists $P \in V(I)$ such that*

$$f \in F[x_1, \ldots, x_n],\ f(P) = 0 \Rightarrow f \in I.$$

Proof. Choose a system of generators $f_1, \ldots, f_m$ for I. By adjoining the coefficients of the f_i to F we may assume $f_i \in F[x_1, \ldots, x_n]$ for all i without destroying the assumptions. Set $I_0 := I \cap F[x_1, \ldots, x_n]$. This is again a prime ideal, so $F[x_1, \ldots, x_n]/I_0$ is a finitely generated F-algebra which is an integral domain. Denoting by F_0 its fraction field, we have a finitely generated field extension $F_0|F$, so $(*)$ implies the existence of an F-embedding $\phi : F_0 \hookrightarrow k$. Denote by $\bar{x}_i$ the image of x_i in F_0 and set $a_i := \phi(\bar{x}_i)$, $P = (a_1, \ldots, a_n)$. By construction $f_i(P) = 0$ for all i, so $P \in V(I)$, and for $f \in F[x_1, \ldots, x_n] \setminus I_0$ one has $f(\bar{x}_1, \ldots, \bar{x}_n) \neq 0$, and therefore $f(P) \neq 0$. □

Proof of the Nullstellensatz assuming $(*)$*:* Pick $f \in I(V(I))$ and $F \subset k$ satisfying $(*)$ such that $f \in F[x_1, \ldots, x_n]$. If $f \notin I$, then for $P \in V(I)$ as in the above lemma one has $f(P) \neq 0$, which contradicts $f \in I(V(I))$. □

Remark 1.6. Here is how to eliminate $(*)$ using mathematical logic. If k does not satisfy $(*)$, let $\Omega = k^I/\mathcal{F}$ an *ultrapower* of k which is big enough to satisfy $(*)$. According to the Łoş lemma Ω is again algebraically closed, so the Nullstellensatz holds over it. Again by the Loş lemma we conclude that it holds over k as well.

Corollary 1.7. *The rule $I \mapsto V(I)$ is an order reversing bijection between ideals in $k[x_1, \dots x_n]$ satisfying $I = \sqrt{I}$ and affine varieties in $\mathbf{A}^n$. Maximal ideals correspond to points, so in particular each maximal ideal is of the form $(x_1 - a_1, \dots, x_n - a_n)$.*

Lemma 1.8. *$I \subset k[x_1, \dots, x_n]$ is a prime ideal if and only if $V(I)$ is an irreducible closed subset in $\mathbf{A}^n$.*

Recall that Z is an irreducible closed subset if there exists no decomposition $Z = Z_1 \cup Z_2$ with the Z_i closed and different from Z.

Proof. Look at the decomposition $I = P_1 \cap \cdots \cap P_r$ given by the Noether-Lasker theorem. If I is not a prime ideal, then $r > 1$, whence a nontrivial decomposition $V(I) = \cup V(P_i)$. On the other hand, if $V(I) = Z_1 \cup Z_2$ nontrivially, then by the Nullstellensatz I is the nontrivial intersection of two ideals that equal their own radical, whence r must be > 1. □

Corollary 1.9. *Each affine variety X is a finite union of irreducible varieties. These are uniquely determined, and are called the* irreducible components *of X.*

Proof. In view of the lemma and the Nullstellensatz, this follows from the Noether-Lasker theorem. □

Example 1.10. For $I = (x_1 x_2) \subset k[x_1, x_2]$ one has

$$V(I) = V((x_1)) \cup V((x_2)).$$

Definition 1.11. *If X is an affine variety, the quotient*

$$\mathcal{A}_X := k[x_1, \dots, x_n]/I(X)$$

is called the coordinate ring *of X.*

Note that since $I(X)$ is a radical ideal, the finitely generated k-algebra $\mathcal{A}_X$ is *reduced*, i.e. it has no nilpotent elements.

The elements of $\mathcal{A}_X$ may be viewed as functions on X with values in k; we call them *regular functions*. Among these the images of the x_i are the restrictions of the coordinate functions of $\mathbf{A}^n$ to X, whence the name. Note that X is a variety if and only if $\mathcal{A}_X$ is *reduced* (i.e. has no nilpotents).

Definition 1.12. *Given an affine variety $X \subset \mathbf{A}^n$, by a* morphism *or* regular map *$X \to \mathbf{A}^m$ we mean an m-tuple $\phi = (f_1, \dots, f_m) \in \mathcal{A}_X^m$. Given an affine variety $Y \subset \mathbf{A}^m$, by a morphism $\phi : X \to Y$ we mean a morphism $\phi : X \to \mathbf{A}^m$ with $\phi(P) := (f_1(P), \dots, f_m(P)) \in Y$ for all $P \in X$.*

Lemma 1.13. *A morphism $\phi : X \to Y$ is continuous in the Zariski topology.*

Proof. It is enough to show that the preimage of each basic open set $D(f) \subset Y$ is open. This is true, because $\phi^{-1}(D(f)) = D(f \circ \phi)$, where $f \circ \phi \in \mathcal{A}_X$ is the regular function obtained by composition. □

Definition 1.14. *Given affine varieties $X \subset \mathbf{A}^n$, $Y \subset \mathbf{A}^m$, the Cartesian product $X \times Y \subset \mathbf{A}^n \times \mathbf{A}^m \cong \mathbf{A}^{n+m}$ is called the* product *of X and Y.*

Lemma 1.15. *The product $X \times Y \subset \mathbf{A}^{n+m}$ is an affine variety.*

Proof. If $X = V(f_1, \ldots, f_r)$ and $Y = V(g_1, \ldots, g_s)$, then

$$X \times Y = V(f_1, \ldots, f_r, g_1, \ldots, g_s).$$

□

2. Affine algebraic groups

Definition 2.1. *An* affine (or linear) algebraic group *is an affine variety G equipped with morphisms $m : G \times G \to G$ ('multiplication') and $i : G \to G$ ('inverse') satisfying the group axioms.*

Examples 2.2.

(1) The additive group $\mathbf{G}_a$ of k. As a variety it is isomorphic to $\mathbf{A}^1$, and $(x, y) \to (x + y)$ is a morphism from $\mathbf{A}^1 \times \mathbf{A}^1$ to $\mathbf{A}^1$.

(2) The multiplicative group $\mathbf{G}_m$ of k. As a variety, it is isomorphic to the affine hyperbola $V(xy - 1) \subset \mathbf{A}^2$.

(3) The subgroup of n-th roots of unity $\mu_n \subset \mathbf{G}_m$ for n invertible in k. As a variety, it is isomorphic to $V(x^n - 1) \subset \mathbf{A}^1$. It is *not* irreducible, not even connected (it consists of n distinct points).

(4) The group of invertible matrices GL_n over k is also an algebraic group (note that $\mathrm{GL}_1 = \mathbf{G}_m$). To see this, identify the set $M_n(k)$ of $n \times n$ matrices over k with points of $\mathbf{A}^{n^2}$. Then

$$\mathrm{GL}_n \cong \{(A, x) \in \mathbf{A}^{n^2+1} : \det(A)x = 1\}.$$

This is a closed subset, since the determinant is a polynomial in its entries.

(5) SL_n is also an algebraic group: as a variety,

$$\mathrm{SL}_n \cong \{A \in \mathbf{A}^{n^2} : \det(A) = 1\}.$$

Similarly, we may realise $\mathrm{O}_n, \mathrm{SO}_n$, etc. as affine algebraic groups.

Proposition 2.3. *Let G be an affine algebraic group.*

(1) *All connected components of G (in the Zariski topology) are irreducible. In particular, they are finite in number.*

(2) *The component G° containg the identity element is a normal subgroup of finite index, and its cosets are exactly the components of G.*

Proof. (1) Let $G = X_1 \cup \cdots \cup X_r$ be the decomposition of G into *irrreducible* components. As the decomposition is irredundant, $X_1 \not\subset X_j$ for $j \neq 1$, and therefore $X_1 \not\subset \cup_{j\neq 1} X_j$, as it is irreducible. Therefore there exists $x \in X_1$ not contained in any other X_j. But x may be transferred to any other $g \in G$ by the *homeomorphism* $y \mapsto gx^{-1}y$. This implies that there is a single X_j passing through g. Thus the X_j are pairwise disjoint, and hence equal the connected components.
(2) Since $y \mapsto gy$ is a homeomorphism for all $g \in G$, gG° is a whole component for all g. If here $g \in G^\circ$, then $g \in G^\circ \cap gG^\circ$ implies $gG^\circ = G^\circ$ and thus $G^\circ G^\circ = G^\circ$. Similarly $(G^\circ)^{-1} = G^\circ$ and $gG^\circ g^{-1} \subset G^\circ$ for all $g \in G$, so G is a normal subgroup, and the rest is clear. □

Remark 2.4. A finite connected group must be trivial. On the other hand, we shall see shortly that any finite group can be equipped with a structure of affine algebraic group.

Now we investigate the coordinate ring of affine algebraic groups. In general, if $\phi : X \to Y$ is a morphism of affine varieties, there is an induced k-algebra homomorphism $\phi^* : \mathcal{A}_Y \to \mathcal{A}_X$ given by $\phi^*(f) = f \circ \phi$.

Proposition 2.5. (1) *Given affine varieties X and Y, denote by* $\mathrm{Mor}(X,Y)$ *the set of morphisms $X \to Y$. Then the map $\phi \to \phi^*$ induces a bijection between* $\mathrm{Mor}(X,Y)$ *and* $\mathrm{Hom}(\mathcal{A}_Y, \mathcal{A}_X)$.
(2) *If A is a finitely generated reduced k-algebra, there exists an affine variety X with $A \cong \mathcal{A}_X$.*

Proof. (1) Choose an embedding $Y \hookrightarrow \mathbf{A}^m$, and let $\bar{x}_1, \ldots, \bar{x}_m$ be the coordinate functions on Y. Then $\phi^* \mapsto (\phi^*(\bar{x}_1), \ldots, \phi^*(\bar{x}_m))$ is an inverse for $\phi \mapsto \phi^*$.
(2) There exist $n > 0$ and an ideal $I \subset k[x_1, \ldots, x_n]$ with $I = \sqrt{I}$ and $A \cong k[x_1, \ldots, x_n]/I$. The Nullstellensatz implies that $X = V(I)$ is a good choice. □

Corollary 2.6. *The maps $X \to \mathcal{A}_X$, $\phi \to \phi^*$ induce an anti-equivalence between the category of affine varieties over k and that of finitely generated reduced k-algebras.*

We say that the affine varieties X and Y are *isomorphic* if there exist $\phi \in \mathrm{Mor}(X,Y)$, $\psi \in \mathrm{Mor}(Y,X)$ with $\phi \circ \psi = \mathrm{id}_Y$, $\psi \circ \phi = \mathrm{id}_X$.

Corollary 2.7. *Let X and Y be affine varieties.*
(1) *X and Y are isomorphic as k-varietes if and only if $\mathcal{A}_Y$ and $\mathcal{A}_X$ are isomorphic as k-algebras.*
(2) *X is isomorphic to a closed subvariety of Y if and only if there exists a surjective homomorphism $\mathcal{A}_Y \to \mathcal{A}_X$.*

Proof. (1) is easy. For (2), note that by Lemma 1.2 (1) $X \subset Y$ closed implies $I(X) \supset I(Y)$, so that $I(X)$ induces an ideal $\bar{I} \subset \mathcal{A}_Y$, and there is a surjection $\mathcal{A}_Y \to \mathcal{A}_Y/\bar{I} \cong \mathcal{A}_X$. Conversely, given a surjection $\phi : \mathcal{A}_Y \to \mathcal{A}_X$, setting $I = \mathrm{Ker}\,(\phi)$ and $X' = V(I)$ we obtain $\mathcal{A}_{X'} \cong \mathcal{A}_X$, whence $X \cong X'$ by (1). $\square$

Lemma 2.8. *If X and Y are affine varieties, there is a canonical isomorphism $\mathcal{A}_{X\times Y} \cong \mathcal{A}_X \otimes_k \mathcal{A}_Y$.*

Proof. Define a map $\lambda : \mathcal{A}_X \otimes \mathcal{A}_Y \to \mathcal{A}_{X\times Y}$ by $\lambda(\Sigma f_i \otimes g_i) = \Sigma(f_i g_i)$. This is a surjective map, because the coordinate functions on $X \times Y$ are in the image (to see this, set f_i or g_i to 1), and they generate $\mathcal{A}_{X\times Y}$. For injectivity, assume $\Sigma f_i g_i = 0$, We may assume the f_i are linearly independent over k, but then $g_i(P) = 0$ for all i and $P \in Y$, so that $g_i = 0$ for all i by the Nullstellensatz. Hence $\Sigma f_i \otimes g_i = 0$. $\square$

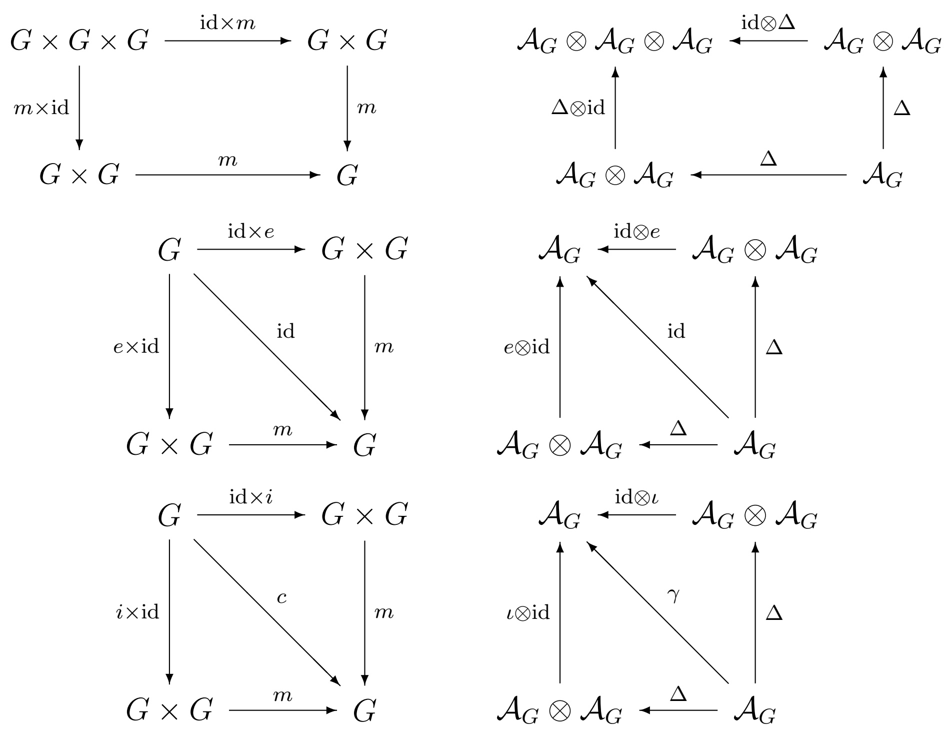
Corollary 2.9. *For an affine algebraic group G the coordinate ring $\mathcal{A}_G$ carries the following additional structure:*

multiplication $m : G\times G \to G \leftrightarrow$ comultiplication $\Delta : \mathcal{A}_G \to \mathcal{A}_G \otimes_k \mathcal{A}_G$

unit $\{e\} \to G \leftrightarrow$ counit $e : \mathcal{A}_G \to k$

inverse $i : G \to G \leftrightarrow$ coinverse $\iota : \mathcal{A}_G \to \mathcal{A}_G$

These are subject to the following commutative diagrams.



where in the last diagram c is the constant map $G \to \{e\}$ and γ the composite $A \to k \to A$.

Definition 2.10. *A k-algebra equipped with the above additional structure is called a* Hopf algebra.

Corollary 2.11. *The maps* $G \to \mathcal{A}_G$*,* $\phi \to \phi^*$ *induce an anti-equivalence between the category of affine algebraic groups over* k *and that of finitely generated reduced Hopf algebras.*

Examples 2.12.

(1) *The Hopf algebra structure on* $\mathcal{A}_{\mathbf{G}_a} = k[x]$ *is given by* $\Delta(x) = x \otimes 1 + 1 \otimes x$*,* $e(x) = 0$*,* $\iota(x) = -x$*.*

(2) *The Hopf algebra structure on* $\mathcal{A}_{\mathbf{G}_m} = k[x, x^{-1}]$ *is given by* $\Delta(x) = x \otimes x$*,* $e(x) = 1$*,* $\iota(x) = x^{-1}$*.*

(3) *The Hopf algebra structure on* $\mathcal{A}_{\mathrm{GL}_n} = k[x_{11}, \ldots, x_{nn}, \det(x_{ij})^{-1}]$ *is given by* $\Delta(x_{ij}) = \Sigma_l\, x_{il} \otimes x_{lj}$*,* $e(x_{ij}) = \delta_{ij}$ *(Kronecker delta),* $\iota(x_{ij}) = y_{ij}$*, where* $[y_{ij}] = [x_{ij}]^{-1}$*.*

Remark 2.13. Given any k-algebra R, the Hopf algebra structure on $\mathcal{A}_G$ induces a group structure on $\mathrm{Hom}(\mathcal{A}_G, R)$. (In particular, we obtain the group structure on $\mathrm{Hom}(\mathcal{A}_G, k) \cong G(k)$ using the Nullstellensatz.) Therefore an affine algebraic group may also be defined as a functor G from the category of k-algebras to the category of groups for which there exists a finitely generated reduced k-algebra A with $G \cong \mathrm{Hom}(A, \ )$ as a *set-valued* functor. Dropping the additional assumptions on A we obtain the notion of an *affine group scheme* over k.

3. Embedding in GL_n

In this section we prove:

Theorem 3.1. *Each affine algebraic group is isomorphic to a closed subgroup of* GL_n *for appropriate* $n > 0$*.*

Because of this theorem affine algebraic groups are also called *linear algebraic groups.*

To give an idea of the proof, we first construct a closed embedding into GL_n for a finite group G. The regular representation is faithful, hence defines an embedding $G \to \mathrm{GL}(k[G])$, where $k[G]$ is the group algebra of G viewed as a k-vector space. The image of G is a finite set, hence Zariski closed.

For an arbitrary affine algebraic group the coordinate ring $\mathcal{A}_G$ could play the role of $k[G]$ in the above argument, but it is not finite dimensional. The idea is to construct a finite dimensional G-invariant subspace.

Construction 3.2. For an affine algebraic group G the map $x \mapsto xg$ is an automorphism of G as an affine variety for all $g \in G$. Thus it induces a k-algebra automorphism $\rho_g : \mathcal{A}_G \to \mathcal{A}_G$. Viewing it as a k-vector space automorphism, we obtain a homomorphism $G \to \mathrm{GL}(\mathcal{A}_G)$ given by $g \mapsto \rho_g$.

Lemma 3.3. *Let* $V \subset \mathcal{A}_G$ *be a* k*-linear subspace.*

(1) $\rho_g(V) \subset V$ *for all* $g \in G$ *if and only if* $\Delta(V) \subset V \otimes_k \mathcal{A}_G$*.*

(2) *If V is finite dimensional, there is a finite dimensional k-subspace $W \subset \mathcal{A}_G$ containing V with $\rho_g(W) \subset W$ for all $g \in G$.*

Proof. (1) Assume $\Delta(V) \subset V \otimes_k \mathcal{A}_G$. Then for $f \in V$ we find $f_i \in V$, $g_i \in \mathcal{A}_G$ with $\Delta(f) = \Sigma f_i \otimes g_i$. Thus $(\rho_g f)(h) = f(hg) = \Sigma f_i(h)g_i(g)$ for all $h \in G$, hence

$$\rho_g f = \sum g_i(g) f_i, \tag{1}$$

so $\rho_g f \in V$, since $f_i \in V$ and $g_i(g) \in k$. Conversely, assume $\rho_g(V) \subset V$ for all g. Let $\{f_i : i \in I\}$ be a basis of V, and let $\{g_j : j \in J\}$ be such that $\{f_i, g_j : i \in I, j \in J\}$ is a basis of $\mathcal{A}_G$. Since $\Delta(f) = \Sigma f_i \otimes u_i + \Sigma g_j \otimes v_j$ for some $u_i, v_j \in \mathcal{A}_G$, we obtain as above $\rho_g f(h) = \Sigma f_i(h)u_i(g) + \Sigma g_j(h)v_j(g)$. By assumption here $v_j(g)$ must be 0 for all $g \in G$, thus $v_j = 0$ for all j.
(2) By writing V as a sum of 1-dimensional subspaces it is enough to consider the case dim $(V) = 1$, $V = \langle f \rangle$. Choosing f_i as in formula (1) above, the formula implies that the finite dimensional subspace W' generated by the f_i contains the $\rho_g f$ for all $g \in G$. Thus the subspace $W := \langle \rho_g f : g \in G \rangle \subset W'$ meets the requirements. □

Proof of Theorem 3.1: Let V be the finite dimensional k-subspace of $\mathcal{A}_G$ generated by a finite system of k-*algebra* generators of $\mathcal{A}_G$. Applying part (2) of the lemma we obtain a k-subspace W invariant under the ρ_g which still generates $\mathcal{A}_G$. Let $f_1, \dots, f_n$ be a k-basis of W. By part (1) of the lemma we find elements $a_{ij} \in \mathcal{A}_G$ with

$$\Delta(f_i) = \Sigma_j f_j \otimes a_{ij} \text{ for all } 1 \le i \le n.$$

It follows that

$$\rho_g(f_i) = \Sigma_j a_{ij}(g) f_j \text{ for all } 1 \le i \le n,\ g \in G. \tag{2}$$

Thus $[a_{ij}]$ is the matrix of ρ_g in the basis $f_1, \dots, f_n$. Define a morphism $\Phi : G \to \mathbf{A}^{n^2}$ by $(a_{11}, \dots, a_{nn})$. Since the matrices $[a_{ij}(g)]$ are invertible, its image lies in GL_n, and moreover it is a group homomorphism by construction. The k-algebra homomorphism $\Phi^* : \mathcal{A}_{\mathrm{GL}_n} \to \mathcal{A}_G$ (see Example 2.12 (3)) is defined by sending x_{ij} to a_{ij}. Since $f_i(g) = \Sigma_j f_j(1)a_{ij}(g)$ for all $g \in G$ by (2), we have $f_i = \Sigma_j f_j(1)a_{ij}$, and therefore Φ^* is surjective, because the f_i generate $\mathcal{A}_G$. So by Corollary 2.7 (2) and Proposition 2.5 (1) Φ embeds G as a closed subvariety in GL_n, thus as a closed subgroup. □

We also prove a more refined version that will be needed later.

Corollary 3.4. *Let G be an affine algebraic group, H a closed subgroup. Then there is a closed embedding $G \subset \mathrm{GL}(W)$ for some finite dimensional W such that H equals the stabilizer of a subspace $W_H \subset W$.*

Proof. Let I_H be the ideal of functions vanishing on H in $\mathcal{A}_G$. In the above proof we may arrange that some of the f_i form a system of generators for I_H. Put $W_H := W \cap I_H$. Observe that

$$g \in H \Leftrightarrow hg \in H \text{ for all}\, h \in H \Leftrightarrow \rho_g(I_H) \subset I_H \Leftrightarrow \rho_g(W_H) \subset W_H,$$

whence the corollary. □

Chapter 2. Jordan Decomposition and Triangular Form

The embedding theorem of the last chapter enables us to apply linear algebra techniques to the study of affine algebraic groups. The first main result of this kind will be a version of Jordan decomposition which is independent of the embedding into GL_n. In the second part of the chapter we discuss three basic theorems that show that under certain assumptions one may put all elements of some matrix group simultaneously into triangular form. The strongest of these, the Lie-Kolchin theorem, concerns connected solvable subgroups of GL_n. As an application of this theorem one obtains a strong structural result for connected nilpotent groups. In the course of the chapter we also describe diagonalizable groups, i.e. commutative groups that can be embedded in GL_n as closed subgroups of the diagonal subgroup.

4. Jordan decomposition

The results of the last section allow us to apply linear algebra techniques in the study of affine algebraic groups. For instance, since k is algebraically closed, in a suitable basis each endomorphism ϕ of an n-dimensional vector space has a matrix that is in *Jordan normal form.* Recall that this means that if $\lambda_1, \dots, \lambda_m$ are the eigenvalues of ϕ, the matrix of ϕ is given by blocks along the diagonal that have the form

$$\begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}.$$

We shall generalise this result to affine algebraic groups *independently* of the embedding into GL_n. First some definitions.

Definition 4.1. *Let V be a finite dimensional vector space. An element $g \in \mathrm{End}(V)$ is* semisimple *(or diagonalizable) if V has a basis consisting of eigenvectors of g. The endomorphism g is* nilpotent *if $g^m = 0$ for some $m > 0$.*

Remark 4.2. Recall from linear algebra that g is semisimple if and only if its minimal polynomial has distinct roots. Consequently, if $W \subset V$ is a g-invariant subspace and g is semisimple, then so is $g|_W$ (because the minimal polynomial of $g|_W$ divides that of g). This fact will be repeatedly used in what follows.

The above statement about matrices can be restated (in a slightly weaker form) as follows:

Proposition 4.3. (Additive Jordan decomposition) *Let V be a finite dimensional vector space, $g \in \mathrm{End}(V)$.*

There exist elements $g_s, g_n \in \mathrm{End}(V)$ with g_s semisimple, g_n nilpotent, $g = g_s + g_n$ and $g_s g_n = g_n g_s$.

Proof. In the basis yielding the Jordan form define g_s by the diagonal of the matrix. □

One has the following additional properties.

Proposition 4.4.

(1) *The elements $g_s, g_n \in \mathrm{End}(V)$ of the previous proposition are uniquely determined.*
(2) *There exist polynomials $P, Q \in k[T]$ with $P(0) = Q(0) = 0$ and $g_s = P(g)$, $g_n = Q(g)$.*
(3) *If $W \subset V$ is a g-invariant subspace, it is invariant for g_s and g_n as well. Moreover, $(g|_W)_s = g_s|_W$ and $(g|_W)_n = g_n|_W$.*

Proof. (1) Let

$$\Phi(T) := \det(T \cdot \mathrm{id}_V - g) = \prod (T - \lambda_i)^{n_i}$$

the characteristic polynomial of g, and set

$$V_i := \{v \in V : (g - \lambda_i)^{n_i} v = 0\}.$$

This is a ϕ-invariant subspace corresponding to the i-th Jordan block of ϕ. By construction $g_s|_{V_i} = \lambda_i \mathrm{id}_{V_i}$.

Now assume $g = g_s' + g_n'$ is another Jordan decomposition. Since $gg_s' = g_s' g$, we have $g_s'(g - \lambda_i \mathrm{id}) = (g - \lambda_i \mathrm{id}) g_s'$, which implies $g_s'(V_i) \subset V_i$ for all i. Since $g - g_s' = g_n'$ is nilpotent, all eigenvalues of $g_s'|_{V_i}$ are equal to λ_i, but then $g_s'|_{V_i} = \lambda_i \mathrm{id}_{V_i}$ as g_s' is semisimple (and hence so is $g|_{V_i}$ – see the above remark). Thus $g_s = g_s'$.
(2) The Chinese Remainder Theorem for polynomial rings gives a direct sum decomposition

$$k[T]/(\Phi) \cong \bigoplus_i k[T]/((T - \lambda_i)^{n_i}),$$

so we find $P \in k[T]$ with $P \equiv \lambda_i \bmod (T - \lambda_i)^{n_i}$ for all i. By construction $P(g) = g_s$, and so $(T - P)(g) = g_n$. Adding a suitable constant multiple of Φ to P we may assume $P(0) = 0$.
The first part of (3) immediately follows from (2). Moreover, as the characteristic polynomial of $g|_W$ divides Φ, $(g|_W)_s = P(g|_W) = g_s|_W$ is a good choice; the statement for $(g|_W)_n$ follows from this. □

Definition 4.5. *An endomorphism $h \in \mathrm{End}(V)$ is* unipotent *if $h - \mathrm{id}_V$ is nilpotent (equivalently, if all eigenvalues of h are 1).*

Corollary 4.6. (Multiplicative Jordan decomposition) *Let V be a finite dimensional vector space, $g \in \mathrm{GL}(V)$.*

(1) *There exist uniquely determined elements $g_s, g_u \in \mathrm{GL}(V)$ with g_s semisimple, g_u unipotent, and $g = g_s g_u = g_u g_s$.*

(2) *There exist polynomials* $P, R \in k[T]$ *with* $P(0) = R(0) = 0$ *and* $g_s = P(g)$*,* $g_u = R(g)$*.*
(3) *If* $W \subset V$ *is a* g*-invariant subspace, it is invariant for* g_s *and* g_u *as well. Moreover,* $(g|_W)_s = g_s|_W$ *and* $(g|_W)_u = g_u|_W$*.*

Proof. Since $g \in \mathrm{GL}(V)$, its eigenvalues are nonzero, hence so are those of the g_s defined in the above proof. Thus g_s is invertible, and $g_u = \mathrm{id}_V + g_s^{-1} g_n$ will do for (1). Then to prove (2) it is enough to see by the proposition that g_s^{-1} is a polynomial in g_s, and hence in g. This is clear, because if $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the minimal polynomial of g_s (note that $a_0 \neq 0$), we have $-a_0^{-1} g_s^{n-1} - a_0^{-1} a_{n-1} g_s^{n-2} - \cdots - a_0^{-1} a_1 = g_s^{-1}$. Statement (3) follows from (2) as above. □

We now consider an infinite dimensional generalisation.

Definition 4.7. Let V be a not necessarily finite dimensional vector space and fix $g \in \mathrm{GL}(V)$. Assume that V is a union of finite dimensional g-invariant subspaces. We say that g is *semisimple (resp. locally unipotent)* if $g|_W$ is semisimple (resp. unipotent) for all finite dimensional g-invariant subspaces W.

Corollary 4.8. *Let* V *be a not necessarily finite dimensional vector space,* $g \in \mathrm{GL}(V)$*. Assume that* V *is a union of finite dimensional* g*-invariant subspaces.*

(1) *There exist uniquely determined elements* $g_s, g_u \in \mathrm{GL}(V)$ *with* g_s *semisimple,* g_u *locally unipotent, and* $g = g_s g_u = g_u g_s$*.*
(2) *If* $W \subset V$ *is a* g*-invariant subspace, it is invariant for* g_s *and* g_u *as well.*

Proof. Using the third statement of the last corollary and the unicity statement of part (1) we may 'glue the $(g|_W)_s$ and $(g|_W)_u$ together' to obtain the required g_s and g_u, whence the third statement. The second one follows from part (3) of the last corollary, once we have remarked that W is also a union of finite dimensional g-invariant subspaces. □

If G is an affine algebraic group, then for all $g \in G$ the action of the 'right translation' $\rho_g \in \mathrm{GL}(\mathcal{A}_G)$ on $\mathcal{A}_G$ satisfies the finiteness condition of the corollary, and thus a unique decomposition $\rho_g = (\rho_g)_s(\rho_g)_u$ exists.

Theorem 4.9. *Let* G *be an affine algebraic group.*

(1) *There exist uniquely determined* $g_s, g_u \in G$ *with* $g = g_s g_u = g_u g_s$ *and* $\rho_{g_s} = (\rho_g)_s$*,* $\rho_{g_u} = (\rho_g)_u$*.*
(2) *In the case* $G = \mathrm{GL}_n$ *the elements* g_s *and* g_u *are the same as those of Corollary 4.6.*
(3) *For each embedding* $\phi : G \to \mathrm{GL}_n$ *we have* $\phi(g_s) = \phi(g)_s$ *and* $\phi(g_u) = \phi(g)_u$*.*

Definition 4.10. *We call* $g \in G$ semisimple (resp. unipotent) *if* $g = g_s$ *(resp.* $g = g_u$*).*

The following lemma already proves part (2) of the theorem.

Lemma 4.11. *Let V be a finite dimensional vector space, $g \in GL(V)$. Then g is semisimple (resp. unipotent) if and only if $\rho_g \in \mathrm{GL}(\mathcal{A}_G)$ is semisimple (resp. locally unipotent).*

Proof. Recall that $\mathcal{A}_{\mathrm{GL}(V)} \cong k[\mathrm{End}(V), 1/D]$, where $D = \det(x_{ij})$ with $x_{11}, \dots, x_{nn}$ the standard basis of $\mathrm{End}(V) \cong M_n(k)$. Regarding $fD^{-m} \in \mathcal{A}_G$ as a function on $\mathrm{GL}(V)$ we have

$$(3) \quad \rho_g(fD^{-m})(x) = f(xg)D^{-m}(x)D^{-m}(g) = D^{-m}(g)(\rho_g(f)D^{-m})(x),$$

where $D^m(g) = \det(g)^m$ for all $m \geq 0$.

Right multiplication by g acts not only on $\mathrm{GL}(V)$, but also on $\mathrm{End}(V) \cong \mathbf{A}^{n^2}$, and by the above formula fD^{-m} is an eigenvector for ρ_g on $\mathcal{A}_{\mathrm{GL}(V)}$ for all m if and only if f is an eigenvector for ρ_g on $\mathcal{A}_{\mathrm{End}(V)}$. Thus ρ_g is semisimple on $\mathcal{A}_{\mathrm{GL}(V)}$ if and only if it is semisimple on $\mathcal{A}_{\mathrm{End}(V)}$. The same holds with 'semisimple' replaced by 'locally unipotent'. (Indeed, note that the formula $\rho_g(D) = D(g)D$ implies that D is an eigenvector for ρ_g, so if ρ_g is unipotent, $D(g) = 1$ and (3) yields 'if'; the converse is trivial.)

Thus it is enough to prove the lemma for $\mathcal{A}_{\mathrm{GL}(V)}$ replaced by $\mathcal{A}_{\mathrm{End}(V)}$. Observe that

$$\mathcal{A}_{\mathrm{End}(V)} \cong k[x_{11}, \dots, x_{nn}] \cong Sym(\mathrm{End}(V)^\vee),$$

where $\vee$ denotes the dual vector space and

$$Sym(\mathrm{End}(V)^\vee) := \bigoplus_{m=0}^{\infty} (\mathrm{End}(V)^\vee)^{\otimes m} / \langle x \otimes y - y \otimes x \rangle.$$

The action of ρ_g on $\mathrm{End}(V)^\vee$ is given by $(\rho_g f)(x) = f(xg)$, and the action on $\mathcal{A}_{\mathrm{End}(V)}$ is induced by extending this action to $Sym(\mathrm{End}(V)^\vee)$. If $\phi \in \mathrm{End}(V)$ is semisimple or unipotent, then so is $\phi^{\otimes m}$ for all m, so using the fact that $\mathrm{End}(V)^\vee$ is the direct summand of degree 1 in $Sym(\mathrm{End}(V)^\vee)$ we see that ρ_g is semisimple (resp. unipotent) on $\mathrm{End}(V)^\vee$ if and only if it is semisimple (resp. locally unipotent) on $\mathcal{A}_{\mathrm{End}(V)}$. Thus we are reduced to showing that $g \in \mathrm{GL}(V)$ is semisimple (resp. unipotent) if and only if ρ_g is semisimple (resp. unipotent) on $\mathrm{End}(V)^\vee$. This is an exercise in linear algebra left to the readers. $\square$

Proof of Theorem 4.9: Take an embedding $\phi : G \to GL(V)$. For all $g \in G$ the diagram

$$\begin{array}{ccc} \mathcal{A}_{GL(V)} & \xrightarrow{\phi^*} & \mathcal{A}_G \\ \rho_{\phi(g)} \downarrow & & \downarrow \rho_g \\ \mathcal{A}_{GL(V)} & \xrightarrow{\phi^*} & \mathcal{A}_G \end{array}$$

commutes.

Now there is a unique Jordan decomposition $\phi(g) = \phi(g)_s\phi(g)_u$ in $\mathrm{GL}(V)$. We claim that it will be enough to show for (1) and (3) that $\phi(g)_s, \phi(g)_u \in \phi(G)$. Indeed, once we have proven this, we may define g_s (resp. g_u) to be the unique element in G with $\phi(g_s) = \phi(g)_s$ (resp. $\phi(g_u) = \phi(g)_u$). Since (1) holds for $G = \mathrm{GL}(V)$ by the previous lemma, the above diagram for g_s and g_u in place of g then implies that it holds for G as well. Statement (3) now follows by the unicity statement of (1) and the diagram above.

It remains to prove $\phi(g)_s, \phi(g)_u \in \phi(G)$. Setting $I := \mathrm{Ker}\,(\phi^*)$, observe that for $\bar{g} \in \mathrm{GL}(V)$

$$\bar{g} \in \phi(G) \Leftrightarrow h\bar{g} \in \phi(G) \text{ for all } h \in \phi(G) \Leftrightarrow \rho_{\bar{g}}(I) \subset I, \tag{4}$$

since I consists of the functions vanishing on G. In particular, $\rho_{\phi(g)}$ preserves I, hence so do $(\rho_{\phi(g)})_s$ and $(\rho_{\phi(g)})_u$ by Corollary 4.8 (2). On the other hand, by the lemma above $(\rho_{\phi(g)})_s = \rho_{\phi(g)_s}$ and $(\rho_{\phi(g)})_u = \rho_{\phi(g)_u}$. But then from (4) we obtain $\phi(g)_s, \phi(g)_u \in \phi(G)$, as required. □

We conclude with the following complement.

Corollary 4.12. *Let $\phi : G \to G'$ be a morphism of algebraic groups. For each $g \in G$ we have $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$.*

Proof. Choose an embedding $\rho : G' \to \mathrm{GL}_n$, and apply part (3) of the theorem to ρ and $\rho\circ\phi$. The corollary follows from the unicity of Jordan decomposition. □

5. Unipotent groups

In the next three sections we shall prove three theorems about putting elements of matrix groups simultaneously into triangular form. In terms of vector spaces this property may be formulated as follows. In an n-dimensional vector space V call a strictly increasing chain $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of subspaces a *complete flag*. Then finding a basis of V in which the matrix of all elements of a subgroup $G \subset \mathrm{GL}(V)$ is upper triangular is equivalent to finding a complete flag of G-invariant subspaces in V.

We call a subgroup of an affine algebraic group *unipotent* if it consists of unipotent elements.

Proposition 5.1. (Kolchin) *For each unipotent subgroup $G \subset \mathrm{GL}(V)$ there exists a complete flag of G-invariant subspaces in V.*

Proof. By induction on the dimension n of V it will suffice to show that V has a nontrivial G-invariant subspace. Assume not. Then V is an irreducible representation of G, hence a simple n-dimensional $k[G]$-module. By Schur's lemma $D = \mathrm{End}_{k[G]}(V)$ is a division algebra over k, hence $D = k$ because k is algebraically closed (and each $\alpha \in D \setminus k$ would generate a commutative subfield of finite degree). But then by the density theorem the natural map $k[G] \to \mathrm{End}_k(V) = \mathrm{End}_D(V)$ is surjective. In other words, the elements of G generate $\mathrm{End}_k(V)$. Each element $g \in G$ has trace n because the trace of a nilpotent matrix is 0 and $g-1$ is nilpotent. It follows that for $g, h \in G$ $Tr(gh) = Tr(g)$, or in other words $Tr((g-1)h) = 0$. Since the $h \in G$ generate $\mathrm{End}_k(V)$, we obtain $Tr((g-1)\phi) = 0$ for all $\phi \in \mathrm{End}_k(V)$. Fixing a basis of V and applying this to those ϕ whose matrix has a single nonzero entry we see that this can only hold for $g = 1$. But g was arbitrary, so $G = 1$, in which case all subspaces are G-invariant, and we obtain a contradiction. □

Corollary 5.2. *Each unipotent subgroup $G \subset \mathrm{GL}_n$ is conjugate to a subgroup of U_n, the group of upper triangular matrices with 1's in the diagonal.*

Proof. This follows from the proposition, since all eigenvalues of a unipotent matrix are 1. □

Corollary 5.3. *A unipotent algebraic group is nilpotent, hence solvable (as an abstract group).*

Recall that a group G is nilpotent if in the chain of subgroups

$$G = G^0, G^1 = [G, G], \ldots, G^i = [G, G^{i-1}], \ldots$$

we have $G^n = \{1\}$ for some n. It is solvable if in the chain of subgroups

$$G = G^{(0)}, G^{(1)} = [G, G], \ldots, G^{(i)} = [G^{(i-1)}, G^{(i-1)}], \ldots$$

we have $G^{(n)} = \{1\}$ for some n.

Proof. A subgroup of a nilpotent group is again nilpotent, so by the previous corollary it is enough to see that the group U_n is nilpotent. This is well-known (and easy to check). □

6. Commutative groups

We begin the study of commutative linear algebraic groups with the following well-known statement.

Lemma 6.1. *For each set S of pairwise commuting endomorphisms of a finite dimensional vector space V, there is a complete flag of S-invariant subspaces of V. If all elements of S are semisimple, there is a basis of V consisting of common eigenvectors of the elements in S.*

Proof. The lemma is easy if all elements in S act by scalar multiplications. Otherwise there is $s \in S$ that has a nontrivial eigenspace $V_\lambda \subsetneq V$. For all $t \in S$ and $v \in V_\lambda$ one has $stv = tsv = \lambda tv$, i.e. $tv \in V_\lambda$, so V_λ is S-invariant. The first statement then follows by induction on dimension. For the second, choose s and V_λ as above, and let W be the direct sum of the other eigenspaces of s. As above, both V_λ and W are stable by S, and we again conclude by induction on dimension (using Remark 4.2). □

Given an affine algebraic group G, write G_s (resp. G_u) for the set of its semisimple (resp. unipotent) elements. Note that G_u is always a closed subset, for applying the Cayley-Hamilton theorem after embedding G into some GL_n we see that all elements $g \in G_u$ satisfy equation $(g-1)^n = 0$, which implies n^2 polynomial equations for their matrix entries. The subset G_s is not closed in general.

Theorem 6.2. *Let G be a* commutative *affine algebraic group. Then the subsets G_u and G_s are closed subgroups of G, and the natural map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.*

Proof. We may assume that G is a closed subgroup of some $\mathrm{GL}(V)$. The second statement of the previous lemma shows that in this case G_s is a subgroup, and its first statement implies that G_u is a subgroup as well (since a triangular unipotent matrix has 1-s in the diagonal). Now use the second statement again to write V as a direct sum of the *common* eigenspaces V_λ of the elements in G_s. Each V_λ is G-invariant (again by the calculation $stv = tsv = \lambda tv$), so applying the first statement of the lemma to each of the V_λ we find a closed embedding of G into GL_n in which all elements map to upper triangular matrices and all semisimple elements to diagonal ones. This shows in particular that $G_s \subset G$ is closed (set the off-diagonal entries of a triangular matrix to 0), and for G_u we know it already. The group homomorphism $G_s \times G_u \to G$ is injective since $G_s \cap G_u = \{1\}$, and surjective by the Jordan decomposition. It is also a morphism of affine varieties, so it remains to see that the inverse map is a morphism as well. The same argument that proves the closedness of G_s shows that the map $g \mapsto g_s$ given by Jordan decomposition is a morphism, hence so is $g \mapsto g_u = g_s^{-1}g$ and finally $g \mapsto (g_s, g_u)$. □

We now investigate commutative semisimple groups. By the above these are closed subgroups of some group D_n of diagonal matrices with invertible entries, hence they are also called *diagonalizable* groups. A diagonalizable group is called a *torus* if it is actually isomorphic to some D_n, and hence to the direct power $\mathbf{G}_m^n$.

To classify diagonalizable groups we need the notion of the *character group.* Given a not necessarily commutative algebraic group G, a *character* of G is a morphism of algebraic groups $G \to \mathbf{G}_m$. These obviously form an abelian group, denoted by $\widehat{G}$. Moreover, the map $G \mapsto \widehat{G}$ is a contravariant functor: each morphism $G \to H$ of algebraic groups induces a group homomorphism $\widehat{H} \to \widehat{G}$ by composition.

Proposition 6.3. *If G is diagonalizable, then $\widehat{G}$ is a finitely generated abelian group having no elements of order* $\mathrm{char}\,(k)$.

The proof uses the following famous lemma.

Lemma 6.4. (Dedekind) *Let G be an abstract group, k a field and $\phi_i : G \to k^\times$ group homomorphisms for $1 \leq i \leq m$. Then the ϕ_i are linearly independent in the k-vector space of functions from G to k.*

Proof. Assume $\Sigma\lambda_i\phi_i = 0$ is a linear relation with $\lambda_i \in k^\times$ that is of minimal length. Then $\Sigma\lambda_i\phi_i(gh) = \Sigma\lambda_i\phi_i(g)\phi_i(h) = 0$ for all $g, h \in G$. Fixing g with $\phi_1(g) \neq \phi_i(g)$ for some i (this is always possible after a possible renumbering) and making h vary we obtain another linear relation $\Sigma\lambda_i\phi_i(g)\phi_i = 0$. On the other hand, multiplying the initial relation by $\phi_1(g)$ we obtain $\Sigma\lambda_i\phi_1(g)\phi_i = 0$. The difference of the two last relations is nontrivial and of smaller length, a contradiction. □

Next recall from Example 2.2(2) that $\mathcal{A}_{\mathbf{G}_m} \cong k[T, T^{-1}]$, with the Hopf algebra structure determined by $\Delta(T) = T \otimes T$. It follows that any character $\chi : G \to \mathbf{G}_m$ is determined by the image of T by χ^*, which should satisfy $\Delta(\chi^*(T)) = \chi^*(T) \otimes \chi^*(T)$. Thus we have proven:

Lemma 6.5. *The map $\chi \mapsto \chi^*(T)$ induces a bijection between $\widehat{G}$ and the set of elements $x \in \mathcal{A}_G$ satisfying $\Delta(x) = x \otimes x$.*

The elements $x \in \mathcal{A}_G$ with $\Delta(x) = x \otimes x$ are called *group-like elements.* The two previous lemmas imply:

Corollary 6.6. *The group-like elements are k-linearly independent in $\mathcal{A}_G$.*

Proof of Proposition 6.3: In the case $G = \mathbf{G}_m$ a group-like element in $\mathcal{A}_{\mathbf{G}_m}$ can only be T^n for some $n \in \mathbf{Z}$ (for instance, by the above corollary), so $\widehat{\mathbf{G}_m} \cong \mathbf{Z}$. Next we have $\widehat{\mathbf{G}_m^n} \cong \widehat{\mathbf{G}_m}^n \cong \mathbf{Z}^n$ and also $\mathcal{A}_{\mathbf{G}_m^n} \cong k[T, T^{-1}]^{\otimes n} \cong k[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]$ using Lemma 2.8. If $\phi : G \to \mathbf{G}_m^n$ is a closed embedding, the induced surjection $\phi^* : k[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}] \to \mathcal{A}_G$ is a map of Hopf algebras, so in particular it sends group-like elements to group like ones. Since the elements $T_1^{m_1} \ldots T_n^{m_n}$ are group-like and generate $\mathcal{A}_{\mathbf{G}_m^n}$, the $\phi^*(T_1^{m_1} \ldots T_n^{m_n})$ must generate $\mathcal{A}_G$. But then by the previous corollary a group-like element in $\mathcal{A}_G$ should be one of the $\phi^*(T_1^{m_1} \ldots T_n^{m_n})$, so the natural map $\mathbf{Z}^n = \widehat{\mathbf{G}_m^n} \to \widehat{G}$ is surjective. Finally, if $\mathrm{char}\,(k) = p > 0$ and $\chi \in \widehat{G}$ has order dividing p, then $\chi^p(g) = \chi(g)^p = 1$ for all $g \in G$, but since $k^\times$ has no p-torsion, $\chi(g) = 1$ and so $\chi = 1$. □

Theorem 6.7. *The functor $G \to \widehat{G}$ induces an anti-equivalence of categories between diagonalizable algebraic groups over k and finitely generated abelian groups having no elements of order* $\mathrm{char}\,(k)$*. Here tori correspond to free abelian groups.*

Proof. Construct a functor in the reverse direction by writing a finitely generated abelian group as a direct sum $\mathbf{Z}^r \oplus \mathbf{Z}/m_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/m_n\mathbf{Z}$ and sending it to $\mathbf{G}_m^r \times \mu_{m_1} \times \ldots \mu_{m_n}$. It is easy to check using the argument of the previous proof that the two functors are inverse to each other (up to isomorphism). □

Remark 6.8. The theory of tori is more interesting over non-algebraically closed fields. In this case the character group has an extra structure: the action of the absolute Galois group of the base field, which gives rise to 'much more' different tori.

7. The Lie-Kolchin theorem

We now come to the third main triangularisation theorem. In the special case when G is a connected solvable complex Lie group it was proven by Lie.

Theorem 7.1. (Lie-Kolchin) *Let V be a finite dimensional k-vector space and $G \subset \mathrm{GL}(V)$ a connected solvable subgroup. Then there is a complete flag of G-invariant subspaces in V.*

Remarks 7.2.

(1) This is not really a theorem about algebraic groups, for we have not assumed that G is closed. It is just a connected subgroup equipped with the induced topology which is solvable as an abstract group. In the case when G is also closed, we shall see later (Corollary 15.5) that the commutator subgroup $[G, G]$ is closed as well (this is not true in general when G is not connected!), so all subgroups G^i in the finite commutator series of G are closed connected algebraic subgroups, by Lemma 7.3 below.

(2) The converse of the theorem also holds, even without assuming G connected: if there is a complete G-invariant flag, then G is solvable (because the group of upper triangular matrices is).

(3) However, the connectedness assumption is necessary even for closed subgroups, as the following example shows. Let $G \subset \mathrm{GL}_2(k)$ be the group of matrices with a single nonzero entry in each row and column. It is not connected but a closed subgroup, being the union of the diagonal subgroup D_2 with the closed subgroup A_2 of invertible matrices with zeros in the diagonal. It is also solvable, because D_2 is its identity component and $G/\mathrm{D}_2 \cong \mathbf{Z}/2\mathbf{Z}$. The only common eigenvectors for D_2 are of the form $\begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ a \end{pmatrix}$, but these are not eigenvectors for the matrices in A_2. Thus there is no common eigenvector for G.

Lemma 7.3. *If G is a connected topological group, then the commutator subgroup $[G, G]$ is connected as well.*

Proof. Write ϕ_i for the map $G^{2i} \to G$ sending $(x_1, \ldots, x_i, y_1, \ldots y_i)$ to $[x_1, y_1] \ldots [x_i, y_i]$. Since G is connected, so is $\mathrm{Im}\,(\phi_i)$. Thus $\mathrm{Im}\,(\phi_1) \subset \mathrm{Im}\,(\phi_2) \subset \ldots$ is a chain of connected subsets and $[G, G]$ is their union, hence it is connected. □

Proof of Theorem 7.1: By induction on $\dim V$ it suffices to show that the elements of G have a common eigenvector v, for then the image of G in $\mathrm{GL}(V/\langle v\rangle)$ is still connected and solvable, and thus stabilizes a complete flag in $V/\langle v\rangle$ whose preimage in V yields a complete flag together with $\langle v\rangle$. We may also assume that V is an irreducible $k[G]$-module, i.e. there is no nontrivial G-invariant subspace in V, for if V' were one, we would find a common eigenvector by looking at the image of G in $\mathrm{GL}(V')$, again by induction on dimension.

We now use induction on the smallest i for which $G^i = \{1\}$. By Lemma 7.3 $[G,G]$ is a connected normal subgroup in G with $[G,G]^{i-1} = \{1\}$, so by induction its elements have a common eigenvector. Write W for the span of *all* common eigenvectors of $[G,G]$ in V. We claim that $W = V$. By the irreducibility of V for this it is enough to see that W is G-invariant. Pick $h \in [G,G]$ and a common eigenvector v of $[G,G]$. Then $hv = \chi(h)v$, where $\chi(h) \in k^\times$ is a constant depending on h (in fact $\chi : [G,G] \to \mathbf{G}_m$ is a character of $[G,G]$, but we shall not use this). Since $[G,G] \subset G$ is normal, we have

$$hgv = g(g^{-1}hg)v = g(\chi(g^{-1}hg)v) = \chi(g^{-1}hg)gv.$$

As h was arbitrary, we conclude that $gv \in W$. So W has a G-invariant basis, and therefore it is G-invariant, i.e. equals V.

We conclude that there is a basis of V in which the matrix of each $h \in [G,G]$ is diagonal. This holds in particular for the conjugates $g^{-1}hg \in [G,G]$ with $g \in G$, and thus conjugation by g corresponds to a permutation of the finitely many common eigenvalues of the $g^{-1}hg$. In particular, each $h \in [G,G]$ has a finite conjugacy class in G. In other words, for fixed $h \in [G,G]$ the map $G \to G, g \mapsto g^{-1}hg$ has finite image. Since G is connected and the map is continuous, it must be constant. This means that $[G,G]$ is contained in the center of G.

Next observe that each $h \in [G,G]$ must act on V via multiplication by some $\lambda \in k^\times$. Indeed, if V_λ is a nontrivial eigenspace of h with eigenvalue λ, it must be G-invariant since h is central in G, but then $V_\lambda = V$ by irreducibility of V. On the other hand, h is a product of commutators, so its matrix has determinant 1. All in all, the matrix should be of the form $\omega \cdot \mathrm{id}$, where ω is a $\dim V$-th root of unity. In particular, $[G,G]$ is finite, but it is also connected, hence $[G,G] = 1$ and G is commutative. We conclude by Lemma 6.1. □

Corollary 7.4. *If G is connected and solvable, then G_s and G_u are closed subgroups, and G_u is normal.*

Proof. By the theorem we find an embedding $G \subset \mathrm{GL}_n$ so that the elements of G map to upper triangular matrices. Then $G_u = G \cap \mathrm{U_n}$, and moreover it is the kernel of the natural morphism of algebraic groups $G \to \mathrm{D_n}$, where D_n is the subgroup of diagonal matrices. So G_u is a closed normal subgroup. On the other hand, the Jordan decomposition shows that $G_s = G \cap \mathrm{D}_n$, so it is also a closed subgroup. □

As an application we give a structure theorem in the nilpotent case.

Theorem 7.5. *Let G be a connected nilpotent affine algebraic group. Then the subsets G_u and G_s are closed normal subgroups of G, and the natural map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.*

Proof. By the previous corollary it suffices to see that G_s is normal, for then the second statement will follow as in Theorem 6.2. We shall prove more, namely that $G_s \subset Z(G)$. Assume this is false, and choose $g \in G_s$ and $h \in G$ that do not commute. Embed G into some $\mathrm{GL}(V)$, and apply the Lie-Kolchin theorem to find a complete flag of G-invariant subspaces. We find a largest subspace V_i in the flag on which g and h commute but they do not commute on the next subspace $V_{i+1} = V_i \oplus \langle v\rangle$. As G_s acts diagonally, v is an eigenvector for g, i.e. $gv = \lambda v$ for some $\lambda \neq 0$. On the other hand, the G-invariance of V_{i+1} shows that there is $w \in V_i$ with $hv = \mu v + w$.

Put $h_1 := h^{-1}g^{-1}hg$. We now show that g and h_1 do not commute. Indeed, noting $g^{-1}v = \lambda^{-1}v$ and $h^{-1}v = \mu^{-1}v - \mu^{-1}h^{-1}w$ we have

$$\begin{aligned} h_1gv &= h^{-1}g^{-1}hg^2v = \lambda^2h^{-1}g^{-1}hv = \lambda^2h^{-1}g^{-1}(\mu v + w) = \\ &= \lambda\mu h^{-1}v + \lambda^2h^{-1}g^{-1}w = \lambda v - \lambda h^{-1}w + \lambda^2h^{-1}g^{-1}w \end{aligned}$$

and

$$\begin{aligned} gh_1v &= gh^{-1}g^{-1}hgv = \lambda gh^{-1}g^{-1}hv = \lambda gh^{-1}g^{-1}(\mu v + w) = \\ &= \mu gh^{-1}v + \lambda gh^{-1}g^{-1}w = \lambda v - gh^{-1}w + \lambda gh^{-1}g^{-1}w. \end{aligned}$$

Since g and h commute on V_i, we have $\lambda gh^{-1}g^{-1}w = \lambda h^{-1}w$, so by substracting the two equations we obtain

$$(h_1g - gh_1)v = \lambda^2h^{-1}g^{-1}w - 2\lambda h^{-1}w + gh^{-1}w = h^{-1}g^{-1}(\lambda - g)^2w.$$

But $gw \neq \lambda w$, for otherwise we would have $ghv = \lambda\mu v + \lambda w = hgv$, and g and h and would commute on the whole of V_{i+1}. Thus h_1 and g do not commute. Repeating the argument with h_1 in place of h and so on we obtain inductively $h_j \in G^j$ that does not commute with g (recall that $G^0 = G$ and $G^j = [G, G^{j-1}]$). But $G^j = \{1\}$ for j large enough by the nilpotence of G, a contradiction. □

The direct product decomposition of the theorem shows that G_s is a homomorphic image of G, hence it is connected. Thus it is a torus, and we shall see later that it is the largest torus contained as a closed subgroup in G. For this reason it is called the *maximal torus* of G.

If G is only connected and solvable, there exist several maximal tori in G (i.e. tori contained as closed subgroups and maximal with respect to this property). We shall prove in Section 21 that the maximal tori are all conjugate in G, and G is the semidirect product of G_u with a maximal torus.

8. A glimpse at Lie algebras

In order to define the Lie algebra associated with an algebraic group, we first have to discuss tangent spaces.

Let first $X = V(I) \subset \mathbf{A}^n$ be an affine variety, and $P = (a_1, \dots a_n)$ a point of X. The *tangent space* $T_P(X)$ *of* X *at* P is defined as the linear subspace of $\mathbf{A}^n$ given by the equations

$$\sum_{i=1}^{n} (\partial_{x_i} f)(P)(x_i - a_i) = 0,$$

for all $f \in I$, where the x_i are the coordinate functions on $\mathbf{A}^n$. Geometrically this is the space of lines tangent to X at P. If $I = V(f_1, \dots, f_m)$, then we may restrict to the finitely many equations coming from the f_j in the above definition, because if $\Sigma_j g_j f_j$ is a general element of I, then

$$\begin{aligned} \partial_{x_i}(\sum_j g_j f_j)(P) &= \sum_j ((\partial_{x_i} g_j)(P) f_j(P) + g_j(P)(\partial_{x_i} f_j)(P)) \\ &= \sum_j g_j(P) \partial_{x_i} f_j(P), \end{aligned}$$

and so

$$\sum_{i=1}^{n} \partial_{x_i}(\sum_j g_j f_j)(P)(x_i - a_i) = \sum_j g_j(P) \sum_{i=1}^{n} \partial_{x_i} f_j(P)(x_i - a_i) = 0.$$

A drawback of this definition is that it depends on the choice of the embedding of X into $\mathbf{A}^n$. We can make it intrinsic at follows. Let M_P be the maximal ideal in $\mathcal{A}_X$ consisting of functions vanishing at P. Denote by $T_P(X)^*$ the dual k-vector space to $T_P(X)$, and define a map $\partial_P : M_P/M_P^2 \to T_P(X)^*$ by

$$\partial_P(\bar{f}) := \text{restriction of } \sum_{i=1}^{n} (\partial_{x_i} f)(P)(x_i - a_i) \text{ to } T_P(X),$$

where f is a polynomial in $k[x_1, \dots, x_n]$ with $f(P) = 0$ that maps to $\bar{f}$ in M_P/M_P^2. The map does not depend on the choice of f, because it vanishes on elements of $I(X)$ by definition of $T_P(X)$, and it also vanishes on elements of M_P^2 by a calculation using the Leibniz rule as above.

Lemma 8.1. (Zariski) *The map* $\partial_P : M_P/M_P^2 \to T_P(X)^*$ *is an isomorphism of* k*-vector spaces.*

Proof. For surjectivity one sees that all elements of $T_P(X)^*$ can be written as a sum $\Sigma\alpha_i(x_i - a_i)$ with suitable α_i, and these linear functions are preserved by ∂_P. For injectivity, $\partial_P(\bar{f}) = 0$ means that for a representative f one has

$$\sum_{i=1}^{n}(\partial_{x_i}f)(P)(x_i-a_i)=\sum_j \alpha_j \sum_{i=1}^{n}(\partial_{x_i}g_j)(P)(x_i-a_i)$$

for some $g_1,\dots,g_r \in I(X)$ and $\alpha_j \in k$. Replacing f by $f-\Sigma_j a_j g_j$ we may thus assume $\sum_{i=1}^{n}(\partial_{x_i}f)(P)(x_i-a_i)=0$, i.e. f has no linear term in the x_i-a_i, and therefore its image in M_P lies in M_P^2. □

For the above reason one calls M_P/M_P^2 the *Zariski cotangent space* of X at P, and its dual the *Zariski tangent space*.

Let now G be an affine algebraic group, and look at the tangent space $T_1(G)$ at the unit element 1. The maximal ideal $M_1 \subset \mathcal{A}_G$ corresponding to 1 is none but the kernel of the counit map $e: \mathcal{A}_G \to k$, and $\mathcal{A}_G$ decomposes as a direct sum $\mathcal{A}_G \cong M_1 \oplus k$ via e. Given an element of $T_1(G)$, identified with a k-linear map $\phi: M_1/M_1^2 \to k$, the above decomposition allows us to view it as a map $\mathcal{A}_G \to k$ vanishing on M_1^2 and k; in fact we get a bijection between elements of $T_1(G)$ and k-linear maps $\mathcal{A}_G \to k$ with this property. Using this bijection we may introduce a *Lie bracket* on $T_1(G)$ by setting $[\phi,\psi] := (\phi\otimes\psi-\psi\otimes\phi)\circ\Delta$, where $\Delta: \mathcal{A}_G \to \mathcal{A}_G \otimes_k \mathcal{A}_G$ is the comultiplication map. This is a k-bilinear function that satisfies $[\phi,\phi]=0$ and the *Jacobi identity*

$$[\phi,[\psi,\chi]]+[\psi,[\chi,\phi]]+[\chi,[\phi,\psi]]=0.$$

Thus $T_1(G)$ is equipped with a *Lie algebra structure.*

Definition 8.2. *Let G be an affine algebraic group. the Lie algebra* $\mathrm{Lie}(G)$ *of G is the (Zariski) tangent space of G at 1 equipped with the above Lie algebra structure.*

Examples 8.3.

(1) $\mathrm{Lie}(\mathrm{GL}_n)$ is the Lie algebra $\mathfrak{gl}_n$ of all $n\times n$ matrices. Indeed, the tangent space of GL_n at 1 is the same as that of $\mathbf{A}^{n^2}$, namely $\mathbf{A}^{n^2}$.

(2) $\mathrm{Lie}(\mathrm{SL}_n)$ is the Lie algebra $\mathfrak{sl}_n$ of all $n\times n$ matrices of trace 0. This is because $\partial_{x_{ij}}(\det(x_{ij})-1)(\delta_{ij})(x_{ij}-\delta_{ij}) = \delta_{ij}(x_{ij}-1)$ (Kronecker delta). In this way we obtain those matrices M where $M-1$ has trace 0; translating from 1 to 0 we get $\mathfrak{sl}_n$ embedded as a vector subspace in $\mathbf{A}^{n^2}$.

A morphism $G \to G'$ of algebraic groups induces a homomorphism $\mathrm{Lie}(G) \to \mathrm{Lie}(G')$ of Lie algebras. In particular, any representation $G \to \mathrm{GL}_n$ induces a Lie algebra representation $\mathrm{Lie}(G) \to \mathfrak{gl}_n$.

Remark 8.4. The theorems of Kolchin and Lie-Kolchin have analogues (in fact, predecessors) in Lie algebra theory. *Engel's theorem* states that if $\rho: L \to \mathfrak{gl}_n$ is a Lie algebra representation such that $\rho(x)$ is a nilpotent matrix for all $x \in L$, then ρ stabilizes a complete flag in k^n. *Lie's theorem* says that a similar conclusion holds for representations of solvable Lie algebras. Here one calls a Lie algebra L solvable if the subalgebras $D^i(L)$ defined inductively by $D^0(L) = L$ and $D^i(L) = [D^{i-1}(L), D^{i-1}(L)]$ satisfy $D^i(L) = 0$ for all i large enough.

One may show that if G is solvable as a group, then Lie(G) is solvable as a Lie algebra (the converse also holds in characteristic 0, but not in characteristic $p > 0$!), and hence deduce Lie's theorem for Lie(G) from the Lie-Kolchin theorem on G°. Similarly, one may show that Kolchin's theorem for unipotent G implies that of Engel for Lie(G).

Chapter 3. Flag Varieties and the Borel Fixed Point Theorem

One way to rephrase the Lie-Kolchin theorem is the following: the natural action of a connected solvable subgroup of $\mathrm{GL}(V)$ on the set of complete flags in V has a fixed point. In the first part of this chapter we show that this set carries an additional structure, namely that of a projective variety. In the simplest case when $\dim V = 2$ the problem reduces to classifying lines through the origin in V which indeed correspond to points of the projective line $\mathbf{P}^1$. The higher-dimensional case is more difficult, however, and gives rise to the so-called flag varieties. With this basic example at hand one gains more insight into the main theorem of this section, the Borel fixed point theorem. It states that quite generally the action of a connected solvable group on a projective variety has a fixed point. For the proof we shall have to develop some foundational material from the theory of quasi-projective varieties.

9. Quasi-projective varieties

We identify points of *projective n-space* over our algebraically closed base field k with

$$\mathbf{P}^n(k) := (\{P = (a_0, \dots, a_n) : a_i \in k\} \setminus \{(0, \dots, 0)\}) \,/ \sim,$$

where $\sim$ is the equivalence relation for which $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if and only if there exists $\lambda \in k^\times$ with $a_i = \lambda b_i$ for all i.

Definition 9.1. *Let $f_1, \dots, f_m \in k[x_0, \dots, x_n]$ be* homogeneous *polynomials. Define*

$$V(f_1, \dots, f_m) := \{P \in \mathbf{P}^n(k) : \, f_i(P) = 0, \, 1 \le i \le m\}.$$

A subset of $\mathbf{P}^n(k)$ of this form is called a projective variety*. These subsets are the closed subsets of a topology on $\mathbf{P}^n$ called the Zariski topology.*

Remark 9.2. We say that a non-homogeneous polynomial vanishes at a point of $\mathbf{P}^n(k)$ if it vanishes on all of its representatives. If $X \subset \mathbf{P}^n$ is a projective closed subset, then the ideal $I(X) \subset k[x_0, \dots, x_n]$ of polynomials vanishing on X has the following additional property: each homogeneous component of a polynomial in $I(X)$ is contained in $I(X)$. (Indeed, if $f = \Sigma f_d$ with f_d homogeneous of degree d, then $0 = f(\lambda a_0, \dots, \lambda a_n) = \Sigma \lambda^d f_d(a_0, \dots, a_n)$ can only hold for all $\lambda \neq 0$ if $f_d(a_0, \dots, a_n) = 0$ for all d, because k is infinite.) Ideals in $k[x_0, \dots, x_n]$ having this property are called *homogeneous* ideals.

There is the following analogue of the Nullstellensatz for projective varieties: if $I \subset k[x_0, \dots, x_n]$ is a homogeneous ideal that equals its radical and does not contain the ideal $(x_0, \dots, x_n)$, then $I(V(I)) = I$. This is not hard to derive from the affine Nullstellensatz. Note that $V(x_0, \dots, x_n) = \emptyset$, so the additional condition is necessary.

Consider the Zariski open subsets

$$D_+(x_i) := \mathbf{P}^n \setminus V(x_i) = \{P = (x_0, \dots x_n) \in \mathbf{P}^n(k) : x_i \neq 0\}$$

for all $0 \leq i \leq n$. These form an open covering of $\mathbf{P}^n$. The points of $D_+(x_i)$ are in bijection with those of $\mathbf{A}^n$ via the maps

$$(x_0, \dots, x_n) \mapsto (\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i})$$

and

$$(t_1, \dots, t_n) \mapsto (t_1, \dots, t_{i-1}, 1, t_i \dots, t_n).$$

Here if $X = V(J)$ is a projective variety, then

$$X^{(i)} := X \cap D_+(x_i) = V(J^{(i)})$$

is an affine variety in $D_+(x_i) \cong \mathbf{A}^n$ for all i, where $J^{(i)} \subset k[t_1, \dots, t_n]$ is the ideal formed by the polynomials

$$f^{(i)}(t_1, \dots, t_n) = f(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_n)$$

for all $f \in J$. Conversely, if $X_i = V(I_i) \subset D_+(x_i)$ is an affine variety, its *projective closure* $\overline{X}_i \subset \mathbf{P}^n$ is defined as its closure in $\mathbf{P}^n$ for the Zariski topology. It can be described as $V(I)$, where I is the homogeneous ideal formed by all polynomials arising as

$$G = x_i^d g(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i})$$

for some $g \in I_i$ of degree d. One has $(\overline{X}_i)^{(i)} = X_i$, so the inclusion map $D_+(x_i) \to \mathbf{P}^n$ is a homeomorphism in the Zariski topology.

Example 9.3. For the conic $X = V(x_1x_2 - x_0^2)$ in $\mathbf{P}^2$ the subsets $X^{(1)}$ and $X^{(2)}$ are affine parabolas of equations $x_2 = x_0^2$ and $x_1 = x_0^2$, respectively, whereas $X^{(0)}$ is the affine hyperbola $x_1x_2 = 1$.

Definition 9.4. *A* quasi-projective *variety is a Zariski open subset of a projective variety.*

This is a common generalisation of affine and projective varieties (any affine variety is open in its closure). Another example of a quasi-projective variety is the complement in $\mathbf{P}^n$ of a projective variety. It need not be affine in general.

We now define products of quasi-projective varieties, beginning with the product of two projective spaces $\mathbf{P}^n$ and $\mathbf{P}^m$. The image of the naive map $\mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{n+m}$ is not closed, so we need another kind of embedding to realise it as a projective variety.

Definition 9.5. *The* Segre embedding $S^{n,m} : \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N$ *(where* $N := nm + n + m$*) is the (set-theoretic) map defined by*

$$S^{n,m}((a_0, \dots, a_n), (b_0, \dots, b_m)) = (a_0b_0, a_0b_1, \dots, a_ib_j, \dots, a_nb_{m-1}, a_nb_m),$$

the products a_ib_j *being listed in lexicographic order.*

It is clear from the definition that $S^{n,m}$ is injective.

Lemma 9.6. *The image of* $S^{n,m}$ *is a closed subvariety of* $\mathbf{P}^N$.

Proof. To ease notation, denote the coordinate functions on $\mathbf{P}^N$ by w_{ij} $(0 \leq i \leq n,\ 0 \leq j \leq m)$. We claim that the closed subvariety

$$W := V(w_{ij}w_{kl} - w_{kj}w_{il}\ :\ 0 \leq i,k \leq n,\ 0 \leq j,l \leq m) \subset \mathbf{P}^N$$

is exactly the image of $S^{n,m}$. The inclusion $\mathrm{Im}\,(S^{n,m}) \subset W$ is obvious. For the converse, pick $Q = (q_{00}, \dots, q_{nm}) \in W$. By permuting the coordinates if necessary we may assume $q_{00} \neq 0$. Then $Q = S^{n,m}((q_{00}, \dots, q_{n0}), (q_{00}, \dots, q_{0m}))$, because $q_{i0}q_{0l} = q_{00}q_{il}$ according to the equations of W. □

Remark 9.7. In the above proof we have in fact shown the equality $W \cap D_+(w_{00}) = S^{n,m}(D_+(x_0) \times D_+(x_0))$. This holds in general: $W \cap D_+(w_{ij}) = S^{n,m}(D_+(x_i) \times D_+(x_j))$. Thus W has a standard affine open covering by copies of $D_+(x_i) \times D_+(x_j) \cong \mathbf{A}^{n+m}$. The construction of the above proof also shows that the map $\mathbf{A}^n \times \mathbf{A}^m \to S^{n,m}(D_+(x_i) \times D_+(x_j))$ is a homeomorphism in the Zariski topology.

Definition 9.8. *If* $X \subset \mathbf{P}^n$ *and* $Y \subset \mathbf{P}^m$ *are quasi-projective varieties, the product* $X \times Y$ *is defined as* $S^{n,m}(X \times Y) \subset \mathbf{P}^N$.

Lemma 9.9. *The product* $X \times Y$ *is a quasi-projective variety. If* X *and* Y *are projective, then so is* $X \times Y$.

Proof. This follows from the above remark and the following easy topological statement (applied to the $D_+(x_i)$ and the $D_+(x_i) \times D_+(x_j)$): *If a topological space* X *has an open covering by subspaces* U_i*, then* $Z \subset X$ *is open (resp. closed) if and only if each* $Z \cap U_i$ *is open (resp. closed) in* U_i. □

10. Flag varieties

We now show that the set of complete flags in an n-dimensional vector space V may be endowed with the structure of a projective variety. As a first step we consider subspaces of fixed dimension d.

Recall that the exterior algebra $\Lambda(V) = \bigoplus \Lambda^d(V)$ is defined by

$$\Lambda(V) := \bigoplus_{d=0}^{\infty} V^{\otimes d}/\langle x \otimes x \rangle.$$

The image of $v_1 \otimes \cdots \otimes v_d$ in $\Lambda^d(V)$ is denoted by $v_1 \wedge \cdots \wedge v_d$. If $e_1, \dots, e_n$ is a basis of V, then the elements $e_{i_1} \wedge \cdots \wedge e_{i_d}$ form a basis of $\Lambda^d(V)$ for $i_1 < \cdots < i_d$. Thus $\Lambda^d(V)$ has dimension $\binom{n}{d}$; in particular, it has dimension 1 for $n = d$. In this case, given vectors $v_1, \dots, v_d$ with $v_i = \Sigma a_{ij}e_j$, one has $v_1 \wedge \cdots \wedge v_d = \det(a_{ij})e_1 \wedge \cdots \wedge e_d$.

Now denote by $\mathrm{Gr}_d(V)$ the set of d-dimensional subspaces in V. The *Plücker embedding* $p_d\ :\ \mathrm{Gr}_d(V) \to \mathbf{P}(\Lambda^d(V))$ is defined by sending a dimension d subspace S to $\Lambda^d(S)$. Explicitly the map can be described as follows: if $e_1, \ldots, e_n$ is a basis of V, then giving a basis $v_1, \ldots, v_d$ for S is the same as giving an $n \times d$ matrix with coefficients in k. Then $p_d(S)$ is the point of $\mathbf{P}^{\binom{n}{d}-1}$ given by the $d \times d$ minors of this matrix.

Lemma 10.1. *The map p_d is injective.*

Proof. Assume S_1 and S_2 are two subspaces of dimension d in V. We may choose bases of S_1 and S_2 as follows: $e_1, \ldots, e_d$ is a basis of S_1, and $e_r, \ldots, e_{r+d-1}$ is a basis of S_2. Now $p_d(S_1) = p_d(S_2)$ is equivalent to $e_1 \wedge \cdots \wedge e_d = \lambda e_r \wedge \cdots \wedge e_{r+d-1}$ for some $\lambda \in k^\times$, which can only hold with $r = 1$. □

Example 10.2. The simplest nontrivial case is when $n = 4$, $d = 2$. If $e_0, \ldots e_3$ is a basis of V and $v_1 = \Sigma a_i e_i$, $v_2 = \Sigma b_i e_i$ generate a 2-dimensional subspace, the image $p_2(\langle v_1, v_2 \rangle) \in \mathbf{P}^{\binom{4}{2}-1} = \mathbf{P}^5$ is the point $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$ with $p_{ij} = a_i b_j - b_i a_j$. Denote the homogeneous coordinates on $\mathbf{P}^5$ by $x_{01}, \ldots, x_{23}$ as above. One may then check that the image of p_2 is the projective hypersurface of equation $x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12} = 0$, called the *Plücker quadric.*

In general, we have:

Proposition 10.3. *The image of p_d is a closed subvariety of $\mathbf{P}^{\binom{n}{d}-1}$ for $0 \leq d \leq n$.*

Proof. The point of $\mathbf{P}^{\binom{n}{d}-1}$ defined by a vector $w \in \Lambda^d(V)$ is in the image of p_d if and only if w is of the form $w = \lambda v_1 \wedge \cdots \wedge v_d$ with $v_i \in V$ and $\lambda \in k^\times$. We first show that this happens if and only if the kernel V_w of the map $V \to \Lambda^{d+1}(V)$, $v \mapsto v \wedge w$ is of dimension d, and otherwise $\dim V_w < d$. Indeed, choose a basis $v_1, \ldots, v_m$ of V_w, and extend it to a basis of V by adding vectors $v_{m+1}, \ldots, v_n$. Then w is expressed as a linear combination of terms of the form $v_{i_1} \wedge \cdots \wedge v_{i_d}$, $i_1 < \cdots < i_d$. For each $1 \leq i \leq d$ we have $v_{i_1} \wedge \cdots \wedge v_{i_d} \wedge v_i = 0$ if $i = i_j$ for some j, and otherwise these are linearly independent $(d+1)$-vectors. Since $w \wedge v_i = 0$ for $1 \leq i \leq m$, this implies that each v_i must be one of these v_{i_j}'s. Thus $m \leq d$, with equality if and only if $w = \lambda v_1 \wedge \cdots \wedge v_d$.

Now embed $\Lambda^d(V)$ in $\mathrm{Hom}_k(V, \Lambda^{d+1}(V)$ via the map $w \mapsto (v \mapsto v \wedge w)$. This induces a closed embedding $\mathbf{P}(\Lambda^d(V)) \hookrightarrow \mathbf{P}(\mathrm{Hom}_k(V, \Lambda^{d+1}(V))$. By the above observation here the points of $\mathrm{Im}\,(p_d)$ come from linear maps whose image has dimension $\leq n-d$. Choosing bases, this means that the $(n-d+1) \times (n-d+1)$ minors of the matrix of the map vanish. These give rise to homogeneous polynomials on $\mathbf{P}(\mathrm{Hom}_k(V, \Lambda^{d+1}(V))$, and hence exhibit $\mathrm{Im}\,(p_d)$ as the intersection of $\mathbf{P}(\Lambda^d(V))$ with a Zariski closed subset. □

The above projective variety is called a *Grassmann variety* or a *Grassmannian.*

Now we come to flag varieties. These parametrise flags in finite dimensional vector spaces. So let V be an n-dimensional vector space, and denote by $\mathrm{Fl}(V)$ the set of complete flags in V. Define a map $p_V : \mathrm{Fl}(V) \to \mathrm{Gr}_0(V) \times \cdots \times \mathrm{Gr}_n(V)$ by sending a flag $V_0 \subset \cdots \subset V_n$ to $(p_0(V_0), \ldots, p_n(V_n))$. The map is obviously injective.

Proposition 10.4. *The image of p_V is Zariski closed, and hence p_V realises* $\mathrm{Fl}(V)$ *as a projective variety.*

Proof. It will be enough to show that the subset $Z_d \subset \mathrm{Gr}(V_d) \times \mathrm{Gr}(V_{d+1})$ consisting of pairs $(p_d(V_d), p_{d+1}(V_{d+1}))$ satisfying $V_d \subset V_{d+1}$ is closed. Indeed, then the image of p_V will arise as the intersection of the closed subsets $\mathrm{Gr}_0(V) \times \cdots \times \mathrm{Gr}_{d-1}(V) \times Z_d \times \mathrm{Gr}_{d+2}(V) \times \cdots \times \mathrm{Gr}_n(V)$.

Putting $w_d := p_d(V_d)$, $w_{d+1} := p_{d+1}(V_{d+1})$ it comes out from the previous proof that $V_d = V_{w_d}$ and $V_{d+1} = V_{w_{d+1}}$. Thus we are dealing with the condition $V_{w_d} \subset V_{w_{d+1}}$, which holds if and only if the kernel of the map $V \to \Lambda^{d+1}(V) \times \Lambda^{d+2}(V)$, $v \mapsto (v \wedge w_d, v \wedge w_{d+1})$ is exactly V_{w_d}. Again by the above proof, this is the same as requiring that the image has dimension $\leq n - d$, which is again a determinant condition. □

We call the above projective variety the *variety of complete flags* in V.

Remark 10.5. Of course, one may also study flag varieties parametrising non-complete flags (increasing chains of subspaces of fixed dimensions). These arise from the above by projection to the product of some components of $\mathrm{Gr}_0(V) \times \cdots \times \mathrm{Gr}_n(V)$.

11. Function fields, local rings and morphisms

We next discuss rational functions and morphisms for quasi-projective varieties. *We assume everywhere that our varieties are irreducible.* This is not a serious restriction, because in our applications all varieties will be either irreducible or finite disjoint unions of irreducibles (think of algebraic groups). The definition will generalise in a straightforward manner to the latter case.

First assume X is affine. The *function field* $k(X)$ of X is the quotient field of the coordinate ring $\mathcal{A}_X$ of X, which is an integral domain by the irreducibility assumption on X. Its elements are represented by quotients of regular functions f/g. If $P \in X$ is a point, the *local ring* $\mathcal{O}_{X,P}$ is the subring of $k(X)$ consisting of functions that have a representative with $g(P) \neq 0$. It is the same as the localisation of $\mathcal{A}_X$ by the maximal ideal corresponding to P. One thinks of it as the ring of functions 'regular at P'.

Lemma 11.1. *For an affine variety X one has* $\mathcal{A}_X = \bigcap_P \mathcal{O}_{X,P}$.

Proof. To show the nontrivial inclusion, pick $f \in \cap_P \mathcal{O}_{X,P}$, and choose for each P a representation $f = f_p/g_P$ with $g_P(P) \neq 0$. The ideal $I := \langle g_P : P \in X \rangle \subset \mathcal{A}_X$ satisfies $V(I) = \emptyset$ by our assumption on f, so by the Nullstellensatz $I = \mathcal{A}_X$. In particular, there exist $P_1, \ldots, P_r \in X$ with $1 = g_{P_1} h_{P_1} + \ldots g_{P_r} h_{P_r}$ with some $h_{P_i} \in \mathcal{A}_X$. Thus

$$f = \sum_{i=1}^{r} f g_{P_i} h_{P_i} = \sum_{i=1}^{r} (f_{P_i}/g_{P_i}) g_{P_i} h_{P_i} = \sum_{i=1}^{r} f_{P_i} h_{P_i} \in \mathcal{A}_X.$$

□

Next we define the function field $k(X)$ for a *projective variety* X as follows. Consider the ring

$$\mathcal{R}_X := \{\frac{f}{g} : f, g \in k[x_0, \ldots, x_n] \text{ homogeneous}, \; g \notin I(X), \deg f = \deg g\}.$$

The $f/g \in \mathcal{R}_X$ with $f \in I(X)$ form a maximal ideal $\mathcal{M}_X$, because $f/g \notin \mathcal{M}_X \Rightarrow g/f \in \mathcal{M}_X$. Therefore $k(X) := \mathcal{R}_X/\mathcal{M}_X$ is a field, the *function field* of X. Its elements, called *rational functions*, are represented by quotients of homogeneous polynomials of the same degree.

Now consider the standard affine open covering of X.

Lemma 11.2. *For each i one has $k(X^{(i)}) \cong k(X)$.*

Proof. Define maps

$$\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)} \in k(X) \;\mapsto\; \frac{f(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)}{g(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)} \in k(X^{(i)})$$

and

$$\frac{f^{(i)}(t_1, \ldots, t_n)}{g^{(i)}(t_1, \ldots, t_n)} \in k(X^{(i)}) \;\mapsto\; x_i^{e-d} \frac{x_i^d f^{(i)}(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})}{x_i^e g^{(i)}(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})},$$

where $d = \deg(f^{(i)})$, $e = \deg(g^{(i)})$. The reader will check that the two maps are inverse to each other. □

As in the affine case, one defines the *local ring* $\mathcal{O}_{X,P}$ at P is the subring of $k(X)$ consisting of functions that have a representative with $g(P) \neq 0$.

Corollary 11.3. *For each i with $P \in X^{(i)}$ one has $\mathcal{O}_{X^{(i)},P} \cong \mathcal{O}_{X,P}$. Therefore $\mathcal{O}_{X^{(i)},P} \cong \mathcal{O}_{X^{(j)},P}$ for $P \in X^{(i)} \cap X^{(j)}$.*

Thus it makes sense to define the function field (resp. local ring at a point) for a quasi-projective variety as the function field (resp. local ring) of its projective closure; this agrees with the definition made for affine varieties.

We now define morphisms for quasi-projective varieties. Unfortunately the same definition as in the affine case does not work, for we shall see in Corollary 13.3 below that on an irreducible projective variety there are no nonconstant everywhere regular functions.

We proceed as follows. First, if X is a quasi-projective variety, and $U \subset X$ is an open subset, we define the *ring of regular functions on* U by

$$\mathcal{O}(U) := \bigcap_{P \in U} \mathcal{O}_{X,P},$$

the intersection being taken inside $k(X)$. Next, we define a *morphism* $\phi : X \to Y$ *of quasi-projective varieties* as a continuous map such that for all open $U \subset Y$ and all $f \in \mathcal{O}(U)$ one has $f \circ \phi \in \mathcal{O}(\phi^{-1}(U))$.

Examples 11.4.

(1) If X and Y are affine, this is the same notion as before. Indeed, it is enough to check this in the case $Y = \mathbf{A}^m$. Then if $f_1, \dots, f_m \in \mathcal{A}_X$, then $\phi = (f_1, \dots, f_m)$ has the above property, and conversely, if ϕ is as above, then for $U = \mathbf{A}^m$ the functions $f_i := t_i \circ \phi$ are in $\mathcal{A}_X$ and define a morphism in the old sense, where the t_i are the coordinate functions.

(2) If $X \subset \mathbf{P}^n$ is projective, and $F_1, \dots, F_m$ are homogeneous polynomials of the same degree d with $V(F_0, \dots, F_m) \cap X = \emptyset$, then $\phi(P) := (F_0(P), \dots, F_m(P)) \in \mathbf{P}^m$ defines a morphism of X into $\mathbf{P}^m$. Indeed, note first that ϕ is everywhere defined by the assumption $V(F_0, \dots, F_m) \cap X = \emptyset$. Over each $X^{(i)}$ it coincides with the map $(F_1/x_i^d, \dots, F_m/x_i^d)$ which is given by everywhere regular functions. Thus ϕ restricts to a morphism on each affine variety $X^{(i)}$, and thus it is a morphism on X because the definition of morphisms is local (note that if a rational function is regular on an open covering, then it is regular).

(3) If X is a quasi-projective variety and $U \subset X$ is an open subset, then the inclusion map $U \to X$ is a morphism.

The following lemma will be used many times in what follows.

Lemma 11.5. *Let X be a quasi-projective variety, and $P \in X$. Then P has an affine open neighbourhood isomorphic (as a quasi-projective variety) to an affine variety. Hence X has an open covering by affine varieties.*

Proof. By cutting X with some $D_+(x_i)$ containing P we may assume that X is a Zariski open subset in some affine variety Y. Since a basis of the Zariski topology of Y is given by open subsets of the form $D(f)$, we only have to prove that each $D(f)$ is isomorphic to an affine variety. To prove this we reduce to the case $Y = \mathbf{A}^n$. Then $D(f)$ is isomorphic to the closed subset $V(x_{n+1}f - 1) \subset \mathbf{A}^{n+1}$ (a trick we already used in realising GL_n as an affine variety). We leave it to the readers to check that the map $(x_1, \dots, x_n) : V(x_{n+1}f - 1) \to D(f)$ is indeed an isomorphism of quasi-projective varieties. □

12. Dimension

Once we have defined function fields, we can introduce the concept of dimension in algebraic geometry.

Definition 12.1. *The dimension* $\dim X$ *of an irreducible quasi-projective variety* X *is the transcendence degree of* $k(X)$ *over* k*. In general it is the maximum of the dimensions of the irreducible components.*

Recall that the transcendence degree means the maximal number of algebraically independent elements in $k(X)$. The definition generalises the notion of dimension for vector spaces, as $k(\mathbf{A}^n) = k(t_1, \dots, t_n)$, and so $\dim \mathbf{A}^n = n$. Similarly, $k(\mathbf{P}^n) = k(D_+(x_i)) = k(t_1, \dots, t_n)$, and therefore $\dim \mathbf{P}^n = n$.

In most of this text we shall get away with some very coarse properties of dimension.

Lemma 12.2. *If* $\phi : X \to Y$ *is a surjective morphism, then* $\dim Y \leq \dim X$*.*

Proof. We may assume that X is irreducible, and hence so is Y, being its continuous image. Then ϕ induces a homomorphism of function fields $\phi^* : k(Y) \to k(X)$ via $f \mapsto f \circ \phi$. Since ϕ^* must be an injection, the lemma follows from the definition of dimension. □

Remark 12.3. In fact, the lemma holds (with the same proof) under the weaker assumption that $\mathrm{Im}\,(\phi)$ is dense in Y. Note, however, that some restrictive assumption is needed on $\mathrm{Im}\,(\phi)$, because e.g. if ϕ is a constant map to $P \in Y$, then $\phi^*(f)$ is only defined if $f \in \mathcal{O}_{Y,P}$.

Proposition 12.4. *If* X *is irreducible, and* $Y \subset X$ *is a closed subvariety with* $Y \neq X$*, then* $\dim Y < \dim X$*.*

Proof. We may assume that Y is irreducible and moreover (by taking the projective closure and then cutting with a suitable $D_+(x_i)$) that X and Y are affine. Then $\mathcal{A}_Y \cong \mathcal{A}_X/P$ with a nonzero prime ideal P, and the proposition results from the following purely algebraic lemma. □

Lemma 12.5. *Let* A *be an integral domain which is a finitely generated* k*-algebra, and* P *a nonzero prime ideal in* A*. Then the transcendence degree of* A/P *is strictly smaller than that of* A*.*

Here the transcendence degree of an integral domain is defined as that of its quotient field.

Proof. Let $\bar{t}_1, \dots, \bar{t}_d$ be a maximal algebraically independent subset in A/P. Lift the $\bar{t}_i$ to elements $t_i \in A$. We show that for any nonzero $t_0 \in P$ the elements $t_0, t_1, \dots t_d$ are algebraically independent in A. If not, there is a polynomial $f \in k[x_0, \dots, x_d]$ with $f(t_0, \dots, t_d) = 0$. We may assume f is irreducible (because A is an integral domain) and that it is not a polynomial in x_0 only (because k is algebraically closed). It follows that reducing modulo P we obtain a nontrivial relation $f(0, \bar{t}_1, \dots, \bar{t}_d) = 0$, a contradiction. □

The following more subtle properties of dimension will not be used until Section 22.

Proposition 12.6. *Let $X \subset \mathbf{A}^n$ be an irreducible affine variety of dimension d, and $f \in k[x_1, \dots, x_n]$ a polynomial vanishing at some point of X. Then each irreducible component of the intersection $X \cap V(f)$ has dimension at least $d - 1$.*

Proof. This is a form of Krull's principal ideal theorem. For the algebraic version, see e.g. Chapter 11 of Atiyah–Macdonald, Introduction to Commutative Algebra; for the geometric version, §I.7 of Mumford's Red Book of Varieties and Schemes. □

Corollary 12.7. *Let $\phi : X \to Y$ be a morphism of quasi-projective varieties with dense image. Then for each point $P \in \mathrm{Im}\,(\phi)$ the fibre $\phi^{-1}(P)$ has dimension at least* $\dim X - \dim Y$.

Proof. Up to replacing Y with an affine open subset containing P, we may assume that Y is affine and embed it as a closed subvariety in some $\mathbf{A}^m$. Choose a polynomial $f_1 \in k[x_1, \dots, x_m]$ vanishing at P but not on the whole of Y. By Propositions 12.4 and 12.6 an irreducible component Z of $Y \cap V(f_1)$ passing through P has dimension exactly $s - 1$. Replacing Y by an affine open subset containing P and disjoint from the other components of $Y \cap V(f_1)$ we may assume $Y \cap V(f_1) = Z$. Repeating the procedure with Z and shrinking Y again, after s steps we arrive at polynomials $f_1, \dots, f_s$ with $Y \cap V(f_1, \dots, f_s) = \{P\}$. Then $\phi^{-1}(P) = \{Q \in X : \phi^* f_1(Q) = \cdots = \phi^* f_s(Q) = 0\}$. The corollary now follows from an inductive application of the proposition over an affine open covering of X. □

We shall see later (Proposition 15.9) that in fact the fibre dimension is exactly $\dim X - \dim Y$ over a dense open subset of Y. Our assumption that $\mathrm{Im}\,(\phi)$ is dense in Y was needed only in order to ensure (via Remark 12.3) that $\dim X - \dim Y$ is a nonnegative integer.

13. Morphisms of projective varieties

We shall now prove the following fundamental theorem.

Theorem 13.1. *Let $\phi : X \to Y$ be a morphism of quasi-projective varieties. If X is projective, then $\phi(X)$ is Zariski closed in Y.*

Here are some immediate corollaries.

Corollary 13.2. *If X is irreducible and projective, and Y is affine, then any morphism $X \to Y$ is constant.*

Proof. By embedding Y into some affine space we may assume $Y = \mathbf{A}^n$. By composing with the natural coordinate projections $\mathbf{A}^n \to \mathbf{A}^1$ we reduce to the case $n = 1$. Composing with the inclusion map $\mathbf{A}^1 \to \mathbf{P}^1$ we obtain a morphism $\widetilde{\phi} : X \to \mathbf{P}^1$ (by Example 11.4 (3) and the obvious fact that a composition of two morphisms is a morphism). By the theorem and the continuity of $\widetilde{\phi}$ the subset $\widetilde{\phi}(X) \subset \mathbf{A}^1$ is a closed and irreducible subset of $\mathbf{P}^1$, hence must be a point. □

Corollary 13.3. *Any regular function on an irreducible projective variety is constant.*

Proof. This is the special case $Y = \mathbf{A}^1$ of the previous corollary. □

The theorem will follow from the following statement:

Theorem 13.4. *Let X be a projective, Y a quasi-projective variety. Then the second projection $p_2 : X \times Y \to Y$ is a closed mapping, i.e. maps closed subsets to closed subsets.*

To see that Theorem 13.4 implies Theorem 13.1, one first proves

Lemma 13.5. *If Y is a quasi-projective variety, then the diagonal subvariety $\Delta(Y) \subset Y \times Y$ defined by $\{(P, P) : P \in Y\}$ is closed in $Y \times Y$.*

Proof. By covering Y with affine varieties we may assume Y is affine. Embedding Y into some $\mathbf{A}^n$ we see that $\Delta(Y) = (Y \times Y) \cap \Delta(\mathbf{A}^n)$, so it is enough to consider the case $Y = \mathbf{A}^n$ which is obvious. □

Proof of Theorem 13.1. Consider the *graph* $\Gamma_\phi \subset X \times Y$ of ϕ defined by $\{(P, \phi(P)) : P \in X\}$. It is the inverse image of $\Delta(Y)$ by the morphism $(\phi, \mathrm{id}) : X \times Y \to Y \times Y$, so it is closed by Lemma 13.5 and the continuity of (ϕ, id). But $\phi(X)$ is the image of Γ_ϕ by $p_2 : X \times Y \to Y$, so we conclude by Theorem 13.4. □

Remarks 13.6.

(1) The property of Lemma 13.5 is called the *separatedness* property of quasi-projective varieties, and that of Theorem 13.4 the *properness* of projective varieties. In older terminology proper varieties are also called complete.

(2) In classical parlance Theorem 13.4 is called the 'Main Theorem of Elimination Theory'. This is because (in the case when Y is affine) the equations for the image of a closed subset of $X \times Y$ in Y were found in the old times by an explicit procedure which may be regarded as a higher degree analogue of Gaussian elimination. The proof below, due to Grothendieck, will be nonconstructive but quicker.

Grothendieck's proof of Theorem 13.4 uses a form of Nakayama's Lemma:

Lemma 13.7. *Let R be a commutative ring with unit, $M \subset R$ a maximal ideal and N a finitely generated R-module. If $MN = N$, then there exists $f \in R \setminus M$ with $fn = 0$ for all $n \in N$.*

Proof. Let $n_1, \ldots, n_m$ be a generating system of N. By assumption for each $1 \leq i \leq m$ we may find $m_{ij} \in M$ with $n_i = \Sigma_j m_{ij} n_j$. Let $[m_{ij}]$ be the $n \times n$ matrix formed by the m_{ij}, and $[n_j]$ the column vector formed by $n_1, \ldots, n_m$. Then $(\mathrm{id} - [m_{ij}])[n_j] = 0$, and multiplication by the adjoint matrix yields $\det(\mathrm{id} - [m_{ij}])N = 0$ using Cramer's rule. So the element $f := \det(\mathrm{id} - [m_{ij}])$, which lies in $1 + M \subset R \setminus M$, is a suitable one. □

Proof of Theorem 13.4. First, by embedding X to some $\mathbf{P}^n$ we reduce to the case when $X = \mathbf{P}^n$. Next, by taking an open covering of Y by affine varieties (Lemma 11.5) we reduce to the case when Y is affine; denote by R its coordinate ring. Given a closed subset $Z \subset \mathbf{P}^n \times Y$, it is enough to find for all $P \in Y \setminus p_2(Z)$ some $f \in R$ with $f(P) \neq 0$ but $f(Q) = 0$ for $Q \in p_2(Z)$, because then $D(f)$ is an affine open subset containing P but disjoint from $p_2(Z)$.

Now $\mathbf{P}^n \times Y$ has an affine open covering by the $D_+(x_i) \times Y$, which have coordinate ring $R_i := R[x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots x_n/x_i]$. The intersection $Z_i := Z \cap (D_+(x_i) \times Y)$ is a closed subvariety of $D_+(x_i) \times Y$. Write $S = R[x_0, \ldots, x_n]$ and $S_d \subset S$ for the R-submodule of homogeneous polynomials of degree d. Define

$$I_d := \{f \in S_d : f(x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots x_n/x_i) \in I(Z_i) \text{ for all } i\}.$$

Then $I := \oplus_d I_d$ is a homogeneous ideal in $S = \oplus_d S_d$. We show that for d large enough there is $f \in R \setminus M$ with $fS_d \subset I_d$, where M is the ideal of P in R. This will do the job, because then $fx_i^d \in I_d$ for all i, which by definition of I_d means $f \in I(Z_i)$ for all i, and hence f as a function on $\mathbf{P}^n \times Y$ vanishes on Z, i.e. f as a function on Y vanishes on $p_2(Z)$.

By the above lemma applied to S_d/I_d it will be enough to show that $S_d = I_d + MS_d$ for d large enough. Since Z_i is disjoint from $D_+(x_i) \times \{P\}$, i.e. $V(I(Z_i)) \cap V(MR_i) = \emptyset$, we have $I(Z_i) + MR_i = R_i$ by the Nullstellensatz for $D_+(x_i) \times Y$. Thus we find $f_i \in I(Z_i)$, $m_{ij} \in M$ and $g_{ij} \in R_i$ with $1 = f_i + \Sigma m_{ij} g_{ij}$. For d sufficiently large $g_{ij} x_i^d \in S_d$ for all i. It will now suffice to show that $f_i x_i^d \in I_d$ for d large enough, for then the equation $x_i^d = f_i x_i^d + \Sigma m_{ij} g_{ij} x_i^d$ will show $x_i^d \in I_d + MS_d$, and therefore for d even larger all degree d monomials in the x_i will be in $I_d + MS_d$, and these generate S_d. To find d with $f_i x_i^d \in I_d$, observe that for d large enough $f_i x_i^d \in S_d$ and it vanishes on $Z_i = Z \cap D_+(x_i)$. But then $f_i x_i^{d+1}$ vanishes on Z_i and on $V(x_i)$, so on the whole of Z, and in particular on the other Z_j as well. Hence $f_i x_i^{d+1} \in I_{d+1}$. □

14. The Borel fixed point theorem

Let G be an affine algebraic group, Y a quasi-projective variety. A (left) *action* of G on Y is a group action $G \times Y \to Y$ such that for each $g \in G$ the associated map $\phi_g : Y \to Y$ is an isomorphism of varieties. The *orbit* of $P \in Y$ under G is the set $\{gP : g \in G\}$. An orbit consisting of a single point is a fixed point.

One way to phrase the Lie-Kolchin theorem is to say that the natural action of a connected solvable subgroup of $\mathrm{GL}(V)$ on the projective variety $\mathrm{Fl}(V)$ of complete flags in V has a fixed point. In this section we prove the following vast generalisation.

Theorem 14.1. (Borel fixed point theorem) *An action of a connected solvable affine algebraic group G on a projective variety X has a fixed point.*

The proof below is due to Steinberg. It begins by solving the following particular case.

Proposition 14.2. *The theorem holds in the case when $G \subset \mathrm{GL}(V)$ with a finite dimensional vector space V, and $X \subset \mathbf{P}(V)$ is a closed subset stabilised by the induced action of G on $\mathbf{P}(V)$.*

Proof. The proposition states that the elements of G have a common eigenvector whose image in $\mathbf{P}(V)$ lies in X. We prove the proposition by induction on the dimension n of V, the case $n = 1$ being obvious. If $n = 2$, then $\mathbf{P}(V) \cong \mathbf{P}^1$, so there are two cases. Either X is the whole of $\mathbf{P}(V)$ and we are done by the Lie-Kolchin theorem. Or X is a finite set of points, but since G is connected, it must fix each of these points, and we are again finished. Now assume $n > 2$. By the Lie-Kolchin theorem the elements of G have a common eigenvector $v \in V$. We may assume $v \notin X$, for otherwise we are done. Then the restriction of the map $\mathbf{P}(V) \to \mathbf{P}(V/\langle v\rangle)$ to X is a morphism, and the image $X' \subset \mathbf{P}(V/\langle v\rangle)$ of X is closed by Theorem 13.1. Thus by induction G has a fixed point P in X'. Let w be a preimage of P in V, and $W = \langle v, w\rangle$. By construction W is G-invariant and we have $X' \cap \mathbf{P}(W) \neq \emptyset$, so we are done by the case $n = 2$. □

The proof in the general case proceeds by reduction to the above proposition. It uses two very useful standard lemmas from the theory of algebraic groups. Here is the first one.

Lemma 14.3. (Chevalley) *Let G be an affine algebraic group, H a closed subgroup. Then there is a morphism of algebraic groups $G \to \mathrm{GL}(V)$ for some finite dimensional V such that H is the stabilizer of a 1-dimensional subspace L in the induced action of G on V.*

Proof. Apply Corollary 3.4 to obtain an embedding $G \subset \mathrm{GL}(V)$ such that H is the stabilizer of a subspace $V_H \subset V$. Let $d = \dim V_H$. We claim that H is the stabilizer of the 1-dimensional subspace $L := \Lambda^d(V_H)$ in $\Lambda^d(V)$ equipped with its natural G-action (which is defined by setting $g(v_1\wedge\cdots\wedge v_d) = g(v_1)\wedge\cdots\wedge g(v_d)$ for $g \in G$). Indeed, H obviously stabilizes L. For the converse, let $g \in G$ be an element stabilizing L. We may choose a basis $e_1, \dots, e_n$ of V in such a way that $e_1, \dots, e_d$ is a basis of V_H and moreover $e_{m+1}, \dots, e_{m+d}$ is a basis of gV_H. We have to show $m = 0$, for then $gV_H = V_H$ and $g \in H$. If not, then $e_1 \wedge \cdots \wedge e_d$ and $e_{m+1} \wedge \cdots \wedge e_{m+d}$ are linearly independent in $\Lambda^d(V)$. On the other hand, we must have $e_1 \wedge \cdots \wedge e_d, e_{m+1} \wedge \cdots \wedge e_{m+d} \in L$ since g stabilizes L, a contradiction. □

The other statement we need is a lemma from algebraic geometry whose proof is postponed to the next section.

Lemma 14.4. *Let $\phi : X \to Y$ be a morphism of quasi-projective varieties with Zariski dense image. Then $\phi(X)$ contains a nonempty open subset of Y.*

The lemma will be used in the proof of theorem 14.1 via the following corollaries.

Corollary 14.5. *Let G be an affine algebraic group acting on a quasi-projective variety Y. Each orbit of G is open in its closure.*

Proof. Let O_P be the orbit of a point $P \in Y$ and Z its Zariski closure. As O_P is the image of the morphism $G \to Z$ sending $g \in G$ to $gP \in Z$, the lemma implies that O_P contains an open subset $U \subset Z$. Since it is the union of the gU for all $g \in G$, it is open in Z. □

Corollary 14.6. (Closed orbit lemma) *If Y is projective*[1]*, an orbit of minimal dimension is closed.*

Proof. Let O_P be such an orbit, Z its closure. Then Z is the union of orbits of G, because if $Q \in Z$ has an open neighbourhood U_Q containing $P' \in O_P$, then the open neighbourhood gU_Q of gQ contains gP'. By the lemma $Z \setminus O_P$ is a closed subset. It does not contain any irreducible component of Z, because Z is the union of the closures of the irreducible components of O_P which are themselves irreducible. From Proposition 12.4 applied to each irreducible component of Z we thus get that $Z \setminus O_P$ is a union of orbits of smaller dimension, and hence must be empty. □

Proof of Theorem 14.1. An orbit of G in X that has minimal dimension is closed by the above corollary, so it is also projective. Thus replacing X by this orbit we may assume there is a single G-orbit in X. Take $P \in X$, and let $G_P \subset G$ be its stabilizer. It is a closed subgroup, being the preimage of P by the morphism $g \mapsto gP$. Thus by Lemma 14.3

[1]This assumption is needed only to ensure that each orbit is a quasi-projective variety (by virtue of the previous corollary). The statement holds in a more general setting.

we find a representation of G on some finite dimensional V with G_P stabilizing a one-dimensional subspace in V, hence fixing a point Q in the induced action of G on the projective space $\mathbf{P}(V)$. Let Y be the orbit of Q in $\mathbf{P}(V)$ and Z that of (P, Q) in $X \times \mathbf{P}(V)$ (equipped with the product action). The natural projections $Z \to X$ and $Z \to Y$ are bijective G-morphisms, so it is enough to find a fixed point in Y (which must then be the whole of Y). For this it is enough to see that Y is closed in $\mathbf{P}(V)$, for then we may conclude by Proposition 14.2 applied to the image of G in $GL(V)$ (which is again connected and solvable; it is also closed by Corollary 15.4 below, but this was not used in the proof of the proposition). Now the closedness of Y follows from that of Z by Theorem 13.4. To prove the latter fact, observe that any G-orbit in $X \times \mathbf{P}(V)$ must project onto X by the projection $X \times \mathbf{P}(V) \to X$, because X is a single G-orbit. Thus the dimension of each G-orbit in $X \times \mathbf{P}(V)$ is at least $\dim X = \dim Z$, hence Z is an orbit of minimal dimension, and as such closed by Corollary 14.6. □

Chapter 4. Homogeneous Spaces and Quotients

We now arrive at a basic question that cannot be circumvented any longer: how to put a *canonical* structure of a quasi-projective variety on the set of (left) cosets of a closed subgroup H in an affine algebraic group G? The emphasis is on the adjective 'canonical', for if we can show that under some additional assumption the variety thus obtained is unique up to unique isomorphism, we have the right to call it 'the' quotient of G by H. In the case when H is normal it turns out that G/H is affine and carries the structure of a linear algebraic group. In general, however, the quotient will only be a quasi-projective variety. This construction will use the last dose of foundational inputs from algebraic geometry that we require in this text.

15. A generic openness property

Most of this section is devoted to the following technical statement, which was already used in a weaker form in the previous section (Lemma 14.4).

Proposition 15.1. *Let $\phi : X \to Y$ be a morphism of irreducible quasi-projective varieties with Zariski dense image. Then X contains a nonempty open subset U such that $\phi|_U$ is an open mapping.*

We shall also use the proposition for *disjoint* unions of irreducible varieties; the extension of the statement is straightforward.

We start the proof with some lemmas.

Lemma 15.2. *If Y is an affine variety, the projection $p_1 : Y \times \mathbf{A}^1 \to Y$ is an open mapping.*

Proof. It will be enough to prove that $p_1(D(f))$ is open in Y for each regular function $f \in \mathcal{A}_{Y\times\mathbf{A}^1} \cong \mathcal{A}_Y[t]$. Write $f = \Sigma f_i t^i$ with $f_i \in \mathcal{A}_Y$. We contend that $p_1(D(f)) = \bigcup D(f_i)$. Indeed, if $(P, \alpha) \in Y \times \mathbf{A}^1$ with $f(P, \alpha) \neq 0$, we must have $f_i(P) \neq 0$ for some i. Conversely, if $f_i(P) \neq 0$ for some i, then the polynomial $\Sigma f_i(P)t^i \in k[t]$ is nonzero, so we find $\alpha \in k$ with $f(P, \alpha) = \Sigma f_i(P)\alpha^i \neq 0$. □

Now recall that when X and Y are affine, a morphism ϕ as in the proposition induces a homomorphism $\phi^* : \mathcal{A}_Y \to \mathcal{A}_X$ which is in fact injective (because so is the induced morphism on function fields; cf. Remark 12.3).

Lemma 15.3. *The proposition holds in the case when X and Y are affine and ϕ^* induces an isomorphism $\mathcal{A}_X \cong \mathcal{A}_Y[f]$ with some $f \in \mathcal{A}_X$.*

Proof. In the case when f is transcendental over $k(Y)$ we have $X \cong Y \times \mathbf{A}^1$, and we are done by the previous lemma. So we may assume f is algebraic over $k(Y)$. Let $F \in k(Y)[t]$ be its minimal polynomial, and let $a \in \mathcal{A}_Y$ be a common denominator of its coefficients. Replacing

Y by the affine open subset $D(a)$ and X by an affine open subset of its preimage we may assume $a = 1$, i.e. $F \in \mathcal{A}_Y[t]$. Then $\mathcal{A}_X \cong \mathcal{A}_Y[t]/(F)$, because if $G \in \mathcal{A}_Y[t]$ satisfies $G(f) = 0$, we find $H, R \in \mathcal{A}_Y[t]$ with $G = HF + R$ and $\deg(R) < \deg(f)$ (observe that F is monic!), so that $R(f) = 0$ and hence $R = 0$ by minimality of $\deg(F)$. It follows that $\mathcal{A}_X$ is a free $\mathcal{A}_Y$-module of rank $d = \deg(f)$.

We now show that for the X and Y just obtained ϕ is an open mapping, i.e. for $f \in \mathcal{A}_X$ the image of the basic open set $D(f)$ by ϕ is open. Let $\Phi = t^d + f_{d-1}t^{d-1} + \cdots + f_0 \in \mathcal{A}_Y[t]$ be the characteristic polynomial of multiplication by f on the free $\mathcal{A}_Y$-module $\mathcal{A}_X$. We show that $\phi(D(f)) = \bigcup D(f_i)$. On the one hand, if P is a maximal ideal of $\mathcal{A}_X$ not containing f (this corresponds to a point of $D(f)$), then P does not contain all the f_i, for otherwise the equation $\Phi(f) = 0$ (Cayley-Hamilton theorem) would imply $f^d \in P$ and hence $f \in P$ by primeness of P, a contradiction. Conversely, if $Q \subset \mathcal{A}_Y$ is a maximal ideal coming from a point of one of the $D(f_i)$, it suffices to show that the radical R of the ideal $Q\mathcal{A}_X$ does not contain f. Indeed, by the Nullstellensatz R is the intersection of the maximal ideals containing it, so we find a maximal ideal P with $f \notin P$ and $P \cap \mathcal{A}_Y = Q$, which in turn corresponds to a point of $D(f)$ in the preimage of $D(f_i)$. To prove our claim about R, assume $f \in R$, i.e. $f^m \in Q$ for some $m > 0$. But then in the k-vector space $\mathcal{A}_X/Q \cong (\mathcal{A}_Y/Q)^d \cong k^d$ the image of f mod Q defines a nilpotent endomorphism, whereas by assumption its characteristic polynomial, which is Φ mod Q, is not of the form t^d, a contradiction. □

Proof of Proposition 15.1: Let U be an affine open subset (Lemma 11.5) of Y, and V an affine open subset of $\phi^{-1}(U)$. By the irreducibility of X the subset V is dense in $\phi^{-1}(U)$, hence so is $\phi(V)$ in U. Thus we may assume X and Y are affine by replacing them with V and U, respectively. In this case $\mathcal{A}_X$ is finitely generated as an $\mathcal{A}_Y$-algebra via the embedding $\phi^* : \mathcal{A}_Y \to \mathcal{A}_X$ (as it is already finitely generated over k), so we may write $\mathcal{A}_X = \mathcal{A}_Y[f_1, \ldots, f_n]$ for suitable f_i. Now consider the factorisation of ϕ^* into the sequence of morphisms

$$\mathcal{A}_Y \to \mathcal{A}_Y[f_1] \to \mathcal{A}_Y[f_1, f_2] \to \cdots \to \mathcal{A}_Y[f_1, \ldots, f_n] = \mathcal{A}_X.$$

By Proposition 2.5 each intermediate map here corresponds to a morphism of affine varieties, so we obtain a factorisation of ϕ into a composite of morphisms to which the above lemma applies. □

The full statement of Proposition 15.1 will be used in the next section. Here are some other important corollaries which already follow from the weaker form (Lemma 14.4).

Corollary 15.4. *Let $\phi : G \to G'$ be a morphism of affine algebraic groups. Then $\phi(G)$ is a Zariski closed subgroup in G'.*

Proof. Let $H \subset G'$ be the Zariski closure of $\phi(G)$. We first show that it is a subgroup of G' (this is in fact true for the closure of a subgroup in any topological group). Indeed, if $x \in H$ and U is an open neighborhood of x containing $h \in \phi(G)$, then $h'U$ is an open neighbourhood of $h'x$ containing $h'h$ for all $h' \in \phi(G)$. Thus $\phi(G)H \subset H$, and continuing the argument shows that $HH \subset H$. The inclusion $H^{-1} \subset H$ is checked in a similar way. Now by the Lemma Lemma 14.4 $\phi(G)$ contains a Zariski open subset U. Since the open sets $\phi(g)U$ cover $\phi(G)$ for $g \in G$, it follows that $\phi(G)$ is open and dense in H. If $h \in H$, then $h\phi(G) \cap \phi(G)$ is an intersection of dense open subsets, hence nonempty. It follows that $h \in \phi(G)\phi(G)^{-1} \subset \phi(G)$. □

Corollary 15.5. *Let G be a connected affine algebraic group. Then $[G, G]$ is closed and connected.*

Proof. Let $\phi_i : G^{2i} \to G$ be the morphisms considered in the proof of Lemma 7.3. By the previous corollary and the connectedness of G each element in the chain $\mathrm{Im}\,(\phi_1) \subset \mathrm{Im}\,(\phi_2) \subset \dots$ is a closed and connected, hence irreducible subgroup. It follows by dimension reasons that the chain must stabilize, so its union $[G, G]$ is closed and connected. □

We close this section by two statements needed later that are proven by a method similar to that of Proposition 15.1. The first of these is:

Proposition 15.6. *Let $\phi : X \to Y$ be an* injective *morphism of quasi-projective varieties with Zariski dense image. If the induced field extension $k(X)|k(Y)$ is separable, then it is an isomorphism.*

Using the arguments of the proof of Proposition 15.1, we see that the proposition is a consequence of the following lemma.

Lemma 15.7. *Assume that $\phi : X \to Y$ is a morphism of affine varieties and ϕ^* induces an isomorphism $\mathcal{A}_X \cong \mathcal{A}_Y[f]$ with f separable over $k(Y)$. Then there is an open subset $V \subset Y$ such that each point of V has exactly $[k(X) : \phi^*k(Y)]$ preimages in X.*

Proof. As in the proof of Lemma 15.3 we may assume $\mathcal{A}_X \cong \mathcal{A}_Y[t]/(F)$, where $F \in \mathcal{A}_Y[t]$ is the minimal polynomial of f over $k(Y)[t]$. The degree of F equals that of the field extension $\phi(X)|\phi^*k(Y)$; let us denote it by d. As F is a separable polynomial, its derivative F' is prime to F in the ring $k(Y)[t]$. Hence we find polynomials $A, B \in k(Y)[t]$ satisfying $AF + BF' = 1$. Multiplying with a common denominator in $g \in \mathcal{A}_Y$ of the coefficients of A and B we obtain polynomials $C = gA, D = gB \in \mathcal{A}_Y[t]$ with $CF + DF' = g$. We claim that $V = D(g)$ is a good choice. Assume Q is a maximal ideal in $\mathcal{A}_Y$ with $g \notin Q$. The image $\bar{F}$ of F in $(\mathcal{A}_Y/Q)[t] \cong k[t]$ has d distinct roots in k, for reducing $CF + DF' = g \bmod Q$ we obtain $\bar{C}\bar{F} + \bar{D}\bar{F}' \neq 0$, so $\bar{F}(\alpha) = 0$ implies $\bar{F}'(\alpha) \neq 0$. Thus $\bar{F}$ is a product of d distinct linear factors, and therefore by the Chinese Remainder Theorem

$\mathcal{A}_X/Q\mathcal{A}_X \cong k[t]/(\bar{F}) \cong k^d$. In particular, this ring is reduced, so the ideal $Q\mathcal{A}_X$ equals its radical. Now the preimages of the point of $D(g)$ defined by Q correspond to the maximal ideals $P_1, \dots, P_r \subset \mathcal{A}_X$ containing the radical ideal $Q\mathcal{A}_X$, so by the Nullstellensatz $Q\mathcal{A}_X = \cap P_i$ and by the Chinese Remainder Theorem $\mathcal{A}_X/Q\mathcal{A}_X \cong \oplus(\mathcal{A}_X/P_i) \cong k^r$. Thus $r = d$, as required. □

Remarks 15.8.

(1) An analysis of the above proof shows that when the polynomial F is not necessarily separable, at least one obtains that each point of Y has at most $d = [k(X) : \phi^* k(Y)]$ preimages in X.

(2) In the jargon of algebraic geometry the lemma claims that the morphism ϕ is *étale* over an open subset of Y, or in other words it is *generically étale.* Similarly, in the proof of Lemma 15.3 we have first proven that ϕ is *generically faithfully flat*, and then that a finite flat morphism is an open mapping. Generic faithful flatness is a key property in the theory of group schemes that is used for the construction of quotients in a more general setting than ours.

The second and last statement we prove in this section is:

Proposition 15.9. *Let $\phi : X \to Y$ be a morphism of irreducible quasi-projective varieties with Zariski dense image. There exists a dense open subset $U \subset Y$ such that for each $P \in U$ the fibre $\phi^{-1}(P)$ has dimension* $\dim X - \dim Y$.

Recall (Corollary 15.9) that each irreducible component of a nonempty fibre has dimension at least $\dim X - \dim Y$.

Proof. As in the proof of Proposition 15.1 we reduce to the case when X and Y are affine and $\mathcal{A}_X \cong \mathcal{A}_Y[f_1, \dots, f_n]$. We may assume that $f_1, \dots, f_r$ are algebraically independent over $k(Y)$, and $f_{r+1}, \dots, f_n$ are algebraic over $k(Y)(f_1, \dots, f_r)$. Here we must have $r = \dim X - \dim Y$, because the extension $k(X)|k(Y)(f_1, \dots, f_r)$ is finite. On the other hand, Corollary 2.6 implies that ϕ factors as $X \to Z \to Y$, where Z is the variety with coordinate ring $\mathcal{A}_Y[f_1, \dots, f_r]$. From the proof of Lemma 15.3 we see that the map $X \to Z$ has finite fibres, whereas the fibres of $Z \to Y$ have dimension r, since $Z \cong Y \times \mathbf{A}^r$ by construction. □

Corollary 15.10. *Given a morphism $\phi : G \to G'$ of connected algebraic groups, we have* $\dim G = \dim \mathrm{Im}\,(\phi) + \dim \mathrm{Ker}\,(\phi)$.

Proof. Using Corollary 15.4 we see that $\mathrm{Im}\,(\phi)$ is a connected algebraic group. Since ϕ is a morphism of algebraic groups, each fibre $\phi^{-1}(g)$ for $g \in \mathrm{Im}\,(\phi)$ is a coset of $\mathrm{Ker}\,(\phi)$, hence isomorphic to $\mathrm{Ker}\,(\phi)$ as a closed subvariety of G. In particular, they all have the same dimension, and the corollary follows from the proposition. □

16. Homogeneous spaces

As a first step towards the construction of quotients we study *homogeneous spaces.*

Definition 16.1. A (left) homogeneous space for an algebraic group G is a quasi-projective variety on which G acts *transitively* (on the left).

If $H \subset G$ is a closed subgroup, then clearly any reasonable definition of the quotient G/H should include the fact that G/H is a homogeneous space for G.

Lemma 16.2. *The irreducible components of a homogeneous space are the same as its connected components. They are all isomorphic as quasi-projective varieties.*

Proof. Same proof as in the special case of G as a homogeneous space under itself (Proposition 2.3 (1).) □

Lemma 16.3. *Let G be an algebraic group, X and Y irreducible homogeneous spaces under G, and $\phi : X \to Y$ a morphism compatible with the action on G. Then ϕ is an open mapping.*

Proof. By Proposition 15.1 there exists $U \subset X$ such that $\phi|_U$ is open. Then for all $g \in G$ the restriction $\phi|_{gU}$ must be open as well, because $x \mapsto gx$ is a homeomorphism of X onto itself. But the gU for all $g \in G$ form an open covering of X, whence the lemma. □

The following result will be the key step in the construction of the quotient of an affine algebraic group by a closed subgroup.

Proposition 16.4. *Let G be an affine algebraic group, and $H \subset G$ a closed subgroup. There exists a homogeneous space X for G together with a point P in X such that P is the stabilizer of H and the fibres of the natural surjection $\rho : G \to X$ given by $g \mapsto gP$ are exactly the left cosets gH of H.*

Proof. By Lemma 14.3 there is a morphism of algebraic groups $G \to \mathrm{GL}(V)$ such that H is the stabilizer of a 1-dimensional subspace $\langle v \rangle$ in V. Let X be the orbit of $P = \langle v \rangle$ in the projective space $\mathbf{P}(V)$. By Corollary 14.5 X is open in its Zariski closure, hence it is a quasi-projective variety and therefore a homogeneous space for G. It manifestly satisfies the other requirements of the proposition. □

In the case of a normal subgroup one can prove more:

Proposition 16.5. *If $H \subset G$ is a closed normal subgroup, there exists a finite dimensional vector space W and a morphism $\rho : G \to \mathrm{GL}(W)$ of algebraic groups with kernel H.*

By Corollary 15.4 $\mathrm{Im}\,(\rho)$ is a closed subgroup of $\mathrm{GL}(W)$, hence in this case G/H may be identified with not only a homogeneous space for G, but actually with an algebraic group.

Proof. Again start with a representation $\phi : G \to \mathrm{GL}(V)$, where H is the stabilizer of a 1-dimensional subspace $\langle v \rangle$ as in Lemma 14.3. In other words, v is a common eigenvector of the $h \in H$. Let V_H be the span of all common eigenvectors of the $h \in H$ in V. As in the proof of the Lie-Kolchin theorem, the fact that H is normal implies that V_H is G-invariant, so we may as well assume $V_H = V$. Thus V is the *direct* sum of the finitely many common eigenspaces $V_1, \dots, V_n$ of H.

Let $W \subset \mathrm{End}(V)$ be the subspace of endomorphisms that leave each V_i invariant; it is the direct sum of the $\mathrm{End}(V_i)$. There is an action of G on $\mathrm{End}(V)$ by $g(\lambda) = \phi(g) \circ \lambda \circ \phi(g)^{-1}$. This action stabilizes W, because if V_i is a common eigenspace for H, then so is $\phi(g)^{-1}(V_i)$ because H is normal, and is therefore preserved by λ. We thus obtain a morphism $\rho : G \to \mathrm{GL}(W)$ of algebraic groups.

It remains to show $H = \mathrm{Ker}\,(\rho)$. As W is the direct sum of the $\mathrm{End}(V_i)$ and each $h \in H$ acts on V_i by scalar multiplication, we have $\phi(h) \circ \lambda \circ \phi(h)^{-1} = \lambda$ for all $\lambda \in W$, i.e. $H \subset \mathrm{Ker}\,(\rho)$. Conversely, $g \in \mathrm{Ker}\,(\rho)$ means that $\phi(g)$ lies in the center of W, which is the direct sum of the centers of the $\mathrm{End}(V_i)$. Thus g acts on each V_i by scalar multiplication. In particular, it preserves the 1-dimensional subspace $\langle v \rangle$, i.e. it lies in H. □

The problem with the above constructions is that they are not canonically attached to the pair $H \subset G$. In the next two sections we carry out the extra work needed for making it canonical.

17. Smoothness of homogeneous spaces

We now bring into play an important local property of varieties. Recall that the tangent space of a point P on an affine variety X was defined in Section 8. Using Lemma 8.1 the definition immediately extends to arbitrary quasi-projective varieties.

Definition 17.1. *On an irreducible quasi-projective variety a point $P \in X$ is a* smooth point *if $\dim T_P(X) = \dim X$, otherwise it is a* singular point*. The variety is smooth if all of its points are smooth, otherwise it is singular.*

The definition obviously extends to finite disjoint unions of irreducible varieties, so in particular to algebraic groups and their homogeneous spaces.

Proposition 17.2. *A homogeneous space under an algebraic group G is a smooth variety. In particular, G itself is smooth.*

Proof. For a homogeneous space X the map $x \mapsto gx$ is an isomorphism of X with itself for each $g \in G$. On the other hand, an isomorphism takes smooth points to smooth points (this follows e.g. from Lemma 8.1 and the fact that the isomorphism preserves the maximal ideals of the points). So taking the transitivity of the G-action on X and Lemma 16.2 into account the proposition follows from the lemma below. $\square$

Lemma 17.3. *An irreducible quasi-projective variety has a smooth point.*

Proof. The proof consists of two steps.
Step 1. The lemma is true for an affine hypersurface $V(f) \subset \mathbf{A}^n$ defined by an irreducible polynomial $f \in k[x_1, \dots, x_n]$ with $\partial_{x_n} f \neq 0$.[2] Indeed, we find $P \in V(f)$ with $\partial_{x_n} f(P) \neq 0$, for otherwise by the Nullstellensatz the irreducible polynomial f would divide some power $(\partial_{x_n} f)^m$, hence $\partial_{x_n} f$, which is impossible. Then $T_P(V(f))$ is defined by a single nonzero linear equation and hence has dimension $n-1$, just like $V(f)$. This proves Step 1, and moreover shows that all points of the open subset $D(\partial_{x_n} f) \subset V(f)$ are smooth.
Step 2. An irreducible variety X contains an open subset U isomorphic to an open subset $V \subset V(f)$ for suitable f as above. This will prove the lemma, for irreducibility of $V(f)$ implies $V \cap D(\partial_{x_n} f) \neq \emptyset$. To prove Step 2 we may assume, by intersecting with some $D_+(x_i)$, that X is an open subset of some affine variety $\bar{X} \subset \mathbf{A}^m$, and then that $X = \bar{X}$. By a general theorem in algebra we find algebraically independent elements $x_1, \dots, x_{n-1} \in k(X)$ so that $k(X) = k(x_1, \dots, x_{n-1}, x_n)$ with x_n satisfying an irreducible polynomial $f \in k[x_1, \dots, x_{n-1}, x]$ with $\partial_x f \neq 0$. In particular, $k(X) \cong k(V(f))$. Choosing an open $U \subset X$ such that all x_i are regular on X we obtain a morphism $U \to V(f)$ defined by $(x_1, \dots, x_n)$. In the same way, the restrictions of the coordinate functions $y_1, \dots, y_m$ of $\mathbf{A}^m$ to X define a morphism $V \to X$ for suitable $V \subset V(f)$. The reader will check that these maps are inverse to each other whenever both are defined, so after possibly shrinking U and V we are done. $\square$

Remark 17.4. In the language of algebraic geometry, in Step 2 of the above proof we have shown that X is *birational* to the affine hypersurface $V(f)$.

Finally, we need for later use the following fact.

Lemma 17.5. *If P is a smooth point on a variety X, then the local ring $\mathcal{O}_{X,P}$ is a unique factorisation domain.*

Proof. Recall that we have identified $T_P(X)$ with the dual k-vector space of M_P/M_P^2, where M_P is the maximal ideal of $\mathcal{O}_{X,P}$. It follows that P is a smooth point if and only if $\dim_k M_P/M_P^2 = \dim X = \dim \mathcal{O}_{X,P}$. In commutative algebra a local ring with this property is called a *regular local ring*, and it is quite generally true that these rings are UFD's (see e.g. Matsumura, Commutative ring theory, Theorem

[2]This condition is automatic in characteristic 0, but not in characteristic $p > 0$: think of the polynomial $x_1^p + \cdots + x_n^p$.

20.3). There is also a direct proof of the special case we need which goes back to Zariski. It proceeds by comparing $\mathcal{O}_{X,P}$ with its completion which is a power series ring, hence a UFD; one shows that the UFD property 'descends' from the completion to $\mathcal{O}_{X,P}$. □

The lemma will be used through the following corollary.

Corollary 17.6. *Let f be a rational function on a quasi-projective variety X which is not regular at a smooth point $Q \in X$. Then there is a point $P \in X$ where $1/f$ is regular and $(1/f)(P) = 0$.*

Proof. By replacing X with an affine open subset containing Q we may assume X is affine, and may choose a representation $f = g/h$ with $g, h \in \mathcal{A}_X$. Since $\mathcal{O}_{X,Q}$ is a UFD which is a localisation of $\mathcal{A}_X$, in $\mathcal{O}_{X,Q}$ we may write $g = up_1^{a_1} \dots p_r^{a_r}$ and $h = vq_1^{b_1} \dots q_s^{b_s}$ with p_i, q_j irreducible elements of $\mathcal{O}_{X,Q}$ lying in $\mathcal{A}_X$ and u, v units in $\mathcal{O}_{X,Q}$; moreover, h is not a unit since $f \notin \mathcal{O}_{X,Q}$. By unique factorisation we may assume that there is no equality $p_i = wq_j$ with w a unit in $\mathcal{O}_{X,Q}$. Now if we find P with $g(P) \neq 0$ but $h(P) = 0$, we are done. Otherwise $h(P) = 0$ implies $g(P) = 0$ for all P, i.e. $g \in I(V(h))$. By the Nullstellensatz we thus have $g^m \in (h)$ for some $m > 0$, i.e. h divides g^m in $\mathcal{A}_X$, and hence in the local ring $\mathcal{O}_{X,Q}$ as well. This contradicts our assumptions that h is not a unit and there is no equation of the form $p_i = wq_j$. □

Remarks 17.7.

(1) The corollary is false in general. Consider the function y/x on the affine plane curve $y^2 = x^3$. It is not regular at the singular point $(0, 0)$, but x/y does not vanish anywhere on the curve.

(2) If $\dim X = 1$, one may choose $P = Q$ in the corollary. Indeed, in this case the local ring $\mathcal{O}_{X,Q}$ is a *discrete valuation ring*, and such rings always contain either f or $1/f$ for an f in their fraction field. However, for $\dim X > 1$ the ring $\mathcal{O}_{X,Q}$ does not have this property, so the corollary does not hold with $P = Q$.

18. Quotients

We now turn to the canonical construction of quotients.

Let G be an affine algebraic group, and $H \subset G$ a closed subgroup. Consider pairs (X, ρ) with X a homogeneous space of G and $\rho : G \to X$ is a surjective morphism of G-homogeneous spaces such that the fibres of ρ are unions of left cosets of H in G.

Definition 18.1. *The pair (X, ρ) is the quotient of G by H if for any other pair (X', ρ') as above there is a morphism $\phi : X \to X'$ of G-homogeneous spaces with $\phi \circ \rho = \rho'$.*

By general abstract nonsense a quotient (X, ρ) is unique up to unique isomorphism.

Lemma 18.2. *Assume that (X, ρ) is a pair as above such that*

(1) *each fibre of ρ is a left coset of H in G;*
(2) *for each open set $U \subset X$ the map ρ^* induces an isomorphism of $\mathcal{O}(U)$ with the ring of those $f \in \mathcal{O}(\rho^{-1}(U))$ that satisfy $f(hP) = f(P)$ for all $h \in H$ and $P \in \rho^{-1}(U)$.*

Then (X, ρ) is the quotient of G by H.

Proof. If $P \in X$, define $\phi(P) := \rho'(g)$, where $g \in G$ is such that $\rho(g) = P$. The map ϕ is well defined by property 1 and compatible with the action of G because X and X' are homogeneous spaces. It is also continuous, because ρ' is continuous and ρ is open by Lemma 16.3. Finally it is a morphism of quasi-projective varieties by property 2. □

Remark 18.3. The above concepts are special cases of more general ones. If Y is a quasi-projective variety equipped with an action by an algebraic group H, by a *geometric quotient* of Y by H one means a quasi-projective variety X together with a surjective morphism $\rho : Y \to X$ such that the fibres of ρ are the orbits of H on Y and property 2 of the above lemma holds. If moreover ρ is an open mapping, then a universal property like in Definition 18.1 above holds, and X is called a *categorical quotient*. One of the main theorems of geometric invariant theory is that a categorical quotient always exists if H is reductive (see the next section for this notion) and the variety is affine. It is the key to the construction of coarse moduli spaces.

We can finally prove the existence of quotients. For simplicity we restrict ourselves to the connected case, the only one we shall need.

Theorem 18.4. *Let G be a connected affine algebraic group, H a closed subgroup. Then the quotient of G by H exists.*

Proof. Let (X, ρ) be the pair constructed in Proposition 16.4. To show that it is a quotient it remains to check property 2 of Lemma 18.2. It is enough to check this property for an affine open subset $U \subset Y$. Pick $f \in \mathcal{O}(\rho^{-1}(U))$ constant on the left cosets of H, and consider the composite map $\rho^{-1}(U) \xrightarrow{(\rho,f)} U \times \mathbf{A}^1 \to U$, where the last map is the natural projection. Note that $\rho^{-1}(U)$ is connected (because ρ is open) and dense in G (because G is irreducible). Let Z be the closure of $\mathrm{Im}\,(\rho, f)$ in $Y \times \mathbf{A}^1$; it is an affine variety. Let $V \subset Z$ be a dense open subset contained in $\mathrm{Im}\,(\rho, f)$ (which exists by Lemma 14.4); it is quasi-projective. We may view the projection $V \to \mathbf{A}^1$ as a regular function $\bar{f}$ on V; it satisfies $\bar{f} \circ (\rho, f) = f$. We have to show that $\bar{f} = p^*g$ for some $g \in \mathcal{O}(U)$, for then $f = \rho^*g$. Since f is constant on H-orbits, the projection $p : V \to U$ is injective, and it has dense image. This implies that we must have $\dim V = \dim U$. Hence the induced field extension $[k(V) : p^*k(U)]$ is finite; moreover, it is separable.[3] Hence

[3]Separability is automatic in characteristic 0. It also holds in characteristic $p > 0$, though it is not obvious to prove; one has to study the induced morphism on

by Proposition 15.6 the map $p^* : k(U) \to k(V)$ is an isomorphism, so $\bar{f} = p^* g$ for some $g \in k(U)$. It remains to see that g is regular on U. For this we use Corollary 17.6 (with U in place of X and g in place of f), which applies by virtue of Proposition 17.2. It shows that if g is not regular, then $1/g$ vanishes somewhere on U, but then $1/f = \rho^*(1/g)$ should vanish somewhere on $\rho^{-1}(U)$, a contradiction. □

Similarly, one proves starting from Proposition 16.5:

Theorem 18.5. *In the previous theorem assume moreover that H is normal. Then G/H is an affine algebraic group, and $\rho : G \to G/H$ is a morphism of algebraic groups.*

The theorem allows us to give other classical examples of linear algebraic groups.

Example 18.6. If G is an affine algebraic group, then $G/Z(G)$ is also an affine algebraic group by the theorem. For example, in the case $G = \mathrm{GL}_n$ we obtain the projective general linear group PGL_n, and for $G = \mathrm{SL}_n$ the projective special linear group PSL_n.

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tangent spaces, and use the differential criterion of separability. We omit the details of the argument, which uses the explicit form of X as constructed in Proposition 16.4.

Chapter 5. Borel Subgroups and Maximal Tori

We can now harvest the fruits of our labours in the previous two chapters, and prove the remaining general structural results for affine algebraic groups. These concern *Borel subgroups*, i.e. maximal closed connected solvable subgroups, and *maximal tori*, i.e. tori embedded as closed subgroups that are maximal with respect to this property. The main theorems state that in a connected group all Borel subgroups (resp. maximal tori) are conjugate.

19. Parabolic subgroups and Borel subgroups

In the remaining part of these notes, G will always denote a *connected* affine algebraic group.

Definition 19.1. *A* parabolic *subgroup in G is a closed subgroup with G/P a projective variety. A* Borel subgroup *is a maximal connected solvable closed subgroup in G.*

Here 'maximal' means a maximal element in the set of connected closed solvable subgroups partially ordered by inclusion. Such elements exist by dimension reasons.

Proposition 19.2. *All Borel subgroups are parabolic.*

We begin the proof by two lemmata.

Lemma 19.3. *If P is is parabolic and B is a Borel subgroup in G, then P contains a conjugate of B.*

Proof. There is a natural left action of G on the projective variety G/P given by $(g, g'P) \mapsto gg'P$. Restricting to B we get a left action on G/P to which the Borel fixed point theorem applies. It yields $g \in G$ with $BgP = gP$. In particular $Bg \subset gP$, so that $g^{-1}Bg \subset P$. □

Lemma 19.4. *If P is parabolic in G and Q is parabolic in P, then Q is parabolic in G.*

Proof. Let $i : G/Q \to \mathbf{P}^n$ be an embedding realising G/Q as a quasi-projective variety. We have to show that $\mathrm{Im}\,(i)$ is closed. Observe that $\mathrm{Im}\,(i)$ equals the image of the map $\phi : P/Q \times G \to \mathbf{P}^n$ given by $(p \bmod Q, g) \mapsto i(gp \bmod Q)$. The graph $\Gamma_\phi \subset P/Q \times G \times \mathbf{P}^n$ of ϕ is closed (see the proof of Theorem 13.4), hence so is its image Z by the natural projection $P/Q \times G \times \mathbf{P}^n \to G \times \mathbf{P}^n$ by this same theorem applied to P/Q. By construction $Z = \{(gp, i(g)) : p \in P\}$, so it induces a subset $Z' \subset G/P \times \mathbf{P}^n$ which is closed, the projection $G \to G/P$ being a surjective open mapping. The projection of Z' to $\mathbf{P}^n$ is closed by Theorem 13.4 applied to G/P, and it equals $\mathrm{Im}\,(i)$. □

Proof of Proposition 19.2. We may assume G is not solvable, for otherwise the claim is obvious. Choose a closed embedding $G \subset \mathrm{GL}(V)$, and consider the action of G on $\mathbf{P}(V)$. Let $P = \langle v \rangle \in \mathbf{P}(V)$ be a point whose orbit O_P is of minimal dimension, hence closed (Corollary 14.6), and let $G_P \subset G$ be its stabiliser. By construction G/G_P is isomorphic to O_P as a quasi-projective variety, so G_P is parabolic. If $G_P = G$, we replace V by $V/\langle v \rangle$ and repeat the argument. After finitely many steps we must arrive at some $G_P \neq G$, for otherwise we would obtain a complete G-invariant flag in V, which is only possible for G solvable (Remark 7.2). Thus $G_P \subset G$ is a proper parabolic subgroup, hence it contains a conjugate gBg^{-1} of B by Lemma 19.3. Since gBg^{-1} is also a Borel subgroup in G_P, it is parabolic in G_P by induction on dimension. Hence it is parabolic in G by Lemma 19.4. □

We can now easily prove:

Theorem 19.5. *In a connected affine algebraic group G any two Borel subgroups are conjugate.*

Proof. Let B and B' be Borel subgroups. Since B' is parabolic, we find $g \in G$ with $g^{-1}Bg \subset B'$ by Lemma 19.3. Then $g^{-1}Bg$ is a Borel subgroup, and hence by definition it must equal B'. □

We also obtain the following interesting criterion for parabolicity.

Theorem 19.6. *A closed subgroup $P \subset G$ is parabolic if and only if it contains a Borel subgroup.*

Proof. Since a conjugate of a Borel subgroup is again a Borel subgroup, the 'only if' part follows from Lemma 19.3. For the 'if' part assume P is a closed subgroup containing a Borel subgroup B. Then there is a natural surjective morphism $G/B \to G/P$. Embed G/P into some $\mathbf{P}^n$ as a quasi-projective variety. By Theorem 13.1 the composite map $G/B \to G/P \to \mathbf{P}^n$ has closed image as G/B is projective, but the image is G/P, which is thus projective as well. □

Observe that the theorem characterizes Borel subgroups by a geometric and not a group theoretic property: they are the minimal parabolic subgroups. Another formulation is that the Borel subgroups are exactly the solvable parabolic subgroups.

Examples 19.7.

(1) In the case $G = \mathrm{GL}_n$ the Borel subgroups are the conjugates of the subgroup T_n of upper triangular matrices (by the Lie-Kolchin theorem). The quotient $\mathrm{GL}_n/\mathrm{T_n}$ is the variety of complete flags constructed in Section 9. For this reason for general G and B the projective variety G/B is often called a *(generalised) flag variety.* Examples of non-solvable parabolic subgroups in GL_n are given by stabilizers of non-complete flags (cp. Remark 10.5).

(2) In the case $G = \mathrm{SL}_n$ the Borel subgroups are the conjugates of the subgroup U_n of unipotent upper triangular matrices, again by the Lie-Kolchin theorem.
(3) It can be shown using the theory of quadratic forms that the Borel subgroups in SO_n are the stabilizers of those flags of subspaces $V_0 \subset V_1 \subset \cdots \subset k^n$ that are maximal with respect to the property that the restriction of the quadratic form to each V_i is trivial (these flags have length $[n/2]$).

Here is an important consequence.

Corollary 19.8. *The identity component $R(G)$ of the intersection of the Borel subgroups in G is the largest closed connected solvable normal subgroup in G.*

Proof. By Theorem 19.5 $R(G)$ is a normal subgroup; it is also closed, connected and solvable by construction. On the other hand, a closed connected solvable normal subgroup N must be contained in a Borel subgroup by the definition of Borel subgroups, hence in all of them by Theorem 19.5 and the normality of N. By connectedness it is then contained in $R(G)$. □

Definition 19.9. *The subgroup $R(G)$ of the last corollary is called the* radical *of G. The group G is* semisimple *if $R(G) = \{1\}$, and it is* reductive *if $R(G)$ is a torus.*

Example 19.10. The group GL_n is reductive. To see this, observe that the group T_n of upper triangular matrices is a Borel subgroup, and so is its transpose L_n of lower triangular matrices. Their intersection is the diagonal subgroup D_n, so $R(G)$ is diagonalizable and hence a torus. The same argument shows that SL_n is semisimple.

Finally, we use the theory of Borel subgroups to establish some basic properties of low-dimensional groups.

Proposition 19.11. *A connected affine algebraic group G of dimension ≤ 2 is solvable.*

The proof uses a lemma.

Lemma 19.12. *Let G be a connected affine algebraic group, and $B \subset G$ a Borel subgroup. If B is nilpotent, then $G = B$.*

Proof. We use induction on the dimension of B. If $\dim B = 0$, then $G = G/B$ is at the same time projective, affine and connected, hence it must be a point. For general B the center $Z(B)$ is nontrivial (by the nilpotence assumption), hence so is its identity component Z°. Given $z \in Z^\circ$, the inner automorphism $g \mapsto zgz^{-1}$ of G is trivial on B, hence induces a morphism of varieties $G/B \to G$. Such a map is constant, because G is affine connected and G/B is projective, so z is central in G. Thus $Z^\circ \subset Z(G)$, and hence Z° is normal in G. The quotient B/Z° is a Borel subgroup in G/Z°, because it is connected, solvable and $(G/Z^\circ)/(B/Z^\circ) \cong G/B$ is projective. By the inductive assumption $G/Z^\circ = B/Z^\circ$, so $G = B$. □

Proof of Proposition 19.11. Let B be a Borel subgroup. If $B = G$, we are done. If $B \neq G$, then $\dim B \leq 1$, so there are two cases. Either $B_u \neq \{1\}$, in which case it is a nontrivial closed subgroup in B by Corollary 7.4, and hence $B = B_u$ by dimension reasons. Otherwise $B_u = 1$, and therefore B is a torus (embed it in $\mathrm{T_n} \subset \mathrm{GL}_n$ using the Lie-Kolchin theorem, and observe that the composite map $B \to \mathrm{T}_n \to \mathrm{D_n}$ is injective, where D_n is the diagonal subgroup). In either case B is nilpotent, which contradicts the proposition. □

Corollary 19.13. *If* $\dim G = 1$*, then* G *is commutative.*

Proof. In any case G is solvable, so its closed commutator subgroup $[G, G]$ cannot equal G. Hence $[G, G]$=1 by dimension reasons. □

Remark 19.14. In fact, one can say more: a connected affine algebraic group of dimension 1 is isomorphic either to $\mathbf{G}_m$ or to $\mathbf{G}_a$. Part of this theorem is easily proven: by dimension reasons we must have $G = G_s$ or $G = G_u$. In the first case G is a torus, and thus must be $\mathbf{G}_m$ by dimension reasons. It then remains to be shown that in the second case G is isomorphic to $\mathbf{G}_a$. In characteristic 0 one can prove this using the formal logarithm (see Remark 21.4 below). The positive characteristic case is much more difficult, however: either one has to develop some analogue of the logarithm in positive characteristic (see Humphreys or Springer), or one has to use some facts about automorphisms of algebraic curves (see Borel).

20. Interlude on 1-cocycles

In this section we collect some very basic facts from the cohomology of groups that will be used in the next section. All groups are abstract groups.

If G is a group, by a *G-module* we mean an abelian group A equipped with a (left) action by G. It is equivalent to giving a left module over the group ring $\mathbf{Z}[G]$.

Definition 20.1. *A* 1-cocycle *of* G *with values in* A *is a map* $\phi : G \to A$ *(of sets) satisfying* $\phi(\sigma\tau) = \phi(\sigma) + \sigma\phi(\tau)$ *for all* $\sigma, \tau \in G$*. These form an abelian group* $Z^1(G, A)$ *under the natural addition. A map* $\phi : G \to A$ *is a* 1-coboundary *if it is of the form* $\phi(\sigma) = a - \sigma(a)$ *for a fixed* $a \in A$*. These form a subgroup* $B^1(G, A) \subset Z^1(G, A)$*, and the quotient* $H^1(G, A) := Z^1(G, A)/B^1(G, A)$ *is the* first cohomology group *of* G *with values in* A*.*

We shall be interested in 1-cocycles because of the following basic example.

Example 20.2. Assume given an extension $1 \to A \to E \xrightarrow{p} G \to 1$ of G by the abelian group A, i.e. a surjective homomorphism $p : E \to G$ with kernel A. In this situation we can give A the structure of a G-module by $\sigma(a) := \tilde{\sigma}a\tilde{\sigma}^{-1}$, where $\tilde{\sigma} \in E$ is any element with $p(\tilde{\sigma}) = \sigma$. Since A is abelian and normal in G, this action is well defined.

A *section* of p is a homomorphism $s : G \to E$ with $p \circ s = \mathrm{id}_G$. Giving a section is equivalent to giving a subgroup $H \subset E$ that is mapped isomorphically onto G by p (set $H = s(G)$).

Now given two sections $s_1, s_2 : G \to E$, the map $\sigma \mapsto s_1(\sigma)s_2(\sigma)^{-1}$ has values in A by definition. Moreover, it is a 1-cocycle because of the calculation

$$\begin{aligned} s_1(\sigma\tau)s_2(\sigma\tau)^{-1} &= s_1(\sigma)s_1(\tau)s_2(\tau)^{-1}s_2(\sigma)^{-1} \\ &= s_1(\sigma)s_2(\sigma)^{-1}(s_2(\sigma)s_1(\tau)s_2(\tau)^{-1}s_2(\sigma)^{-1}) \\ &= s_1(\sigma)s_2(\sigma)^{-1}\sigma(s_1(\tau)s_2(\tau)^{-1}), \end{aligned}$$

where we have used that $p(s_2(\sigma)) = \sigma$.

Assume now that this cocycle is a 1-coboundary, i.e. there is an $a \in A$ with $s_1(\sigma)s_2(\sigma)^{-1} = a\sigma(a)^{-1}$. By the equality $\sigma(a) = s_2(\sigma)as_2(\sigma)^{-1}$ this holds if and only if $s_1(\sigma) = as_2(\sigma)a^{-1}$, so that $s_1s_2^{-1}$ is a 1-coboundary if and only if the s_i are conjugate. It follows that under the assumption $H^1(G, A)$=0 any two sections are conjugate.

We finally derive sufficient conditions for the vanishing of $H^1(G, A)$.

Lemma 20.3. *If G is a finite group of order n, then $nH^1(G, A) = 0$ for all G-modules A.*

Proof. Let ϕ be a 1-coycle with values in A. Fix $\tau \in G$ and consider the map $\phi^\tau : \sigma \mapsto \phi(\sigma\tau) - \phi(\tau)$. By the cocycle relation $\phi(\sigma\tau) - \phi(\tau) - \phi(\sigma) = \sigma\phi(\tau) - \phi(\tau)$, so ϕ^τ differs from ϕ by a 1-coboundary. Therefore it is a 1-cocycle cohomologous to ϕ. Now for all $\sigma \in G$

$$\sum_{\tau\in G} \phi^\tau(\sigma) = \sum_{\tau\in G} \phi(\sigma\tau) - \sum_{\tau\in G} \phi(\tau) = 0,$$

i.e. the sum of the ϕ^τ over all $\tau \in G$ is 0. But this sum is cohomologous to $n\phi$, which proves the lemma. □

Corollary 20.4. *Let G be a finite group of order n, and A a G-module that is either*

- *a $\mathbf{Q}$-vector space; or*
- *a group of finite exponent prime to n.*

Then $H^1(G, A) = 0$.

Proof. By definition of 1-cohomology for each $m > 0$ the multiplication by m map on A induces multiplication by m on $H^1(G, A)$. In the case of a **Q**-vector space this map is an isomorphism on A and hence on $H^1(G, A)$, but for $m = n$ it is the zero map by the lemma, whence the statement in this case. In the second case we obtain that $H^1(G, A)$ is annihilated both by n and the exponent of A which is prime to n, so it is trivial again. □

21. Maximal tori

A *maximal torus* in a connected algebraic group G is a torus contained as a closed subgroup in G. Such a torus exists by dimension reasons.

Example 21.1. In GL_n the maximal tori are the conjugates of the diagonal subgroup $\mathrm{D_n}$. In SL_n they are the conjugates of of the subgroup $\mathrm{D}_n \cap \mathrm{SL}_n$, which is the kernel of the determinant map on D_n. Both of these facts follow from the Lie-Kolchin theorem. Thus for GL_n the maximal tori have dimension n, and for SL_n they have dimension $n-1$.

However, it is not *a priori* clear in general that a maximal torus is a nontrivial subgroup. In any case, it must be contained in a Borel subgroup since it is connected and solvable, so to prove nontriviality it suffices to discuss the case when G is solvable. Recall from Corollary 7.4 (and its proof) that in this case we have a commutative diagram with exact rows and injective vertical maps

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_u & \longrightarrow & G & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mathrm{U}_n & \longrightarrow & \mathrm{T}_n & \longrightarrow & \mathrm{D_n} & \longrightarrow & 1 \end{array}$$

where $G_u \subset G$ is the closed subgroup of unipotent elements. Now that we have constructed quotients, we can deduce that the quotient G/G_u embeds as a closed subgroup into D_n. Hence it is a torus, because G is connected and hence so is G/G_u.

Theorem 21.2. *Let G be a connected solvable affine algebraic group. There exists a torus T contained as a closed subgroup in G that maps isomorphically onto G/G_u via the natural projection $G \to G/G_u$.*

The proof below is that of Grothendieck ([1], exposé on 10/12/1956), which contains several improvements with respect to Borel's original proof. It uses two lemmas.

Lemma 21.3. *Let G be a commutative unipotent algebraic group. If k is of characteristic 0, then G as an abelian group is isomorphic to a* **Q***-vector space. In characteristic $p > 0$ its elements have p-power order.*

Proof. The group U_n of unipotent matrices in GL_n has a composition series $N_1 \supset N_2 \supset \dots$ obtained as follows: first we set a_{12} to 0, then a_{23}, and so on until $a_{n-1,n}$, then $a_{1,3}$, and so on; each successive quotient is isomorphic to $\mathbf{G}_a$. From this one infers in characteristic $p > 0$ that each element of U_n itself has p-power order. Assume now we are in characteristic 0. Note first that a closed subgroup $G \subset \mathrm{U}_n$ must be connected, for G/G° is a finite unipotent group (by Proposition 2.3 and Corollary 4.12) and hence must be trivial (a nontrivial element would have an eigenvalue different from 1). Now one sees that either $N_i \cap G = N_{i+1} \cap G$ or $(N_i \cap G)/(N_{i+1} \cap G) \cong \mathbf{G}_a$. Indeed, each $N_i \cap G$ is closed in U_n, hence connected. Therefore so are their quotients, but the only closed connected subgroups of the 1-dimensional group $\mathbf{G}_a$ are the trivial subgroup or $\mathbf{G}_a$ itself. We thus obtain that G as an abstract group has a chain of normal subgroups with the successive quotients $\mathbf{Q}$-vector spaces. If moreover G is commutative, it is a $\mathbf{Q}$-vector space, because an abelian group that is an extension of $\mathbf{Q}$-vector spaces is itself a $\mathbf{Q}$-vector space. □

Remark 21.4. In characteristic 0 one can in fact show that a commutative unipotent group is isomorphic to a direct power of $\mathbf{G}_a$. This isomorphism is realised using the formal exponential and logarithm series (which are polynomials for nilpotent, resp. unipotent elements).

Lemma 21.5. *Let $s \in G$ be a semisimple element, $Z \subset G$ its centralizer, and $U \subset G$ a closed normal unipotent subgroup. Then the image of the composite map $Z \to G \to G/U$ is exactly the centralizer of the image of g in G/U.*

Proof. We first reduce to the case when U is commutative using induction on the length of the commutator series of the unipotent (hence solvable) group U. (Note that its terms are closed characteristic subgroups in U, hence normal subgroups in G.) Let $U^{(n)}$ be the smallest nontrivial term. It is commutative, so we may assume the lemma holds for it. The statement for U then follows from the inductive hypothesis applied to $G/U^{(n)}$.

So assume U is commutative, and let $S \subset G$ be the closure of the subgroup generated by s. It is a closed subgroup contained in the closed subset G_s of semisimple elements, so it is diagonalizable. By Theorem 6.7 it is thus a product of a torus and a finite abelian group, therefore it is the Zariski closure of the union of its n-torsion subgroups S_n for $n > 0$ (because so is $\mathbf{G}_m$). Let Z_n be the centralizer of S_n; it is a closed subset because the commutation relation with each $s \in S_n$ gives equations for the entries of the matrices in Z_n. Thus the intersection of the Z_n is Z, it is therefore enough to prove the statement for S_n in place of s and Z_n in place of Z.

Let g be an element whose mod U image commutes with the mod U image of S_n. This means that for each $\sigma \in S_n$ there is a unique $\phi(\sigma) \in U$ with $\sigma g \sigma^{-1} = g\phi(\sigma)$. The map $\sigma \mapsto \phi(\sigma)$ is a 1-cocycle of S_n with values in the abelian group U (endowed with the S_n-action given by conjugation), because

$$\phi(\sigma\tau) = g^{-1}\sigma\tau g\tau^{-1}\sigma^{-1} = (g^{-1}\sigma g\sigma^{-1})\sigma(g^{-1}\tau g\tau^{-1})\sigma^{-1} = \phi(\sigma)\sigma\phi(\tau)\sigma^{-1}.$$

Now Lemma 21.3 says that in characteristic 0 U as an abelian group is a $\mathbf{Q}$-vector space, and in characteristic $p > 0$ its elements have p-power order. Thus Corollary 20.4 shows that $H^1(S_n, U) = 0$ (noting that in characteristic $p > 0$ S has no p-torsion). It follows that $\phi(\sigma) = u(\sigma u\sigma^{-1})^{-1}$ for some $u \in U$, so that $\sigma gu\sigma^{-1} = gu$ for all $\sigma \in S_n$, i.e. $gu \in Z_n$, and moreover gu is in the same mod U coset as g. This proves the lemma. □

Proof of Theorem 21.2. We use induction by dimension, the case of dimension 0 being trivial. Assume first that each $s \in G_s$ centralizes G_u. Then $G/[G_u, G_u]$ is a central extension of G/G_u by $G_u/[G_u, G_u]$. Therefore G is nilpotent, because so is G_u (being a subgroup of the nilpotent matrix group $\mathrm{U}_n(k)$). Thus in this case we are done by Theorem 7.5. Otherwise there is an element $s \in G_s$ whose centralizer Z is not the whole of G. This Z is closed in G (same argument as in the previous proof) and it is also solvable, being a subgroup of G. As G/G_u is commutative, the natural map $Z \to G/G_u$ is surjective by the above lemma. Hence so is the map $Z^\circ \to G/G_u$ for the identity component $Z^\circ \subset Z$, because G/G_u is connected. Applying the inductive hypothesis to Z° we obtain the result, because $Z_u^\circ = G_u \cap Z^\circ$. □

Corollary 21.6. *A torus T as in the theorem is a maximal torus and G is a semidirect product of G_u by T.*

Proof. Since $T \cap G_u = \{1\}$, this follows from the Jordan decomposition. □

Theorem 21.7. *Any two maximal tori in a connected affine algebraic group are conjugate.*

Proof. By the remarks at the beginning of this section and the conjugacy of Borel subgroups we may assume G is solvable. Then by the previous theorem it is the semidirect product of a maximal torus T by G_u. We first show that we may assume that G_u is commutative. This is done by a similar induction as in Lemma 21.5: let S be another maximal torus, and let $U^{(n)}$ be the smallest nontrivial term of the commutator series of G_u. By induction applied to $G/U^{(n)}$ we obtain an element $g \in G$ with $gSg^{-1} \subset TU^{(n)}$. But $U^{(n)}$ is commutative and $TU^{(n)}$ is the semidirect product of T with $U^{(n)}$, so by the commutative case we may conjugate gSg^{-1} into T.

So assume henceforth that G_u is commutative. As in the proof of Lemma 21.5 we may write S as the Zariski closure of an increasing chain of finite subgroups S_n. For each $n > 0$ put

$$C_n := \{u \in G_u : uS_nu^{-1} \subset T\}.$$

This is a decreasing chain of closed subsets of G_u whose intersection C_∞ is the set of $u \in G_u$ with $uSu^{-1} \subset T$. The chain must stabilize for dimension reasons, i.e. $C_n = C_\infty$ for n large enough. Thus to prove the theorem it is enough to show that $C_n \neq \emptyset$ for all n. Put $G_n := S_n G_u$; it is a semidirect product. The intersection $T_n := T \cap G_n$ maps isomorphically onto G_n/G_u by construction, so G_n is also the semidirect product of G_u by T_n. But $H^1(S_n, G_u) = 0$ as in the proof of Lemma 21.5, so S_n and T_n are conjugate, i.e. $C_n \neq \emptyset$. □

Remark 21.8. For solvable G the above proof did not use the fact that S is actually a torus; the argument works more generally for any commutative subgroup $S \subset G$ consisting of semisimple elements. Indeed, the closure $\overline{S}$ of such a subgroup is always diagonalizable by Lemma 6.2, so the above argument works for $\overline{S}$, and we obtain that some conjugate of $\overline{S}$ (hence of S) lies in T. In particular, we may choose S to be the cyclic subgroup generated by a semisimple element, and obtain: *In a connected solvable group each semisimple element is contained in a maximal torus.*

The theorem also yields characterisations of nilpotent algebraic groups.

Corollary 21.9. *The following are equivalent for a connected affine algebraic group G.*

(1) *G is nilpotent.*
(2) *All maximal tori are contained in the center of G.*
(3) *G has a unique maximal torus.*

Proof. By Lemma 19.12 we may assume that G is solvable. Then we have seen (1) ⇒ (2) in the proof of Theorem 7.5, and (2) ⇒ (3) follows from the above theorem. To show (3) ⇒ (1), observe first that by the theorem the unique maximal torus T is stable by conjugation, hence so is its n-torsion subgroup T_n for each n. But T_n is finite, so each $t \in T_n$ has finite conjugacy class. As in the proof of the Lie-Kolchin theorem, the connectedness of G implies that T_n is contained in the center $Z(G)$ of G. Hence $T \subset Z(G)$, because T is the closure of the union of the T_n and $Z(G)$ is closed. But then by Corollary 7.4 we have $G \cong G_u \times T$, and therefore G is nilpotent. □

Finally, we use the conjugacy of maximal tori to define a fundamental invariant of a connected affine algebraic group G.

Definition 21.10. *Let T be a maximal torus in G, $N_G(T)$ its normalizer in G and $Z_G(T)$ its centralizer. The quotient $W(G,T) := N_G(T)/Z_G(T)$ is the* Weyl group *of G.*

By the conjugacy of maximal tori the isomorphism class of $W(G,T)$ does not depend on T, hence it is indeed an invariant of G.

Proposition 21.11. *The Weyl group $W(G,T)$ is finite.*

The proof is based on the following very useful lemma.

Lemma 21.12. (Rigidity Lemma) *Let S and T be diagonalizable groups, V a connected variety, and $\phi : V \times S \to T$ a morphism of varieties. If the morphism $\phi_P : S \to T$ given by $s \mapsto \phi(P, s)$ is a morphism of algebraic groups for each $P \in V$, then the map $P \mapsto \phi_P$ is constant.*

Proof. If $s \in S$ is a fixed element of finite order m, the morphism $\phi_s : P \mapsto \phi(P, s)$ has finite image, because T has only finitely many elements of order dividing m. Hence ϕ_s is constant, because V is connected. In other words, we have $\phi_P(s) = \phi_{P'}(s)$ for all $P, P' \in V$. We now use again a trick seen in the proof of Lemma 21.5: S is the Zariski closure of the subgroup of finite order elements, so by continuity $\phi_P = \phi_{P'}$ on the whole of S. □

Proof of Proposition 21.11. We prove more, namely an equality of identity components $N_G(T)^\circ = Z_G(T)^\circ$; this implies the proposition since both identity components have finite index in $N_G(T)$ (resp. $Z_G(T)$). Since $Z_G(T)^\circ \subset N_G(T)^\circ$, it is enough to show $N_G(T)^\circ \subset Z_G(T)$, i.e. that the homomorphism $t \to ntn^{-1}$ is constant for each $n \in N_G(T)^\circ$. This follows from Lemma 21.12 applied with $V = N_G(T)^\circ$, $S = T$ and $\phi(n, t) = ntn^{-1}$. □

22. The union of all Borel subgroups

In this section we prove the following somewhat surprising theorem of Borel.

Theorem 22.1. *Each element of a connected affine algebraic group is contained in a Borel subgroup.*

In view of Theorem 19.5 an equivalent phrasing of the theorem is that if B is a Borel subgroup in a connected group G, then the union of the conjugates gBg^{-1} for all $g \in G$ is the whole of G. In particular:

Corollary 22.2. *If a Borel subgroup B is normal in G, then $B = G$.*

Unfortunately, it is not obvious at all to construct 'by hand' a connected solvable subgroup containing a given element of G. The problem is with the semisimple elements. Still, for those contained in a (maximal) torus one may proceed as follows. If T is a maximal torus, the only maximal torus in the identity component of the centralizer $Z_G(T)$ is T (by Theorem 21.7), hence $Z_G(T)^\circ$ is nilpotent by Corollary 21.9. Thus $Z_G(T)^\circ$ is a connected solvable subgroup containing T.

The main point in the proof Theorem 22.1 will be that the union of the conjugates of $Z_G(T)^\circ$ is already dense in G; the rest will then follow rather easily. We begin with the following general lemma.

Lemma 22.3. *Let G be a connected algebraic group, and H a closed connected subgroup. Denote by X the union of all conjugates gHg^{-1} in G.*

(1) *If H is parabolic, then X is a Zariski closed subset.*
(2) *Assume that H contains an element whose natural left action on G/H has finitely many fixed points. Then X is dense in G.*

Proof. We may view X as the image of the composite morphism $p_2 \circ \phi : G \times G \to G$, where $\phi : G \times H \to G \times G$ is given by $\phi(g, h) = (g, ghg^{-1})$, and $p_2 : G \times G \to G$ is the second projection. Let Y be the image of $\mathrm{Im}\,(\phi)$ by the quotient map $\pi : G \times G \to G \times G/(H \times \{1\}) \cong (G/H) \times G$. Since π is an open surjective mapping (Lemma 16.3) and $\mathrm{Im}\,(\phi)$ is closed in $G \times G$ (Corollary 15.4), we get that Y is closed in $(G/H) \times G$. On the other hand, its image by the second projection $\bar{p}_2 : (G/H) \times G \to G$ is still X by construction. Hence if H is parabolic, X must be closed by Theorem 13.4.

We prove (2) by a dimension count. By Proposition 12.4 it suffices to show that the dimension of the Zariski closure $\overline{X}$ equals that of G. The assumption in (2) means that there is an element $h \in H$ over which the fibre of $\bar{p}_2$ is finite, i.e. of dimension 0. Hence by Corollary 12.7 we must have $\dim Y = \dim \overline{X}$. On the other hand, the first projection $\bar{p}_1 : (G/H) \times G \to G/H$ maps Y onto G/H, and the fibre over a coset gH is isomorphic to gHg^{-1}, so it is of dimension $\dim H$. Thus from Proposition 15.9 we obtain $\dim Y = \dim H + \dim G/H = \dim G$, as required. □

The following proposition verifies condition (2) of the lemma for the identity component of the centralizer of a maximal torus.

Proposition 22.4. *Let T be a maximal torus in G, and $C = Z_G(T)$ its centralizer. There is an element $t \in T$ whose natural left action on G/C° has finitely many fixed points.*

In fact, we shall show in the next section that $C = C^\circ$, but the proof will use Theorem 22.1, so we are not allowed to use this. Before proving the proposition let us first show how it implies Theorem 22.1.

Proof of Theorem 22.1. Applying statement (2) of Lemma 22.3 to the subgroup C° of the proposition we see that its conjugates are dense in G. As remarked at the beginning of this section, C° is nilpotent. Hence it is solvable, and as such is contained in a Borel subgroup B. The union of the conjugates of B is therefore dense in G, and it remains to apply statement (1) of Lemma 22.3 to B. □

It remains to prove the proposition. We need the following elementary lemma.

Lemma 22.5. *If T is a torus embedded as a closed subgroup in a connected group G, there is an element $t \in T$ with $Z_G(T) = Z_G(t)$.*

The proof will in fact show that the t having the required property form a dense open subset in T.

Proof. Choose a closed embedding of G into some GL_n. Up to composing with an inner automorphism of GL_n, we may assume using Lemma 6.1 that the elements of T map to diagonal matrices. A calculation shows that in GL_n the centralizer of a diagonal matrix $\mathrm{diag}\,(d_i)$ consists of those matrices $[c_{ij}]$ where $c_{ij} = 0$ if $d_i \neq d_j$ and c_{ij} is arbitrary otherwise. It follows that we may choose t as any diagonal matrix $\mathrm{diag}\,(t_i)$ where $t_i \neq t_j$ for all $i \neq j$, unless $s_i = s_j$ for all $s = \mathrm{diag}\,(s_i) \in T$. □

Proof of Proposition 22.4. By the previous lemma we find $t \in T$ with $C = Z_G(t)$. Now observe that the class $gC^\circ \in G/C^\circ$ is a fixed point for t if and only if $g^{-1}tg \in C^\circ$. But $g^{-1}tg$ is a semisimple element (by Corollary 4.12), so $g^{-1}tg \in T$, as T is the semisimple part of the nilpotent group C° (by the remarks at the beginning of this section and Theorem 7.5). Hence $T \subset Z_G(g^{-1}tg)^\circ = g^{-1}Z_G(t)^\circ g = g^{-1}C^\circ g$, so that $gTg^{-1} \subset C^\circ$. Since T is the only maximal torus in C°, this forces $gTg^{-1} = T$, i.e. $g \in N_G(T)$. But $N_G(T)^\circ = C^\circ$ by Proposition 21.11 (and its proof), which leaves finitely many possibilities for gC°. □

Finally, we note that Theorem 22.1 together with Remark 21.8 yields:

Corollary 22.6. *In a connected affine algebraic group each semisimple element is contained in a maximal torus.*

23. Connectedness of centralizers

We now turn to the proof of the following theorem.

Theorem 23.1. *For a torus S contained as a closed subgroup in a connected algebraic group G the centralizer $Z_G(S)$ is connected.*

The first reduction is:

Lemma 23.2. *If the theorem holds for connected solvable groups, it holds for arbitrary connected groups.*

Proof. We shall prove that given G and S as in the theorem, for $z \in Z_G(S)$ there is a Borel subgroup B containing both z and S. Since $Z_B(S)^\circ \subset Z_G(S)^\circ$ and by the solvable case z is contained in $Z_B(S)^\circ = Z_B(S)$, the theorem will follow for G. Choose a Borel subgroup B_0 containing z. Then $B = gB_0g^{-1}$ will be a good choice provided $zg \subset gB_0$ and $sg \subset gB_0$ for all $s \in S$. This is equivalent to saying that the coset gB_0 is a common fixed point under the natural left actions of z and S on the projective variety G/B_0. Consider the subset $X \subset G/B_0$ of fixed points under the action of z. This is a nonempty subset of G/B_0 (as $z \in B_0$), and it is closed, being the preimage of the graph of the multiplication-by-z map by the diagonal morphism $G/B_0 \to (G/B_0) \times (G/B_0)$. Thus it is a projective variety. Since z centralizes

S, the natural left action of S on G/B_0 preserves X, so it has a fixed point in X by the Borel fixed point theorem. □

The key lemma is the following one.

Lemma 23.3. *Let G be a connected algebraic group, $U \subset G$ a connected commutative normal unipotent subgroup and $s \in G$ a semisimple element. Then $Z_G(s) \cap U$ is connected.*

Proof. Consider the map $\gamma_s : U \to U$ given by $u \mapsto usu^{-1}s^{-1}$. Since U is commutative and normal, this is a group homomorphism, and its kernel C equals $Z_G(s) \cap U$. Observe now that $C \cap \gamma_s(U) = \{1\}$. Indeed, assume $u \in C$ and $v \in U$ are such that $u = \gamma_s(v)$, or in other words $us = vsv^{-1}$. Here s is semisimple, u is unipotent and commutes with s, so this must be the Jordan decomposition of vsv^{-1}. But vsv^{-1} is also semisimple (by Corollary 4.12), which forces $u = 1$.

By the above property the multiplication map $m : C \times \gamma_s(U) \to U$ is injective. But here $\dim C + \dim \gamma_s(U) = \dim U$ by Corollary 15.9, so by the same corollary the image of the multiplication map $C^\circ \times \gamma_s(U) \to U$ must be the whole of U. Therefore there exists a projection of U onto C° which must map C isomorphically onto C° since $C \cap \gamma_s(U) = \{1\}$. The connectedness of C follows. □

The following lemma is much simpler.

Lemma 23.4. *Let $1 \to G' \to G \xrightarrow{\phi} G'' \to 1$ be an exact sequence of algebraic groups. If G' and G'' are connected, then so is G.*

Proof. By assumption, G° surjects onto G'', so for each $g \in G$ we find $g^\circ \in G^\circ$ with $\phi(g) = \phi(g^\circ)$. But then $g^\circ g^{-1} \in G' \subset G^\circ$, so $g \in G^\circ$. □

Corollary 23.5. *If G is a connected solvable group, then its unipotent subgroup G_u is connected as well.*

Proof. Since G/G_u is diagonalizable, hence in particular commutative, G_u must contain $[G, G]$, which is connected by Lemma 7.3. The quotient $G_u/[G, G]$ is the unipotent subgroup of the commutative group $G^{\mathrm{ab}} := G/[G, G]$ (by Corollary 4.12), so it is connected, being a direct factor of G^{ab} in view of Theorem 6.2. The corollary now follows from the lemma. □

Proof of Theorem 23.1. Using Lemma 22.5 it will suffice to prove that the centralizer of a semisimple element $s \in G$ is connected. Moreover, by Lemma 23.2 we may assume G is solvable. Then G_u is a closed normal subgroup in G, and moreover connected by the previous corollary. We use induction on the length of the commutator series of G_u, the case $G_u = \{1\}$ being obvious. Let U be the smallest nontrivial term in the series; it is closed, commutative and normal in G. It is also connected by an iterated application of Lemma 7.3, so Lemma 23.3 applies and yields the connectedness of $Z_G(s) \cap U$. On the other hand, the image of

$Z_G(s)$ in G/U is exactly the centralizer of s mod U in G/U according to Lemma 21.5, so it is connected by the inductive assumption. The connectedness of $Z_G(s)$ now follows from lemma 23.4. □

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24. The normalizer of a Borel subgroup

We now prove the last important structural result concerning Borel subgroups, which is due to Chevalley. Its proof will use all the major results proven earlier in this chapter.

Theorem 24.1. *Let B be a Borel subgroup in a connecetd affine algebraic group G. Then $N_G(B) = B$, i.e. B equals its own normalizer.*

For the proof we need the following proposition which is interesting in its own right.

Proposition 24.2. *Let S be a torus contained as a closed subgroup in G, and B a Borel subgroup of G. Then $Z_G(S) \cap B$ is a Borel subgroup in $Z_G(S)$.*

Note that the centralizer $Z_G(S)$ is a connected affine algebraic group by Theorem 23.1.

Proof. We have $Z_G(S) \cap B = Z_B(S)$, so it is connected by (the solvable case of) Theorem 23.1. It is also solvable, so by Theorem 19.6 it is enough to see that it is parabolic in $Z_G(S)$. The composite map $Z_G(S) \to G \to G/B$ factors through $Z_G(S)/(Z_G(S) \cap B)$, and maps it isomorphically onto the image of $Z_G(S)$ in G/B. We show that this image is a closed subset of the projective variety G/B. In any case, it is the same as the image of the subgroup $Y = Z_G(S)B$ in G/B, so since the projection $G \to G/B$ is surjective and open (Lemma 16.3), it is enough to show that Y is closed in G. It is certainly connected (being the image of the multiplication map $Z_G(S) \times B \to G$), hence so is its Zariski closure $\overline{Y}$.

Pick now $\bar{y} \in \overline{Y}$; we show that it lies in Y. To do so, we shall find $b \in B$ such that $\bar{y}b^{-1}$ centralizes S. In any case, we know that $\bar{y}^{-1}S\bar{y} \subset B$, because the elements of Y have this property by definition, hence so does $\bar{y}$ by continuity. Now write $T = B/B_u$ (where $B_u \subset B$ is the unipotent subgroup), and apply Lemma 21.12 to the map $\overline{Y} \times S \to T$ sending a pair (y, s) to the image of $y^{-1}sy$ in T. It says that for each y and s the image of $y^{-1}sy$ in T equals that of s. In particular, $\bar{y}^{-1}S\bar{y} \subset SB_u$. But S and $\bar{y}^{-1}S\bar{y}$ are maximal tori in the connected solvable group SB_u, so by Theorem 21.7 we have $\bar{y}^{-1}S\bar{y} = b^{-1}Sb$ for some $b \in B_u$. But then $\bar{y}b^{-1} \in Z_G(S)$, as required. □

Proof of Theorem 24.1. We use induction on the dimension of G, the case of dimension 1 being trivial by Corollary 19.13. Fix a maximal torus $T \subset B$ and an element $x \in N_G(B)$. We shall show that $x \in B$. Conjugation by x maps T onto another maximal torus in B which is of the form yTy^{-1} for some $y \in B$ by Theorem 21.7. Hence up to replacing x by $y^{-1}x$ (which is allowed) we may assume that $xTx^{-1} = T$. Now consider the endomorphism $\rho_x : T \to T$ given by $t \mapsto xtx^{-1}t^{-1}$. We distinguish two cases.

Case 1: ρ_x is not surjective. Then $\mathrm{Im}\,(\rho_x)$ is a proper closed subgroup of T, whence it follows (for example by a dimension count using Corollary 15.10) that the identity component S of $\mathrm{Ker}\,(\rho_x)$ is a nontrivial torus. By construction, x lies in the centralizer $Z_G(S)$ of S. On the other hand, $B \cap Z_G(S)$ is a Borel subgroup in the connected group $Z_G(S)$ by the proposition above, and since $x \in N_G(B)$, it normalizes $B \cap Z_G(S)$. Thus if $Z_G(S) \neq G$, the inductive hypothesis applies to $Z_G(S)$ and shows that $x \in B$. Otherwise S is central in G and hence it is a normal subgroup. But then we may conclude by applying the inductive hypothesis to G/S (which has lower dimension by Corollary 15.10).

Case 2: ρ_x is surjective. This assumption implies that T is contained in the commutator subgroup $[N_G(B), N_G(B)]$. We now use the always handy Lemma 14.3 to find a finite dimensional vector space V and a morphism $G \to \mathrm{GL}(V)$ such that the stabilizer of a 1-dimensional subspace $L \subset V$ is exactly $N_G(B)$. The action of $N_G(B)$ on L is given by a morphism $N_G(B) \to \mathrm{GL}(L) \cong \mathbf{G}_m$. Since $\mathbf{G}_m$ is commutative and semisimple, it follows that both $[N_G(B), N_G(B)]$ and the unipotent part B_u of B act trivially on L. But $T \subset [N_G(B), N_G(B)]$ and $B = TB_u$ (Theorem 21.2), so B acts trivially on L. Therefore if $v \in L$, the map $G \to V$ given by $g \mapsto gv$ factors through G/B. But this is a morphism of the irreducible projective variety G/B into V viewed an as affine space, so it is constant by Corollary 13.2. In particular, the whole of G stabilizes L, so $G = N_G(B)$, i.e. B is normal in G. But then $G = B$ by Corollary 22.2, and the statement to be proven is obvious. □

Corollary 24.3. *The map $g \mapsto gBg^{-1}$ induces a bijection between the points of the flag variety G/B and the set of Borel subgroups in G.*

Proof. The map certainly factors through G/B and it is surjective by Theorem 19.5. Theorem 24.1 now says that its kernel is exactly B. □

Because of the corollary above G/B is also called the *variety of Borel subgroups* in G. The study of this variety as a homogeneous space for G is extremely important. For instance, one has the following difficult theorem, which is one form of the *Bruhat decomposition*:

Theorem 24.4. *Let T be a maximal torus in B. In the natural left action of B on G/B each B-orbit contains a unique fixed point by the action of T.*

A further study of the B-orbits reveals that each of them is locally closed in G/B (i.e. each point has an open neighbourhood on which the trace of the orbit is closed), and moreover isomorphic to some affine space. In this way one obtains a *cellular decomposition* of G/B.

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