HEIDELBERG LECTURES ON FUNDAMENTAL GROUPS

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1. Grothendieck's Fundamental Group

Grothendieck's theory of the algebraic fundamental group is a common generalization of Galois theory and the theory of covers in topology. Let us briefly recall both. The proofs of all statements in this section can be found in [23].

Let k be a field. Recall that a finite dimensional k-algebra A is *étale* (over k) if it is isomorphic to a finite direct product of separable extensions of k. Fix a separable closure $k_s|k$. The $\mathrm{Gal}(k_s|k)$-action on k_s induces a left action on the set of k-algebra homomorphisms $\mathrm{Hom}_k(A, k_s)$. The rule $A \mapsto \mathrm{Hom}(A, k_s)$ is a contravariant functor.

Theorem 1.1. (Main Theorem of Galois Theory – Grothendieck's version) *The contravariant functor $F : A \mapsto \mathrm{Hom}_k(A, k_s)$ gives an anti-equivalence between the category of finite étale k-algebras and the category of finite sets with continuous left* $\mathrm{Gal}(k_s|k)$*-action.*

Note that the functor F depends on the choice of the separable closure k_s. The latter is not a finite étale k-algebra but a *direct limit* of such. Also, one checks that $\mathrm{Gal}(k_s|k)$ is naturally isomorphic to the *automorphism group* of the functor F.

Now to the topological situation. Let X be a connected, locally connected and locally simply connected topological space. Recall that a *cover* of X is a space Y equipped with a continuous map $p : Y \to X$ subject to the following condition: each point of X has an open neighbourhood V for which $p^{-1}(V)$ decomposes as a disjoint union of open subsets U_i of Y such that the restriction of p to each U_i induces a homeomorphism of U_i with V.

Given a point $x \in X$, the fundamental group $\pi_1(X, x)$ has a natural left action on the fibre $p^{-1}(x)$ defined as follows: given $\alpha \in \pi_1(X, x)$ represented by a closed path $f : [0, 1] \to X$ with $f(0) = f(1) = x$ as well as a point $y \in p^{-1}(x)$, we define $\alpha y := \widetilde{f}(1)$, where $\widetilde{f}$ is the unique lifting of the path f to Y with $\widetilde{f}(0) = y$. One checks that this indeed gives a well-defined left action of $\pi_1(X, x)$. It is called the *monodromy action.*

Theorem 1.2. *The functor* Fib_x *sending a cover* $p : Y \to X$ *to the fibre* $p^{-1}(x)$ *equipped with the monodromy action induces an equivalence of the category of covers of* X *with the category of left* $\pi_1(X, x)$*-sets.*

Here again, the functor Fib_x depends on the choice of the point x. It is in fact *representable* by a cover $\pi : \widetilde{X}_x \to X$. It can be constructed as the space of homotopy classes of paths starting from x, the projection π mapping the class of a path to its other endpoint. As a consequence, we have isomorphisms

$$\mathrm{Aut}\,(\widetilde{X}_x) \cong \mathrm{Aut}\,(\mathrm{Fib}_x) \cong \pi_1(X, x).$$

Here is an important consequence. Call a cover $Y \to X$ *finite* if it has finite fibres; for connected X these have the same cardinality, called the *degree* of X.

Corollary 1.3. *For* X *and* x *as in Theorem 1.2, the functor* Fib_x *induces an equivalence of the category of* finite *covers of* X *with the category of finite continuous left* $\widehat{\pi_1(X, x)}$*-sets. Connected covers correspond to finite* $\widehat{\pi_1(X, x)}$*-sets with transitive action and Galois covers to coset spaces of open normal subgroups.*

Here $\widehat{\pi_1(X, x)}$ denotes the *profinite completion* of $\pi_1(X, x)$, i.e. the inverse limit of the natural inverse system of its finite quotients.

We can now come to Grothendieck's common generalization in algebraic geometry. Let S be a connected scheme. Recall that a *finite étale cover* of S is a finite flat surjection $X \to S$ such that each fibre at a point $s \in S$ is the spectrum of a finite étale $\kappa(s)$-algebra. Fix a geometric point $\bar{s} : \mathrm{Spec}\,(\Omega) \to S$. For a finite étale cover $X \to S$ we consider the geometric fibre $X \times_S \mathrm{Spec}\,(\Omega)$ over $\bar{s}$, and denote by $\mathrm{Fib}_{\bar{s}}(X)$ its underlying set. This gives a set-valued functor on the category of finite étale covers of X.

We *define* the *algebraic fundamental group* $\pi_1(S, \bar{s})$ as the automorphism group of this functor. By definition an automorphism of $\mathrm{Fib}_{\bar{s}}$ induces an automorphism of the set $\mathrm{Fib}_{\bar{s}}(X)$ for each finite étale cover X; in this way we obtain a natural left action of $\pi_1(S, \bar{s})$ on the set $\mathrm{Fib}_{\bar{s}}(X)$.

Theorem 1.4. (Grothendieck) *Let* S *be a connected scheme, and* $\bar{s} : \mathrm{Spec}\,(\Omega) \to S$ *a geometric point.*

1. *The group* $\pi_1(S, \bar{s})$ *is profinite, and its action on* $\mathrm{Fib}_{\bar{s}}(X)$ *is continuous for every* X *in* Fet_S.
2. *The functor* $\mathrm{Fib}_{\bar{s}}$ *induces an equivalence of* Fet_S *with the category of finite continuous left* $\pi_1(S, \bar{s})$*-sets.*

Here the fibre functor $\mathrm{Fib}_{\bar{s}}$ is *pro-representable*, which means that there exists a (filtered) inverse system $P = (P_\alpha, \phi_{\alpha\beta})$ of finite étale covers and a functorial isomorphism $\varinjlim \mathrm{Hom}(P_\alpha, X) \cong \mathrm{Fib}_{\bar{s}}(X)$. The automorphism group of each finite étale cover $P_\alpha \to S$ is finite, and $\pi_1(S, \bar{s})$ is their inverse limit; this explains its profiniteness. In fact, Grothendieck showed that one may choose as a pro-representing system the system of all *Galois covers* $P_\alpha \to X$, i.e. those finite étale covers for which $\mathrm{Aut}\,(P_\alpha|S)$ acts transitively on geometric fibres. These are turned into an inverse system by choosing a distinguished point $p_\alpha \in \mathrm{Fib}_{\bar{s}}(P_\alpha)$ for each α; for each pair α, β there is then at most one S-morphism $P_\beta \to P_\alpha$ sending p_β to p_α. We define this map to be $\phi_{\alpha\beta}$ (if it exists).

Remark 1.5. Any two fibre functors on the category of finite étale S-schemes are (non-canonically) isomorphic. One way to prove this is to use pro-representability of the fibre functor which reduces the construction of an isomorphism between functors to the construction of a compatible system of automorphisms of the Galois objects P_α transforming one system of maps $\phi_{\alpha\beta}$ to another. This can be done by means of a compactness argument.

An isomorphism between two fibre functors $\mathrm{Fib}_{\bar{s}}$ and $\mathrm{Fib}_{\bar{s}'}$ is called a *path* from $\bar{s}$ to $\bar{s}'$. It induces an isomorphism of fundamental groups $\pi_1(S, \bar{s}) \xrightarrow{\sim} \pi_1(S, \bar{s}')$. In the topological situation such an isomorphism is induced by the choice of a (usual) path between base points, whence the name in the algebraic situation. As in topology, two isomorphisms $\pi_1(S, \bar{s}) \xrightarrow{\sim} \pi_1(S, \bar{s}')$ induced by diffrent paths differ by an inner automorphism of $\pi_1(S, \bar{s})$.

Historically, the case of a normal scheme was known earlier. In fact:

Proposition 1.6. *Let S be an integral normal Noetherian scheme. Denote by K_s a fixed separable closure of the function field K of S, and by K_S the composite of all finite subextensions $L|K$ of K_s such that the normalization of S in L is étale over S. Then $K_S|K$ is a Galois extension, and $\mathrm{Gal}(K_S|K)$ is canonically isomorphic to the fundamental group $\pi_1(S, \bar{s})$ for the geometric point $\bar{s} : \mathrm{Spec}\,(\overline{K}) \to S$, where $\overline{K}$ is the algebraic closure of K containing K_s.*

The following examples show that the algebraic fundamental group indeed yields a common generalization of the algebraic and topological cases:

Examples 1.7.

1. For $X = \mathrm{Spec}\,(k)$, $\bar{x} : \mathrm{Spec}\,(k_s) \to \mathrm{Spec}\,(k)$ we have
$$\pi_1(X, \bar{x}) \cong \mathrm{Gal}(k_s|k).$$
This holds basically because finite étale $\mathrm{Spec}\,(k)$-schemes are spectra of finite étale k-algebras.
2. For X of finite type over $\mathbf{C}$ and $\bar{x} : \mathrm{Spec}\,(\mathbf{C}) \to X$ there is a canonical isomorphism an isomorphism
$$\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, \bar{x})} \xrightarrow{\sim} \pi_1(X, \bar{x})$$

where on the left hand side we have the profinite completion of the topological fundamental group of X with base point $\mathrm{Im}\,(\bar{x})$, and X^{an} denotes the complex analytic space associated with X.

This isomorphism relies on a deep algebraization theorem for finite topological covers of schemes of finite type over $\mathbf{C}$.

The algebraic fundamental group is functorial with respect to base point preserving morphisms. To construct it, let S and S' be connected schemes, equipped with geometric points $\bar{s} : \mathrm{Spec}\,(\Omega) \to S$ and $\bar{s}' : \mathrm{Spec}\,(\Omega) \to S'$, respectively. Assume given a morphism $\phi : S' \to S$ with $\phi \circ \bar{s}' = \bar{s}$. For a finite étale cover $X \to S$ consider the base change $X \times_S S' \to S'$. The condition $\phi \circ \bar{s}' = \bar{s}$ implies that $\mathrm{Fib}_{\bar{s}}(X) = \mathrm{Fib}_{\bar{s}'}(X \times_S S')$. This construction is functorial in X, and thus every automorphism of the functor $\mathrm{Fib}_{\bar{s}'}$ induces an automorphism of $\mathrm{Fib}_{\bar{s}}$, which defines the required map $\phi_* : \pi_1(S', \bar{s}') \to \pi_1(S, s)$. It is a continuous homomorphism of profinite groups.

Proposition 1.8. *Let X be a quasi-compact and geometrically integral scheme over a field k. Fix an algebraic closure $\overline{k}$ of k, and let $k_s|k$ be the corresponding separable closure. Write $\overline{X} := X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k_s)$, and let $\bar{x}$ be a geometric point of $\overline{X}$ with values in $\overline{k}$. The sequence of profinite groups*

$$1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}(k_s|k) \to 1$$

induced by the maps $\overline{X} \to X$ and $X \to \mathrm{Spec}\,(k)$ is exact.

The group $\pi_1(X, \bar{x})$ acts on its normal subgroup $\pi_1(\overline{X}, \bar{x})$ via inner automorphisms, whence a map $\phi_X : \pi_1(X, \bar{x}) \to \mathrm{Aut}\,(\pi_1(\overline{X}, \bar{x}))$. Inside $\mathrm{Aut}\,(\pi_1(\overline{X}, \bar{x}))$ we have the normal subgroup $\mathrm{Inn}(\pi_1(\overline{X}, \bar{x}))$ of inner automorphisms; the quotient is the group $\mathrm{Out}(\pi_1(\overline{X}, \bar{x}))$ of *outer automorphisms*. By the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\overline{X}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & \mathrm{Gal}(\overline{k}|k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & \mathrm{Aut}\,(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & \mathrm{Out}(\pi_1(\overline{X}, \bar{x})) & \longrightarrow & 1 \end{array}$$

we get an important representation

$$\rho_X : \mathrm{Gal}(k_s|k) \to \mathrm{Out}(\pi_1(\overline{X}, \bar{x})).$$

It will be studied in the section on anabelian geometry.

2. Fundamental Groups of Curves

In this section k denotes an algebraically closed field of characteristic $p \geq 0$, and X a proper smooth curve over k.

Theorem 2.1. **(Grothendieck)** *Let $U \subset X$ be an open subcurve (possibly equal to X), and $n \geq 0$ the number of closed points in $X \setminus U$. Then $\pi_1(U)^{(p')}$ is isomorphic to the profinite p'-completion of the group*

$$\Pi_{g,n} := \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n \mid [a_1, b_1] \ldots [a_g, b_g]\gamma_1 \ldots \gamma_n = 1\rangle.$$

Here $G^{(p')}$ denotes the *maximal prime-to-p quotient* of the profinite group G, i.e. the inverse limit of its finite quotients of order prime to p; for $p = 0$ we define it to be G itself.

For $k = \mathbf{C}$ the theorem follows from the well-known structure of the topological fundamental group; the algebraization theorem lying behind is just the Riemann existence theorem of complex analysis. One deduces the result for k of characteristic 0 by a general rigidity theorem which says that the fundamental group of a smooth curve does not change under extensions of algebraically closed fields of characteristic 0 (this also holds in positive characteristic, but only for proper curves).

In positive characteristic Grothendieck proved the result by first lifting the curve to characteristic 0 and then proving a specialization theorem establishing an isomorphism between maximal prime-to-p quotients of the fundamental groups of the curve and its lifting. Thus this case also relies on the topological result over $\mathbf{C}$. The only case where a proof avoiding the topological argument is known at present is for $k = \overline{\mathbf{F}}_p$. There Wingberg was able to prove that the maximal pro-ℓ quotients (for $\ell \neq p$ a prime) of $\pi_1(U)$ have the above structure using class field theory and delicate group-theoretic arguments.

Remark 2.2. For X proper and $\ell \neq p$ a prime the theorem implies that the maximal abelian pro-ℓ-quotient of $\pi_1(X)$ is isomorphic to $\mathbf{Z}_\ell^{2g}$. On the other hand, for J the Jacobian of X the Tate module $T_\ell(J)$ has the same structure. This is not a coincidence: by a theorem of Serre and Lang every finite étale cover of J of ℓ-power degree is a quotient of some cover given by

$$0 \to {}_{\ell^n}J \to J \xrightarrow{\ell^n} J \to 0;$$

on the other hand, given some standard embedding $X \to J$, the induced map on fundamental groups becomes an isomorphism on the maximal prime-to-p abelian quotient ('abelian prime-to-p covers are obtained via pullback from the Jacobian').

There is also a generalization to open curves but we don't discuss it here.

The *maximal pro-p quotient* $G^{(p)}$ of G is defined as the inverse limit of finite quotients of p-power order, and we have:

Theorem 2.3. *Assume $p > 0$. Then $\pi_1(X)^{(p)}$ is a free pro-p group of finite rank equal to the p-rank of the Jacobian variety of X.*

For an open subcurve $U \neq X$ the group $\pi_1(U)^{(p)}$ is a free pro-p group of infinite rank equal to the cardinality of k.

Here recall that the *p-rank* of an abelian variety A over an algebraically closed field k of characteristic $p > 0$ is the dimension of the $\mathbf{F}_p$-vector space given by the kernel of the multiplication-by-p map on the k-points of A. It is a nonnegative integer bounded by $\dim A$.

The proof of this theorem is based on the group-theoretic fact that a pro-p-group G is free if and only if the Galois cohomology groups $H^i(G, \mathbf{Z}/p\mathbf{Z})$ vanish for $i > 1$. In the case $G = \pi_1(X)^{(p)}$ they can be identified with the étale cohomology groups $H^i_{\text{ét}}(X, \mathbf{F}_p)$ of X using arguments of cohomological dimension and the latter groups are known to vanish for $i > 1$. The rank is then equal to that of the maximal abelian quotient, i.e. the dual of $H^1_{\text{ét}}(X, \mathbf{Z}/p\mathbf{Z})$ and thus can be determined using Artin–Schreier theory.

Remark 2.4. Observe that Theorems 2 and 2.3 do not elucidate completely the structure of the fundamental group of an integral normal curve over an algebraically closed field of positive characteristic; this is still unknown at the present day. The theorems give, however, a good description of its maximal *abelian* quotient: this group is the direct sum of its maximal prime-to-p and pro-p quotients, and hence the previous two theorems together suffice to describe it.

Concerning curves over non-closed fields, a main object of study is the outer Galois representation

$$\rho_X : \operatorname{Gal}(k_s|k) \to \operatorname{Out}(\pi_1(\overline{X}, \bar{x}))$$

over fields of arithmetic interest. One of the most basic results is:

Theorem 2.5. (Matsumoto [12]) *If k is a number field and X is affine such that $\overline{X}$ has non-commutative fundamental group, then ρ_X is injective.*

Recently, Hoshi and Mochizuki proved that the result holds for proper curves of genus > 1 as well. One can easily decide using Theorem which curves have noncommutative geometric fundamental group: those for which $(g, n) \neq (0, 0), (0, 1), (0, 2), (1, 0)$. These are the so-called *hyperbolic* curves: their fundamental groups are center-free and, for $g > 0$, even free.

The case $(g, n) = (0, 3)$ is due to Belyi and is a consequence of this famous theorem stating that every smooth proper curve definable over a number field can be realized as a finite cover of $\mathbf{P}^1$ branched above at most 3 points. The proof of the general case uses different methods.

3. Grothendieck's Section Conjecture

The conjecture concerns the exact sequence

$$1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \xrightarrow{p_*} \operatorname{Gal}(k_s|k) \to 1 \tag{1}$$

where $p : X \to \operatorname{Spec} k$ is the structure map.

Given a k-rational point $y : \operatorname{Spec} k \to X$, it induces by functoriality a map $\sigma_y : \operatorname{Gal}(k_s|k) \to \pi_1(X, \bar{y})$ for a geometric point $\bar{y}$ lying above y. This is not quite a splitting of the exact sequence above because of the difference of base points. But the choice of a path (see Remark 1.5) from $\bar{y}$ to $\bar{x}$ induces an isomorphism $\lambda : \pi_1(X, \bar{y}) \xrightarrow{\sim} \pi_1(X, \bar{x})$. Changing the path is reflected by an inner automorphism of $\pi_1(X, \bar{x})$; moreover, this automorphism induces the trivial automorphism of $\operatorname{Gal}(k_s|k)$, so it is in fact conjugation by an element of $\pi_1(\overline{X}, \bar{x})$. The composite $\lambda \circ \sigma_y$ is then a section of the exact sequence uniquely determined up to conjugation by elements of $\pi_1(\overline{X}, \bar{x})$. We thus obtain a map

$$X(k) \to \{\pi_1(\overline{X}, \bar{x})\text{-conjugacy classes of sections of } p_*\}. \tag{2}$$

The Section Conjecture states:

Conjecture 3.1. (Grothendieck [7]) *If k is finitely generated over $\mathbf{Q}$ and X is a smooth projective curve of genus $g \geq 2$, then the above map is a bijection.*

Injectivity is not hard to prove and was known to Grothendieck:

Proposition 3.2. *If k is finitely generated over $\mathbf{Q}$ and X is a smooth projective curve of genus $g \geq 2$, then the map (2) is injective.*

Proof. Set $\Gamma := \operatorname{Gal}(\overline{k}|k)$. Fix a k-point y_0 of X and denote by s_0 the corresponding section $\Gamma \to \pi_1(X, \bar{x})$. Given another k-point y of X with corresponding section s, the composite map $\Gamma \to \pi_1(X, \bar{x}) \to \pi_1^{\text{ab}}(X)$ induced by $s_0 s^{-1}$ has image in $\pi_1^{\text{ab}}(\overline{X})$ and is a continuous 1-cocycle. We thus get compatible classes in $H^1(\Gamma, \pi_1^{\text{ab}}(\overline{X})/m)$ for all $m > 0$. Denoting by J the Jacobian of X we have a Galois-equivariant isomorphism $\pi_1^{\text{ab}}(\overline{X})/m \cong {}_m\overline{J}$ (see Remark 2.2), so we actually get maps $\operatorname{Div}^0(X) \to H^1(\Gamma, {}_m\overline{J})$ for all m, where $\operatorname{Div}^0(X)$ is the group of degree 0 divisors on C. Moreover, it is an exercise to check the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{Div}^0(X) & \longrightarrow & H^1(\Gamma, {}_m\overline{J}) \\ \downarrow & & \uparrow \\ J(k) & \xrightarrow{\cong} & J(\overline{k})^\Gamma \end{array}$$

where the right vertical map comes from the Kummer sequence

$$J(\overline{k})^\Gamma \xrightarrow{m} J(\overline{k})^\Gamma \to H^1(\Gamma, {}_m\overline{J})$$

By this commutativity, if we assume $s = s_0$, the class of the divisor $y - y_0$ lies in the kernel of the Kummer map $J(\overline{k})^\Gamma \to H^1(\Gamma, {}_m\overline{J})$ for all m, i.e. it is divisible in $J(k)$. But for k finitely generated over $\mathbf{Q}$, the group $J(k)$ is finitely generated by the Mordell–Weil–Lang–Néron theorem and as such has trivial divisible subgroup. We conclude $y = y_0$. □

4. Parshin's Proof of Mordell's Conjecture Over Function Fields

Let B be a smooth projective integral curve over the field $\mathbf{C}$ of complex numbers, and let C be a smooth projective integral curve defined over the function field $\mathbf{C}(B)$ of B. The following statement is usually called the geometric case of Mordell's Conjecture or the Mordell Conjecture for function fields of characteristic 0.

Theorem 4.1. *Assume that there is no finite extension $K|\mathbf{C}(B)$ for which the base changed curve $C \times_{\mathbf{C}(B)} K$ can be defined over $\mathbf{C}$. Then C has only finitely many $\mathbf{C}(B)$-rational points.*

As a consequence, one gets the same result over base fields that are finitely generated field of characteristic 0 (assuming B geometrically integral).

This famous theorem has several proofs. The first one was given by Manin [11]; Coleman later discovered that it contained a gap which he was able to fill in [1]. The first complete published proof seems to be that of Grauert [5]. Parshin himself gave two proofs ([16], [17]); it is the second one that we are going to explain now.

We prove the following equivalent statement.

Theorem 4.2. *Let V be a smooth projective surface equipped with a proper flat morphism $p : V \to B$ with generic fibre C as above. If V as a family over B is non-isotrivial, then the projection p has only finitely many sections.*

Recall that the family $p : V \to B$ is *isotrivial* if there is a finite flat base change $B' \to B$ such that $V \times_B B' \to B'$ is a trivial family (i.e. isomorphic to a direct product).

To see the equivalence of the two statements, note that one may find a smooth projective surface $\widetilde{V}$ over $\mathbf{C}$ whose function field is that of the curve C of Theorem 4.1, by resolution of singularities for surfaces. The inclusion $\mathbf{C}(B) \to \mathbf{C}(\widetilde{V})$ induces a rational map $\widetilde{V} \to B$ with generic fibre C; by elimination of indeterminacy we find a blowup V of $\widetilde{V}$ in finitely many points equipped with a morphism $p : V \to B$ as required. A section of p induces a section on the generic fibre. On the other hand, by properness of V any section of the projection $C \to \operatorname{Spec} \mathbf{C}(B)$ extends uniquely to a section of p.

Strategy of the proof of Theorem 4.2. Choose a Zariski open subset $B_0 \subset B$ such that p is smooth over B_0. Fix a point $b_0 \in B_0$, and denote by F the fibre $p^{-1}(b_0)$. Fixing a base point $v_0 \in F$, we have a homotopy exact sequence of topological fundamental groups

$$1 \to \pi_1^{\text{top}}(F, v_0) \to \pi_1^{\text{top}}(V_0, v_0) \xrightarrow{p_*} \pi_1^{\text{top}}(B_0, b_0) \to 1$$

where $V_0 = p^{-1}(B_0)$. A section $s_0 : B_0 \to V_0$ of p over B_0 meets F in a point v_1, whence a map $s_{0*} : \pi_1^{\text{top}}(B_0, b_0) \to \pi_1^{\text{top}}(V_0, v_1)$. Fixing a path from v_0 to V_1 induces an isomorphism $\pi_1^{\rm top}(V_0, v_1) \xrightarrow{\sim} \pi_1^{\rm top}(V_0, v_0)$; it is unique up to inner automorphism. By composition s_{0*} induces a section of the map p_* above. Therefore we obtain a map

$$S : \{\text{sections of } p|_{v_0} : V_0 \to B_0\} \to \{\text{conjugacy classes of sections of } p_*\}.$$

As any section of p is determined by its restriction to B_0, the theorem follows from the two claims below. □

Claim 4.3. *The map S has finite fibres.*

Claim 4.4. *The map S has finite image.*

We begin with the proof of Claim 4.3. First we recall the notion of $K|k$-trace for abelian varieties. Given a field extension $K|k$ and an abelian variety A over K, the $K|k$-trace ${\rm tr}_{K|k}(A)$ is the k-abelian variety characterized by the property that ${\rm Hom}(B_K, A) \xrightarrow{\sim} {\rm Hom}(B, {\rm tr}_{K|k}(A))$ for all k-abelian varieties B. Its existence is a theorem of Chow; see ([8], Appendix A) or [2] for modern proofs. The cases $B = {\rm tr}_{K|k}(A)$ and $B = {\rm Spec}\, k$ show that there is a canonical map $\tau :\ {\rm tr}_{K|k}(A)_K \to A$; its image is the maximal abelian subvariety of A defined over k. The map τ induces a map ${\rm tr}_{K|k}(A)(k) \to A(K)$ on points which we shall denote in the same way.

Proof of Claim 4.3. The diagram

$$\begin{array}{ccc} C & \longrightarrow & {\rm Spec}\, \mathbf{C}(B) \\ \downarrow & & \downarrow \\ V_0 & \longrightarrow & B_0 \end{array}$$

is Cartesian, so a section $s_0 :\ B_0 \to V_0$ induces a section $s :\ {\rm Spec}\, \mathbf{C}(B) \to C$. On the other hand, a section $\pi_1^{\rm top}(B_0, b_0) \to \pi_1^{\rm top}(V_0, v_0)$ induces a map on profinite completions, i.e. a map $\pi_1(B_0, b_0) \to \pi_1(V_0, v_0)$ of *algebraic* fundamental groups. For some geometric point c_0 of C above v_0 the diagram of groups

$$\begin{array}{ccc} \pi_1(C, c_0) & \longrightarrow & {\rm Gal}(\overline{\mathbf{C}(B)}|\mathbf{C}(B)) \\ \downarrow & & \downarrow \\ \pi_1(V_0, v_0) & \longrightarrow & \pi_1(B_0, b_0) \end{array}$$

coming from the previous diagram of schemes commutes, and moreover the composite map ${\rm Gal}(\overline{\mathbf{C}(B)}|\mathbf{C}(B)) \to \pi_1(C, c_0) \to \pi_1(V_0, v_0)$ coming from the section s above factors through $\pi_1(B_0, b_0)$ as s comes from s_0. Hence it is enough to show that the map

$$C(\mathbf{C}(B)) \to \{\text{conjugacy classes of sections of } \pi_1(C, c_0) \to \Gamma\}$$

has finite fibres, where $\Gamma := {\rm Gal}(\overline{\mathbf{C}(B)}|\mathbf{C}(B))$. This is done as in the injectivity part of the section conjecture. If y_0 is a a $\mathbf{C}(B)$-point of C and y another $\mathbf{C}(B)$-point inducing the same section $\Gamma \to \pi_1(C, c_0)$, then the argument given there shows that the class of the divisor $y - y_0$ is divisible in $J(\mathbf{C}(B))$. But by the Lang–Néron theorem (see [9] for a beautiful short proof) the group $J(\mathbf{C}(B))/\tau(\mathrm{tr}_{\mathbf{C}(B)|\mathbf{C}}(J)(\mathbf{C}))$ is finitely generated and as such has no nontrivial divisible element. Therefore the image of y by the embedding $C \to J$ with base point y_0 lies in the image of the trace $\mathrm{tr}_{\mathbf{C}(B)|\mathbf{C}}(J)$. But if C is non-isotrivial, the whole of C cannot lie in the trace (this can be checked using the explicit construction of the trace in [8]). Their intersection is thus a proper closed, hence finite subset of C, which shows that there can be only finitely many points y inducing the same section as y_0. □

The proof of Claim 4.4 is entirely topological. The idea is to bound the 'size' of sections of p_* in a suitable way. This is accomplished using ideas of complex hyperbolic geometry, of which here are some basic facts (see [10] for more).

Equip the complex unit disc with the Poincaré metric given by $z \mapsto (1-|z|^2)^{-1}$. It defines a distance function d_{hyp} on D which we may use to define the *Kobayashi pseudo-distance* on any complex manifold X:

$$d_X(x, y) = \inf \sum_{i=1}^{r} d_{\mathrm{hyp}}(p_i, q_i)$$

where the infimum is taken over systems of points $p_i, q_i \in D$ $(1 \le i \le r)$ for which there exist holomorphic maps $f_1, \dots, f_r : D \to X$ with $f_1(p_1) = x$, $f_r(q_r) = y$ and $f_i(q_i) = f_{i+1}(p_{i+1})$. The pseudo-distance d_D is identically 0, so d_X does not satisfy $d_X(x, y) \neq 0$ for $x \neq y$ in general. The manifold X is said to be hyperbolic if $d_X(x, y) \neq 0$ for $x \neq y$, and in this case we get a distance function that can be used to define the length of a path in X. If $\phi : X \to Y$ is a holomorphic map, then $d_Y(\phi(x), \phi(y)) \le d_X(x, y)$ (this follows from the case $X = Y = D$, where it is a consequence of Schwarz's lemma).

By a famous theorem of Brody, a compact manifold X is hyperbolic if and only if there is no holomorphic map $\mathbf{C} \to X$. In particular, a compact Riemann surface of genus $g > 1$ is hyperbolic and we may obtain a hyperbolic manifold from any compact Riemann surface after removing finitely may open discs. Also, a fibred complex manifold with base and fibre of this type is again hyperbolic. So we can make a hyperbolic manifold V' out of V_0 by removing the preimage of finitely many open discs in B (which we may assume to contain the finitely many deleted points of B). Write $B' = p(V')$ and assume that the fixed fibre F (and in particular the base point v_0) lies in V'.

Lemma 4.5. *For each $C > 0$ there are only finitely many elements of $\pi_1^{\mathrm{top}}(V_0, v_0)$ that can be represented by paths lying in V' that have length at most C in the hyperbolic metric of V'.*

Proof. Consider the universal cover $\widetilde{V}_0 \to V_0$. Any holomorphic map $D \to V_0$ lifts to $\widetilde{V}_0$, therefore liftings of paths of length $\le C$ starting v_0

stay inside a closed ball of radius C. As V_0 is hyperbolic, so is $\widetilde{V}_0$, and therefore the ball is compact. Closed paths around v_0 lift to paths with endpoints contained in a fixed orbit of $\pi_1^{\mathrm{top}}(V_0, v_0)$. As these orbits are discrete, they intersect the compact ball in finitely many points. □

Now fix generators $x_1, \ldots, x_r$ of the finitely generated group $\pi_1^{\mathrm{top}}(B_0, b_0)$. In view of the lemma, Claim 4.4 is a consequence of:

Proposition 4.6. *There exists a constant $C > 0$ such that for every section $s : B_0 \to V_0$ the images of $x_1, \ldots, x_n$ under the induced map $\pi_1^{\mathrm{top}}(B_0, b_0) \to \pi_1^{\mathrm{top}}(V_0, v_0)$ can be represented by paths lying in V' that have length at most C.*

Proof. We may assume that the x_i are represented by closed paths γ_i lying inside B' (this may require the modification of the deleted discs or of b_0 but that does not affect the proof of the main result). Let $s : V_0 \to B_0$ be a section. As holomorphic maps are distance-decreasing, we have for $x, y \in s(B')$ a sequence of inequalities

$$d_{s(B')}(x, y) \geq d_{V'}(x, y) \geq d_{B'}(p(x), p(y)) \geq d_{s(B')}(x, y)$$

induced by the maps $s(B') \hookrightarrow V' \xrightarrow{p} B' \xrightarrow{s} s(B')$. Thus we have equality throughout, which shows that for each i the length of $s(\gamma_i)$ calculated with respect to $d_{V'}$ is the same as that of γ_i with respect to $d_{B'}$. This gives a uniform bound on the V'-length of the $s(\gamma_i)$. A representative of $s_*(x_i)$ in $\pi_1^{\mathrm{top}}(V_0, v_0)$ is given by $\gamma s(\gamma_i)\gamma^{-1}$, where γ is a path lying in $F \subset V'$ joining v_0 to $s(b_0)$. But F is a compact hyperbolic Riemann surface, so we may join v_0 to any point by a path of length bounded by an absolute constant (e.g. a geodesic). This proves the proposition, and thereby Claim 2. □

5. Anabelian Geometry

By 'anabelian geometry' one refers to a sheaf of conjectures formulated by Grothendieck in a famous letter to Faltings [7]. The rough idea is that a certain category of schemes defined over finitely generated fields should be determined by their geometric fundamental groups together with its outer Galois action.

There are two kinds of motivation for the conjectures. The first one comes from topology.

Fact 5.1. Recall that for a smooth proper curve X of genus ≥ 2 over $\mathbf{C}$ the topological fundamental group has a presentation

$$\Pi = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$$

This group is non-commutative; moreover, it has trivial center.

On the other hand, the universal cover of X is the unit disc D which is contractible. Therefore $\pi_q(X) = \pi_q(D) = 0$ for $q \geq 2$, and so X is the *Eilenberg-MacLane space* $K(\Pi, 1)$, and as such it is determined up to homotopy.

As an algebraic curve, X may be defined over a finitely generated extension k of $\mathbf{Q}$. The hope therefore arises that the extra structure given by Galois action on Π may determine X up to algebraic isomorphism, not just up to homotopy.

The second motivation comes from the Tate conjecture.

Fact 5.2. Let k now be a number field, X_1, X_2 smooth proper curves over k, of genus ≥ 2. Assume for simplicity that both have a k-point. These k-points can be used to embed X_i in its Jacobian J_i. Write $\overline{X}_i := X_i \times_k \overline{k}$ and similarly for J_i. We know that for each prime ℓ and $i = 1, 2$

$$T_\ell(\overline{J}_i) \cong \pi_1^{\mathrm{ab}}(\overline{J}_i)^{(\ell)} \cong \pi_1^{\mathrm{ab}}(\overline{X}_i)^{(\ell)}$$

where T_ℓ stands for the ℓ-adic Tate module (Remark 2.2)

By a fundamental theorem of Faltings (ex Tate conjecture) the natural map

$$\mathrm{Hom}(J_1, J_2) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(\overline{J}_1), T_\ell(\overline{J}_2))^{\mathrm{Gal}(\overline{k}|k)}$$

is an isomorphism. In other words, Galois-invariant homomorphisms $T_\ell(\overline{J}_1) \to T_\ell(\overline{J}_2)$ can be 'approximated ℓ-adically' by morphisms $J_1 \to J_2$.

One can ask here whether working with the whole geometric fundamental group instead of its abelian quotient can give a stronger result: does a Galois-invariant outer homomorphism $\pi_1(\overline{X}_1) \to \pi_1(\overline{X}_2)$ come from a k-morphism $X_1 \to X_2$? Or, even more economically, does a Galois-invariant outer homomorphism $\pi_1(\overline{X}_1)^{(\ell)} \to \pi_1(\overline{X}_2)^{(\ell)}$ between maximal pro-ℓ-quotients come from a map of curves?

Before formulating precise statements, let us elucidate the role of center-freeness (besides the topological motivation). Recall that the representation

$$\rho_X : \mathrm{Gal}(\overline{k}|k) \to \mathrm{Out}(\pi_1(\overline{X}, \overline{x}))$$

is defined using the exact commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\overline{X}, \overline{x}) & \longrightarrow & \pi_1(X, \overline{x}) & \longrightarrow & \mathrm{Gal}(\overline{k}|k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\pi_1(\overline{X}, \overline{x})) & \longrightarrow & \mathrm{Aut}\,(\pi_1(\overline{X}, \overline{x})) & \longrightarrow & \mathrm{Out}(\pi_1(\overline{X}, \overline{x})) & \longrightarrow & 1. \end{array}$$

Observe: when the center of $\pi_1(\overline{X}, \overline{x})$ is trivial, this becomes a pushout diagram. Therefore

$$\pi_1(X, \overline{x}) \cong \mathrm{Aut}\,(\pi_1(\overline{X}, \overline{x})) \times_{\mathrm{Out}(\overline{X}, \overline{x})} \mathrm{Gal}(k),$$

i.e. $\pi_1(X, \overline{x})$ is determined by $\pi_1(\overline{X}, \overline{x})$ and ρ_X. When $k \subset \mathbf{C}$, it thus appears as a "transcendental object" endowed with a Galois action.

We now define a category of profinite groups as follows. Given two profinite groups G_1, G_2 together with morphisms $p_i : G_i \to G$, define $Hom_G^*(G_1, G_2) =$ as the set of morphisms $G_1 \to G_2$ compatible with the p_i up to conjugation by an element of G. This set carries an action of G_1 from the left and of G_2 from the right. The latter defines a finer equivalence, so put $Hom_G^{ext}(G_1, G_2) = Hom_G^*(G_1, G_2)$ modulo action of G_2. Fixing G we thus get a category $\mathbf{Prof}_G^{\mathrm{ext}}$ with objects profinite groups with projections onto G and Hom-sets the $Hom_G^{ext}(G_1, G_2)$. Denote by $\mathbf{Prof}_G^{\mathrm{ext,\,open}}$ the full subcategory with the same objects but with morphisms having open image.

Sending a variety over a field k to its algebraic fundamental group gives a functor

$$\pi_1 : \{k\text{-varieties}\} \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{\mathrm{ext}}$$

where base points do not play a role any more, so we drop them from now on.

Similarly, sending a field to its absolute Galois group yields a contravariant functor

$$\mathrm{Gal} : \{\text{field extensions of } k\} \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{ext}.$$

In his letter to Faltings Grothendieck formulated the following conjecture.

Conjecture 5.3. *Let k be a finitely generated extension of $\mathbf{Q}$. Denote by $\mathbf{Hyp}_k$ the category of hyperbolic k-curves equipped with* dominating *k-morphisms. The functor*

$$\pi_1 : \mathbf{Hyp}_k \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{\mathrm{ext,\,open}}$$

is fully faithful.

Recall that hyperbolic k-curves are the smooth k-curves of genus g with at least $2-2g$ geometric points at infinity. These are precisely the smooth curves with non-trivial center-free π_1. Grothendieck also speculated about extending $\mathbf{Hyp}_k$ by including some higher-dimensional varieties called 'anabelian varieties'. At present there is no precise conjectural characterization of anabelian varieties in dimensions > 1. However, there is a precisely formulated birational analogue:

Conjecture 5.4. *Let k be finitely generated over $\mathbf{Q}$. Denote by $\mathbf{Bir}_k^{\mathrm{dom}}$ the category of fields finitely generated over k together with k-morphisms. Then the contravariant functor*

$$\mathrm{Gal} : \mathbf{Bir}_k^{\mathrm{dom}} \to \mathbf{Prof}_{\mathrm{Gal}(k)}^{ext,\,open}$$

is fully faithful.

Here are the most important known results about these conjectures.

Theorem 5.5. (Mochizuki [13]) *Conjecture 5.3 is true more generally for k sub-p-adic, i.e. a subfield of some finitely generated extension of a $\mathbf{Q}_p$. In fact, over such fields the following holds: for a hyperbolic k-curve X and an arbitrary smooth k-variety V the map*

$$\mathrm{Hom}_k^{dom}(V, X) \to \mathrm{Hom}_{\mathrm{Gal}(k)}^{\mathrm{ext,\,open}}(\pi_1(V), \pi_1(X))$$

is bijective. Here π_1 *may be replaced by its quotient* π_1^p *classifying covers whose base change to* $\overline{k}$ *is of* p*-power degree.*

This is all the more remarkable as the Tate conjecture does not hold over $\mathbf{Q}_p$! Concerning the birational version, we have:

Theorem 5.6.

1. (Pop, [18], [22]) *The isomorphism version of conjecture 5.4 is true, even in positive characteristic. More precisely, if* K*,* L *are finitely generated fields (over the prime field), the natural map*
$$\mathrm{Isom}^i(K, L) \to \mathrm{Isom}^{ext}(\mathrm{Gal}(L), \mathrm{Gal}(K))$$
is bijective, where on the left Isom^i *means "up to a purely inseparable cover".*
2. (Mochizuki [13]) *Conjecture 5.4 is true more generally for* k *sub-*p*-adic.*

Here part (2) has been recently improved by Corry and Pop [3]: one can replace $\mathrm{Gal}(K)$ (and similarly $\mathrm{Gal}(L)$ by its natural quotient obtained as an extension of $\mathrm{Gal}(k)$ by the maximal pro-p quotient of the subgroup $\mathrm{Gal}(K\overline{k})$. Thus one has a birational result that is completely analogous to Theorem 5.5. However, the positive characteristic analogue is not known at present.

On the other hand, Pop's result does not use the augmentation $\mathrm{Gal}(K) \to \mathrm{Gal}(k)$. This hints at the possibility that 'absolute' forms of Grothendieck's conjecture hold true. And indeed, Mochizuki proved by combining Theorem 5.3 and Theorem 5.4 (1):

Theorem 5.7. (Mochizuki [15]) *Let* X *and* Y *be hyperbolic curves defined over some finitely generated extension of* $\mathbf{Q}$ *(not necessarily the same). Then the natural map*
$$\mathrm{Isom}(X, Y) \to \mathrm{Isom}^{ext}(\pi_1(X), \pi_1(Y))$$
is bijective.

Even more surprisingly, 'absolute' results hold over a finite base field:

Theorem 5.8.

1. (Tamagawa [24]) *Let* X *and* Y *be smooth affine curves defined over some finite field (not necessarily the same) with profinite universal covers* $\widetilde{X}$*,* $\widetilde{Y}$*, respectively. Then the natural map*
$$\mathrm{Isom}(\widetilde{X}|X, \widetilde{Y}|Y) \to \mathrm{Isom}(\pi_1(X), \pi_1(Y))$$
is bijective. Here on the left hand side we have the set of commutative diagrams of isomorphisms
$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\cong} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\cong} & Y. \end{array}$$

2. (Mochizuki [15]) *The same statement holds for proper smooth curves of genus ≥ 2 over a finite field.*

Here the profinite universal cover of an normal integral scheme S means its normalization in the field K_S of Proposition 1.6.

Remarks 5.9.

1. Recently Saïdi and Tamagawa [20] proved that in the theorem above one may replace fundamental groups by their maximal prime-to-p quotients (where p is the characteristic of the base field). They also proved results with even smaller quotients but it is not known whether the statement holds for the maximal pro-ℓ quotients of the fundamental groups (where $\ell \neq p$ is a prime).
2. Before the full statement of Theorem 5.5 was proven, Tamagawa and Mochizuki used specialization arguments to derive the statement of Theorem 5.5 for isomorphisms of hyperbolic curves over number fields from Theorem 5.8 (1). Stix [21] used a similar method to prove an isomorphism statement for hyperbolic curves over global fields of positive characteristic.

We have time only for a very brief description of the methods involved. We start with the birational variant. The prototype is the following special case of Theorem 5.6 (1):

Theorem 5.10. (Neukirch) *Let K_1, K_2 be two Galois extensions of $\mathbf{Q}$. Then any isomorphism $\mathrm{Gal}(K_1) \xrightarrow{\sim} \mathrm{Gal}(K_2)$ comes from a unique isomorphism $K_2 \xrightarrow{\sim} K_1$.*

The proof can be divided in two parts.

Local part: There is a purely group-theoretic characterization of subgroups $D \subset \mathrm{Gal}(k)$ that arise as a decomposition group of some ($\mathbf{Q}$-valued) valuation of $\overline{\mathbf{Q}}$. By an old theorem of F. K. Schmidt such a valuation is then unique.

The key tool for this characterization is:

Fact 5.11.

1. *(Hasse) For $K_{\mathfrak{p}}$ $\mathfrak{p}$-adic and a prime ℓ, $H^2(K_{\mathfrak{p}}, \mathbf{Z}/\ell\mathbf{Z})) \cong \mathbf{Z}/\ell\mathbf{Z}$;*
2. *(Albert-Brauer-Hasse-Noether) For K a number field there is an exact sequence (for $\ell \neq 2$)*

$$0 \to H^2(K, \mathbf{Z}/\ell\mathbf{Z})) \to \bigoplus_{\mathfrak{p}} H^2(K_{\mathfrak{p}}, \mathbf{Z}/\ell\mathbf{Z})) \xrightarrow{\Sigma} \mathbf{Z}/\ell\mathbf{Z} \to 0$$

Surjectivity for a finite set of places holds even for infinite algebraic extensions of $\mathbf{Q}$ or $\mathbf{Q}_p$.

Corollary 5.12. *If $\mu_\ell \subset K$, a subgroup $D \subset \mathrm{Gal}(K)$ is the decomposition group of a (unique) valuation if and only if $H^2(D, \mathbf{Z}/l\mathbf{Z}) = \mathbf{Z}/l\mathbf{Z}$.*

There is a refinement for fields not containing μ_ℓ.

Global part: In this case it is easy. In fact, by the local theory we have a bijection of decomposition groups of places. An additional cohomological argument shows that under this bijection places of the same degree correspond. In particular, completely split places correspond, and by a classical theorem of M. Bauer they determine the extensions K_i uniquely.

A purely group-theoretical argument due to Ikeda and Uchida reduces the case of arbitrary number fields to the Galois case.

Now some words about the proof of Theorem 5.6 (1) in characteristic 0. First, there is a local theory establishing a bijection between decomposition groups of discrete valuation rings of fraction field K (resp. L) whose residue field is a function field that has 'one dimension less'. This uses valuation-theoretic results of Engler, Koenigsmann et al. that characterize decomposition groups of such valuations (on higher-dimensional fields) completely group-theoretically.

Then there is an ingenious geometric argument characterizing sets of such valuations which correspond to codimension one points on some normal model U of the function field K (resp. L). Using the fact that the group of units $\mathcal{O}(U)^\times$ is an extension of $k^\times$ by a finitely generated abelian group (a subgroup of $\mathrm{Div}_{X\setminus U}$ for a normal compactification X) one finds finitely many codimension 1 points $P_1, \ldots, P_r$ such that the natural map

$$\mathcal{O}(U)^\times \to \bigoplus_{i=1}^{r} \kappa(P_i)^\times$$

is injective. Thus using the local correspondence and induction on transcendence degree (starting from Neukirch's result) allows one to construct isomorphisms between subgroups of multiplicative groups of the form $\mathcal{O}(U)^\times$.

Finally, again using specialization arguments, one glues these isomorphisms together and proves additivity of the bijection.

We finish with a few words about the famous theorems of Tamagawa and Mochizuki. Let X be a smooth curve over a field k. In Tamagawa's approach one also begins with a characterization of decomposition subgroups of closed points of X in the fundamental group $\pi_1(X, \overline{x})$ (for some base point $\overline{x}$). These can defined as subgroups of the Galois group $\mathrm{Gal}(K_X|k(X))$ of Proposition 1.6 that stabilize a closed point of the universal cover above the given point.

But since we are working with the fundamental group instead of the full Galois group, there is no inertia, and thus every decomposition group is of the form $s(G)$, where $G \subset \mathrm{Gal}(k_s|k)$ is an open subgroup and s is a section of the natural projection $\pi_1(X, \overline{x}) \to \mathrm{Gal}(k_s|k)$ above G

induced by a closed point P defined over the fixed field of G. By the injectivity statement in Grothendieck's Section Conjecture (Proposition 3.2) P is uniquely determined by s. To decide which sections actually come from points there is the following simple but crucial lemma.

Lemma 5.13. (Tamagawa) *Assume one of the following:*

- *k is finite;*
- *k is p-adic*
- *k is finitely generated over $\mathbf{Q}$ and X is proper of genus ≥ 2.*

Given a section s of the projection $\pi_1(X, \overline{x}) \to \mathrm{Gal}(k_s|k)$ above an open subgroup $G \subset \mathrm{Gal}(k_s|k)$, the image $s(G)$ is the decomposition subgroup of a point over the fixed field L of G if and only if for each open subgroup $H \subset \pi_1(X, \overline{x})$ containing $s(G)$ the corresponding cover X_H has a point over L.

Proof. The nontrivial implication follows because under the assumption on k the sets $X_H(L)$ are compact in their natural topology, hence their inverse limit is nonempty. An element of the inverse limit defines a point of the universal cover $\widetilde{X}$ whose image in X induces s. □

For X affine there is a similar characterization of decomposition groups of points 'at infinity' among subgroups *containing* some $s(G)$.

For $k = \mathbf{F}_q$ the criterion of the lemma can be verified using the fundamental group: according to the Lefschetz–Weil trace formula for $L|k$ of degree m we have

$$|X_H(L)| = 1 + q^m - \sum \alpha_i$$

where the α_i are the eigenvalues of the m-th power Frobenius on $\pi_1^{\mathrm{ab}}(\overline{X}_H)^{(\ell)}$, where $\ell \neq p$ is a prime (this action comes from exact sequence (1)). This gives a characterization of decomposition subgroups over finite fields. For X affine one then recovers the multiplicative groups of function fields using class field theory, and uses a difficult ad hoc argument to prove additivity. Mochizuki deduced the proper case of Theorem 5.8 from the affine case by a geometric technique he calls cuspidalization.

Mochizuki's approach to Theorem 5.5 is completely different. Assume for simplicity we want to recover a proper smooth curve X of genus ≥ 2 from $\pi_1(\overline{X})^{(p)}$ together with its Galois action. The starting idea is to use p-adic Hodge theory. By Tate's first results in the area we have an isomorphism

$$((\pi_1(\overline{X}))^{ab,(p)} \otimes_{\mathbf{Z}_p} \mathbf{C}_p)^{\mathrm{Gal}(k)} \cong H^0(X, \omega_{X/K})$$

where K is a finite extension of $\mathbf{Q}_p$ and $\omega_{X/k}$ is the canonical sheaf of X. But if X is not hyperelliptic, it is canonically embedded into the projective space $\mathbf{P}(H^0(X, \omega_{X/K}))$! So to recover X it is enough to characterize the generic point $\operatorname{Spec} k(X) \to \mathbf{P}(H^0(X, \omega_{X/K}))$ group-theoretically (take schematic closure). To do so, Mochizuki embeds $k(X)$ into a "big" field L which is p-adically complete and has no proper tame extensions, and looks at the curve X_L obtained by base change. The question then becomes: which sections of the projection $\Pi^{(p)}(X_L) \to \operatorname{Gal}(L)$ come from dominating maps $\operatorname{Spec} L \to X$?

The method is inspired by Tamagawa's lemma (Lemma 5.13): one looks for L-points on covers of X_L. From the fact that L has no nontrivial tame extensions of one concludes that it is enough to look for divisors (or line bundles) of degree prime to p. There is a group-theoretical characterization for the latter using the p-adic exponential of Bloch–Kato. Of course the actual implementation of the method is highly technical.

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