Schemes: The Beginnings

We give the basic definitions and constructions concerning schemes. The word ‘ring’ will mean *commutative ring with unit.* Also, when referring to compact topological spaces, we do *not* assume that they are Hausdorff spaces.

1. Prime Spectra

Recall that a subset S of a ring A is called *multiplicatively closed* if $1 \in S$, $0 \notin S$ and for any $f, g \in S$ we have $fg \in S$. A *prime ideal* P is an ideal such that the set $A \setminus P$ is multiplicatively closed. An equivalent formulation of this is that the quotient ring A/P should be a (nontrivial) domain, i.e. it should have no zero-divisors. From this formulation it follows easily that any maximal ideal of A (i.e. an ideal contained in no other proper ideal of A than itself) is always a prime ideal since in this case the quotient ring is a field.

We now turn the set of prime ideals of an arbitrary ring A into a topological space.

Definition 1.1 The *prime spectrum* $\operatorname{Spec} A$ of A is the topological space whose points are prime ideals of A and a basis of open sets is given by the sets

$$D(f) := \{P : P \text{ is a prime ideal with} f \notin P\}$$

for all $f \in A$.

For this definition to be correct, we must verify that the system of the sets $D(f)$ is closed under finite intersections. But we have for all $f, g \in A$

$$D(f) \cap D(g) = D(fg) \tag{1}$$

for by definition a prime ideal avoids fg if and only if it avoids f and g.

It follows from the definition that a closed subset in the topology of $\operatorname{Spec} A$ can be described as the set of prime ideals containing some fixed ideal I (generated by a system of elements $\{f_i : i \in J\}$ of A). Thus one-point sets given by maximal ideals are closed; in fact, maximal ideals give the only closed points of the prime spectrum since for any prime ideal P a closed subset containing P contains the maximal ideals containing P as well. This shows that in general the prime spectrum does not satisfy even the weakest of the separation axioms in topology. However, it enjoys a nice topological property:

Proposition 1.2 *For any ring A the prime spectrum* $\operatorname{Spec} A$ *is compact.*

First a lemma we shall also use later.

Lemma 1.3 *A system of elements $\{f_i : i \in I\}$ generates A if and only if the sets $D(f_i)$ give an open covering of* $\operatorname{Spec} A$.

Proof: Indeed, if the f_i generate A, there can be no prime ideal of A containing all of them, which is equivalent to the $D(f_i)$ covering $\operatorname{Spec} A$. If, however, they do not generate A, then they are all contained (by Zorn's Lemma) in some maximal ideal M which thus gives an element of $\operatorname{Spec} A$ not contained in any of the $D(f_i)$. □

Proof of Proposition 1.2: Let $\{U_i : i \in I\}$ be an open covering of $\operatorname{Spec} A$; we may assume that each U_i is in fact some basic open set $D(f_i)$. By the above lemma the f_i generate A. In particular, there is a relation of the form

$$a_1 f_1 + a_2 f_2 + \ldots + a_n f_n = 1 \tag{2}$$

with $a_i \in A$ and $f_1, \ldots, f_n$ chosen among the f_i above. This means, however, that already $f_1, \ldots, f_n$ generate A, i.e. the sets $D(f_1), \ldots, D(f_n)$ cover $\operatorname{Spec} A$. □

Equations of type (2) are sometimes referred to as *algebraic analogues of partitions of unity.*

Examples 1.4 We conclude this section by some easy examples.

1. The prime spectrum of a field consists of a single point, corresponding to the ideal (0).

2. The prime spectrum of $\mathbf{Z}$ is the space consisting of a closed point for each prime p, and a non-closed point corresponding to (0), called the *generic point*, whose closure is the whole space. Other closed subsets are only finite sets of primes; indeed, any ideal in $I \subset \mathbf{Z}$ is of the form $m\mathbf{Z}$ for some positive integer m; the prime ideals containing I are generated by the prime divisors of m.

3. The prime spectrum of $\mathbf{C}[x]$ consists of a closed point for each $a \in \mathbf{C}$, plus a non-closed generic point corresponding to (0). The closed subsets are again finite sets of closed points. Indeed, $\mathbf{C}[x]$ is a principal ideal ring with prime elements the polynomials $x - a$ ($a \in \mathbf{C}$), and we may argue as in the previous case.

4. If A is isomorphic to a finite direct sum $\bigoplus_{i=1}^{n} A_i$, then $\operatorname{Spec} A$ is a disjoint union of clopen sets each of which is homeomorphic to one of the $\operatorname{Spec} A_i$. To see this, observe first that if one writes e_i for the idempotent given by putting 1 at the i-th component and 0 elsewhere, any pairwise product $e_i e_j$ is 0 and hence no prime ideal P of A can avoid both e_i and e_j. However, P cannot contain all of the e_j since the sum of these is 1. Thus we conclude that P concludes all of the e_j except one, say e_i, which implies that P is of the form $A_1 \oplus \ldots A_{i-1} \oplus P_i \oplus A_{i+1} \ldots \oplus A_n$ with a prime ideal P_i of A_i. The required decomposition of $\operatorname{Spec} A$ is then induced by the map $P \mapsto P_i$.

2. Schemes – Mostly Affine

The prime spectrum of a ring is a rather coarse invariant: for instance, it cannot even distinguish between two fields. We shall remedy this by defining some additional structure on the prime spectrum. To motivate the construction to come, let us reconsider the third example from the last section.

Example 2.1 The ring $\mathbf{C}[x]$ is nothing but the ring of holomorphic functions on $\mathbf{C}$ having at worst a pole at infinity. The prime spectrum of this ring can be identified to $\mathbf{C}$ with a generic point (0) added; closed sets are finite sets not containing (0). Obviously one cannot recover $\mathbf{C}[x]$ from these data; we cannot even distinguish between constant functions. Remember, however, that we have seen in the previous chapter that a Riemann surface is uniquely determined by the underlying topological space plus the sheaf of holomorphic functions on it. If we restrict to the sheaf of holomorphic functions on $\mathbf{C}$ having at worst a pole at infinity, we can easily describe its sections over a set of the form $D(f)$ with the generic point removed (this is an open set in the complex topology). For instance, over $D(x)$ (which with the generic point thrown away identifies to $\mathbf{C}^*$) the sections are the rational functions whose denominator is a power of x, for thesections are meromorphic functions on $\mathbf{P}^1(\mathbf{C})$ and hence elements of $\mathbf{C}(T)$; moreover, any denominator other than the x^m has a zero elsewhere. We find an analogous result for $D(x - a)$ for ($a \in \mathbf{C}$); all other $D(f)$ are finite intersections of these, so the sections of the sheaf over $D(f)$ are just the restrictions of the sections over the $D(x-a)$ with $(x-a)$ dividing f.

If we wish to define something analogous to this for any ring A, we first have to extend the notion of a rational function, i.e. give a meaning to fractions of elements in an arbitrary ring A. So let S be a multiplicatively closed subset of A, i.e. a subset $S \subset A \setminus \{0\}$ containing 1 such that $x, y \in S \Rightarrow xy \in S$. We would like to define a ring A_S which is to be the "ring of fractions with numerator in A and denominator in S".

Example 2.2 When A is a domain, this is fairly easy to do since in this case A admits a fraction field K. Elements of K can be represented by fractions f/g with $f, g \in A$, $g \neq 0$, where $f/g = f_1/g_1$ whenever $fg_1 = f_1 g$. We may then take A_S to be the subring of those elements which can be written as fractions with denominators in S; this is indeed a subring as S is multiplicatively closed.

Now to treat the general case, observe first that just as the fraction field K can be defined as the object representing a certain functor, the ring A_S of the previous example is easily seen to represent the set-valued functor F given by

$$F(R) = \{\phi \in \mathrm{Hom}(A, R) : \phi(s) \text{ is a unit in } R \text{ for all } s \in S\}$$

on the category of rings. When A has zero-divisors, A has no fraction field, but the above functor F still exists.

Proposition 2.3 *The functor F is representable by a ring A_S for any ring A and multiplicatively closed subset S.*

The ring A_S is called the *localisation of A with respect to S*. By the Yoneda lemma, it is determined up to unique isomorphism. Moreover, it is equipped with a canonical homomorphism $\phi_S : A \to A_S$ sending elements of S to units which corresponds to the identity map $A_S \to A_S$.

Proof: Define A_S as a set to be the quotient of $A \times S$ by the equivalence relation:

$$(f, s) \sim (f', s') \quad \text{iff there is a } t \in S \text{ with} \quad (fs' - f's)t = 0.$$

One sees that this is indeed an equivalence relation; for transitivity, note that the equations $(fs' - f's)t = 0$ and $(f's'' - f''s')u = 0$ imply $(fs'' - f''s)s'tu = 0$

(multiply the first equation by $s''u$ and the second by st). Denote by f/s the image of (f, s) in A_S and define the addition and multiplication laws as for fractions; one checks that this is independent of the representatives chosen.

Now given a homomorphism $\phi : A \to R$ sending elements of S to units, define a homomorphism $A_S \to R$ by sending f/s to $\phi(f)\phi(s)^{-1}$ (note that units are never zero-divisors, so $\phi(s)^{-1}$ is a well-defined element of R). This is a well-defined map, for if (f', s') is another representative for f/s, we have

$$0 = \phi((fs' - f's)t) = (\phi(f)\phi(s') - \phi(f')\phi(s))\phi(t),$$

whence $\phi(f)\phi(s') = \phi(f')\phi(s)$ as $\phi(t)$ is a unit. Conversely, as any element of S maps to a unit in A_S by the map $\phi_S : A \to A_S$ sending s to $s/1$, homomorphisms $A_S \to R$ induce elements of $F(R)$ by composition with ϕ_S. Thus we have obtained a bijection between $F(R)$ and $\mathrm{Hom}(A_S, R)$ which is immediately seen to be functorial. □

We now wish to compare the prime spectra of A and A_S.

Lemma 2.4 *The map $P \mapsto \phi_S(P)A_S$ defines a canonical bijection between prime ideals P of A avoiding S and prime ideals of A_S.*

Proof: Let P be a prime ideal of A avoiding S. By this last condition, the ideal $\phi_S(P)A_S$ generated by $\phi_S(P)$ does not contain units and hence is different from A_S. Moreover, it is a prime ideal, for if $(f/s)(g/t) \in \phi_S(P)A_S$, then $ufg \in P$ for some $u \in S$, whence f or g is in P and thus (f/s) or (g/t) is in $\phi_S(P)A_S$. For surjectivity, note the easy fact that for any prime ideal Q of A_S the ideal $\phi_S^{-1}(Q)$ is a prime ideal of A avoiding S; the assertion then follows from the equality $\phi_S(\phi_S^{-1}(Q))A_S = Q$. Similarly, injectivity follows from $\phi_S^{-1}(\phi_S(P)A_S) = P$; the verification of these relations is left to the reader. □

Examples 2.5 The two key examples of localisation to be used in the sequel are the following.

1. Let S be the set $\{1, f, f^2, f^3, \ldots\}$ of all powers of f for some $f \in A$. In this case elements of A_S are represented by fractions with numerator in A and denominator a power of f; we shall use the notation A_f for this particular A_S. The previous lemma implies that $\mathrm{Spec}\, A_f$ is naturally homeomorphic to the open set $D(f)$.

2. Let P be a prime ideal of A and take S to be the complement of P; it is multiplicatively closed by primeness of P. Adopting a common

abuse of notation from the literature, we shall denote the localisation of A with respect to S by A_P instead of $A_{A\setminus P}$. The points of $\operatorname{Spec} A_P$ correspond to prime ideals of A contained in P; in particular, A_P has a unique maximal ideal generated by the image of P. Rings having a unique maximal ideal are usually called *local rings*.

This example contains the case of fraction fields: take P to be the ideal (0) in a domain.

Now we may turn to defining a sheaf of rings $\mathcal{O}_X$ on the prime spectrum X of any commutative ring A. In obvious analogy with the example of $\mathbf{C}[x]$ described above, we define $\mathcal{O}_X(D(f)) = A_f$ for all $f \in A$. To proceed further, we need an easy lemma.

Lemma 2.6 *If $f, g \in A$ are such that $D(f) \subset D(g)$, then the image of g in A_f is a unit.*

Proof: Indeed, if g did not give a unit in A_f, it would be contained in a maximal ideal Q. By Lemma 2.4 there is a unique prime ideal P of A whose image in A_f generates Q. This P contains g but not f, a contradiction. □

Combining the lemma with Proposition 2.3, we get for any inclusion $D(f) \subset D(g)$ of basic open sets a canonical restriction homomorphism $A_g \to A_f$. Clearly for a tower of inclusions $D(f) \subset D(g) \subset D(h)$ the map $A_h \to A_f$ thus obtained is the composition of the intermediate maps $A_h \to A_g$ and $A_g \to A_f$. So putting $\mathcal{O}_X(D(f)) = A_f$, we have obtained "something which behaves like a presheaf on basic open sets". That this indeed extends to a presheaf on X follows from the first statement of the following formal lemma (of which we advise the readers to skip the proof in a first reading).

Lemma 2.7 *Let X be a topological space and $\mathcal{V}$ a basis of open sets on X. Assume given for each $V \in \mathcal{V}$ a set (resp. abelian group, ring, etc.) $\mathcal{F}(V)$ and for each inclusion $V' \subset V$ of elements of $\mathcal{V}$ a map (resp. homomorphism) $\rho_{VV'} : \mathcal{F}(V) \to \mathcal{F}(V')$ satisfying $\rho_{VV} = \mathrm{id}_{\mathcal{F}(V)}$ and $\rho_{VV''} = \rho_{V'V''} \circ \rho_{VV'}$ for each tower $V'' \subset V' \subset V$ of elements of $\mathcal{V}$.*

1. *There exists a presheaf of sets (resp. abelian groups, rings, etc.) $\mathcal{F}$ on X whose sections and restriction maps over elements of $\mathcal{V}$ can be canonically identified to those given above.*

2. *Assume moreover that the $\mathcal{F}(V)$ above satisfy the sheaf axioms for all coverings of elements of $\mathcal{V}$ by elements of $\mathcal{V}$. Then there is a* unique

sheaf $\mathcal{F}$ on X whose sections and restriction maps over elements of $\mathcal{V}$ are those given above.

3. *Finally assume given two sheaves $\mathcal{F}$, $\mathcal{G}$ on X and for each $V \in \mathcal{V}$ a map $\phi_V : \mathcal{F}(V) \to \mathcal{G}(V)$ such that for each inclusion $V' \subset V$ of elements of $\mathcal{V}$ the diagram*

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ {\scriptstyle \rho^{\mathcal{F}}_{VV'}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{VV'}} \\ \mathcal{F}(V') & \xrightarrow{\phi_{V'}} & \mathcal{G}(V') \end{array}$$

commutes. Then there is a unique morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ with the ϕ_V given as above.

Proof: For the first statement, consider for a given open set $U \subset X$ the set $\mathcal{V}_U$ of elements of $\mathcal{V}$ contained in U; this set is partially ordered by inclusion. The restriction maps $\phi_{VV'}$ for $V' \subset V \subset U$ turn the system of $\mathcal{F}(V)$ with $V \in \mathcal{V}_U$ into an inverse system. Note that this is a *non-filtered* inverse system. Define $\mathcal{F}(U)$ as the inverse limit of this system. By definition, $\mathcal{F}(U)$ consists of sequences (f_V) indexed by all $V \in \mathcal{V}_U$ with $f_V \in \mathcal{F}(V)$ having the property that $f_{V'} = \phi_{VV'}(f_V)$ whenever $V' \subset V$. If $U' \subset U$, define a restriction map $\rho_{UU'}$ by mapping the sequence (f_V) above to the sequence of those f_V for which $V \subset U'$. There is no difficulty in checking that we have thus defined a presheaf. Moreover, for $W \in \mathcal{V}$, the sections of $\mathcal{F}$ over W can be canonically identified with the elements of the prescribed set $\mathcal{F}(W)$ as in this case the sequences (f_V) defining the inverse limit are given by restrictions of elements of the prescribed $\mathcal{F}(W)$ to all elements of $\mathcal{V}$ contained in W. Thus for any U containing W, the restriction map ρ_{UW} can be identified to the map projecting a sequence (f_V) to f_W; in particular, a section in $\mathcal{F}(U)$ is uniquely determined by its restrictions to each $W \in \mathcal{V}_U$.

For the second statement, note first that unicity follows from the first sheaf axiom since each open $U \subset X$ can be covered by elements of $\mathcal{V}$. So it suffices to show that the presheaf $\mathcal{F}$ we have just defined satisfies the sheaf axioms. By construction of $\mathcal{F}$, for the first sheaf axiom it is enough to see that for any open cover $\{U_i : i \in I\}$ of U, two sections $(f_V), (g_V) \in \mathcal{F}(U)$ restricting to the same section over each U_i restrict to the same section over each $W \in \mathcal{V}_U$. Since $\mathcal{V}$ is a basis of open sets (hence in particular closed under finite intersections), we may write each U_i as a union of some $V_{ij} \in \mathcal{V}_U$ in such a way that W itself is a union of some of the V_{ij}. Now as (f_V) and (g_V) restrict to the same section over each V_{jk}, they must restrict to the same section over W by the assumption. The verification of the second sheaf axiom is similar and is left to the readers.

Finally, the last statement follows from the fact that the maps ϕ_V induce a morphism of the inverse systems defining $\mathcal{F}(U)$ and $\mathcal{G}(U)$ for a general U as above. The map $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is then obtained by passing to the limit: explicitly, it maps a sequence (f_V) to the sequence $(\phi_V(f_V))$. □

Now we are ready to prove:

Theorem 2.8 *For any ring A, there is a unique sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Spec} A$ for which $\mathcal{O}_X(D(f)) = A_f$ for all $f \in A$ and the restriction maps $\mathcal{O}_X(D(g)) \to \mathcal{O}_X(D(f))$ for $D(f) \subset D(g)$ are the natural maps $A_g \to A_f$ described above.*

Proof: We have to check the hypothesis of the previous proposition, i.e. the sheaf axioms over the basic open sets $D(f)$. Notice that since $\operatorname{Spec} A_f$ is naturally homeomorphic to $D(f)$ and $\mathcal{O}_X(D(f)) = A_f$, we may replace A by A_f and assume $f = 1$. Then for the first sheaf axiom we are given a covering of X by basic open sets $D(f_i)$; by compactness of X we may assume the covering is finite, say $X = D(f_1) \cup \ldots \cup D(f_n)$. To give a section of $\mathcal{O}_X(X) = A$ restricting to 0 over each $D(f_i)$ is to give an element $g \in A$ satisfying

$$f_i^{k_i} g = 0 \tag{3}$$

for all $1 \leq i \leq n$ with appropriate positive integers k_i. Now by the definition of prime ideals we have $D(f_i^{k_i}) = D(f_i)$ for all i, so the $D(f_i^{k_i})$ cover X as well and hence by Lemma 1.3 there exist $g_i \in A$ with

$$g_1 f_1^{k_1} + \ldots + g_n f_n^{k_n} = 1 \tag{4}$$

Thus if we multiply each equation in (3) by g_i and take the sum we get $g = 0$, as desired.

For the second sheaf axiom, assume again given a covering of X by basic open sets $D(f_i)$ $(1 \leq i \leq n)$ and elements $a_i / f_i^{k_i} \in A_{f_i}$ (viewed as sections of a would-be sheaf over $D(f_i)$) whose restrictions to the pairwise intersections $D(f_i) \cap D(f_j) = D(f_i f_j)$ coincide. This latter property can be written explicitly as $(a_i f_j^{k_j} - a_j f_i^{k_i})(f_i f_j)^{k_{ij}} = 0$ with some positive integer k_{ij}. By changing the a_i if necessary we may assume $k_i = k$ for all i and $k_{ij} = m$ for all i, j, where m is large enough. Thus

$$a_i f_j^k (f_i f_j)^m = a_j f_i^k (f_i f_j)^m \tag{5}$$

for all $1 \leq i, j \leq n$. Now apply (4) with $k_i = k + m$ for all i to get some g_i with $\sum_i g_i f_i^{k+m} = 1$ and define $a = \sum_i g_i a_i f_i^m$. Using equation (5) we get for all j a chain of equations

$$f_j^{k+m}a = \sum_{i=1}^{n} g_i a_i f_j^k (f_i f_j)^m = \sum_{i=1}^{n} g_i a_j f_i^k (f_i f_j)^m = a_j f_j^m \sum_i g_i f_i^{k+m} = a_j f_j^m$$

which means that the image of a in A_{f_j} coincides with a_j/f_j^k, as required. □

Definition 2.9 An *affine scheme* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X such that $X = \operatorname{Spec} A$ for some ring A and $\mathcal{O}_X$ is the sheaf occuring in the above theorem. We call $\mathcal{O}_X$ the *structure sheaf* of X.

By abuse of notation, we shall frequently write X or $\operatorname{Spec} A$ instead of the pair $(X, \mathcal{O}_X)$. Next an important fact:

Proposition 2.10 *If $X = \operatorname{Spec} A$ is an affine scheme, then the stalk $\mathcal{O}_{X,P}$ of $\mathcal{O}_X$ at any point P of X is canonically isomorphic to the localisation A_P; in particular, it is a local ring.*

Proof: Recall that the stalk at P is defined as the direct limit of the *filtered* direct system of the rings $\mathcal{O}_X(U)$, for U containing P. Since basic open sets $D(f)$ are cofinal in the index set of this direct system, we may restrict to the rings A_f. Then the proposition follows from the fact that the direct limit of these rings is obtained by "dividing out by all $f \notin P$". More precisely, it follows from the construction of $\varinjlim A_f$ that any $f \notin P$ is a unit in $\varinjlim A_f$, hence there is a homomorphism $A_P \to \varinjlim A_f$. If an element $g \in A_P$ maps to zero here, it means $f^n g = 0$ for some $f \notin P$ and $n \geq 0$ and thus $g = 0$ in A_P; surjectivity is equally obvious. □

Now some definitions.

Definition 2.11 A *ringed space* is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is a pair $(\phi, \phi^\sharp)$ consisting of a continuous map $\phi : X \to Y$ and a morphism $\phi^\sharp : \mathcal{G} \to \phi_* \mathcal{F}$ of sheaves of rings on Y.

Ringed spaces thus form a category with the morphisms just defined. Affine schemes are naturally objects of this category enjoying the additional property that the stalks of the structure sheaf are all local rings.

Next notice that given a morphism of ringed spaces $(\phi, \phi^\sharp)$ as above, for any $x \in X$ the morphism $\phi^\sharp$ induces a ring homomorphism $\mathcal{G}_{\phi(x)} \to \mathcal{F}_x$ on the stalks, for by definition $\mathcal{G}_{\phi(x)}$ is the (filtered) direct limit of $\mathcal{G}(U)$ for the open sets U containing $\phi(x)$, whereas $\phi_*\mathcal{F}(U) = \mathcal{F}(\phi^{-1}(U))$ and there is a natural map from the direct limit of the latter sets to $\mathcal{F}_x$, for $\mathcal{F}_x$ is defined as the direct limit of *all* open neighbourhoods containing x.

Definition 2.12 A *locally ringed space* is a ringed space $(X, \mathcal{F})$ such that the stalk $\mathcal{F}_x$ is a local ring for all $x \in X$. A *morphism* of locally ringed spaces is to be a morphism of ringed spaces for which the induced maps $\mathcal{G}_{\phi(x)} \to \mathcal{F}_x$ on stalks described above are *local* homomorphisms, which means that the preimage of the maximal ideal of $\mathcal{F}_x$ is the maximal ideal of $\mathcal{G}_{\phi(x)}$. Thus the category of locally ringed spaces is a subcategory of that of ringed spaces.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that X admits an open covering $\{U_i : i \in I\}$ such that for all i the locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ are isomorphic (in the category of locally ringed spaces) to affine schemes. The category of schemes is defined as the *full* subcategory of that of locally ringed spaces whose objects are schemes.

Construction 2.13 As we have already remarked, for any commutative ring A the affine scheme $X = \operatorname{Spec} A$ is naturally an object of the category of schemes. We now show that the map $A \mapsto \operatorname{Spec} A$ is in fact a *contravariant functor* from the category of rings to that of schemes. For this we have to assign to any ring homomorphism $\phi : A \to B$ a morphism $(\operatorname{Spec}(\phi), \operatorname{Spec}(\phi)^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$ of schemes. The definition of $\operatorname{Spec}(\phi)$ is obvious: it maps a prime ideal $P \in \operatorname{Spec} B$ to $\phi^{-1}(P)$ which is immediately seen to be a prime ideal of A. The map is continuous since the inverse image of a basic open set $D(f)$ is just $D(\phi(f))$. Now for defining $\operatorname{Spec}(\phi)^\sharp$ note that by the third statement of Lemma 2.7 it suffices to consider sections over the basic open sets $D(f)$. By construction of the structure sheaves, over $D(f)$ our task is to define a ring homomorphism $A_f \to B_{\phi(f)}$. But there is a canonical such homomorphism according to Lemma 2.3: it corresponds to the composite $A \to B \to B_{\phi(f)}$.

Now consider the natural question: given an affine scheme $X = \operatorname{Spec} A$, how can we recover A from X? The answer is easy: we have $A = \mathcal{O}_X(X)$. Moreover, the rule $X \mapsto \mathcal{O}_X(X)$ is also a contravariant functor: given a morphism $\phi : X \to Y$ of affine schemes, we have in particular a morphism of sheaves $\phi^\sharp : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$, whence a homomorphism $\mathcal{O}_Y(Y) \to \phi_*\mathcal{O}_X(Y) = \mathcal{O}_X(X)$.

Theorem 2.14 *The functors $A \mapsto \operatorname{Spec} A$ and $X \to \mathcal{O}_X(X)$ are inverse to each other. Thus the category of affine schemes is* isomorphic *to the opposite category of the category of commutative rings with unit.*

Proof: If $Y = \operatorname{Spec} B$ and the scheme morphism $X \to Y$ comes from a homomorphism $\lambda : A \to B$, by construction the map $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is none but λ.

We are left to prove that given a morphism $(\phi, \phi^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$ of schemes, if $\lambda : A \to B$ is the ring homomorphism induced by taking global sections, then $(\phi, \phi^\sharp) = (\operatorname{Spec}(\lambda), \operatorname{Spec}(\lambda)^\sharp)$. For this, we have to show first that for $P \in \operatorname{Spec} B$ we have $\phi(P) = \lambda^{-1}(P)$. Indeed, $\phi^\sharp$ induces a map on the stalks $\phi_P^\sharp : A_{\phi(P)} \to B_P$ which by definition makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \downarrow & & \downarrow \\ A_{\phi(P)} & \xrightarrow{\phi_P^\sharp} & B_P \end{array}$$

commute. But $\phi_P^\sharp$ is a *local* homomorphism, i.e. $\phi(P)A_{\phi(P)} = (\phi_P^\sharp)^{-1}(PB_P)$; on the other hand, the vertical maps are localisation maps, whence the assertion. The equality $\phi^\sharp = \operatorname{Spec}(\lambda)^\sharp$ follows from the analogous commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \downarrow & & \downarrow \\ A_f & \xrightarrow{\phi_{D(f)}^\sharp} & B_{\lambda(f)} \end{array}$$

for sections over basic open sets. □

3. First Examples of Schemes

It is now time for some examples. Let us first take a new look of those of the first section, but this time considering the structure sheaves as well.

Examples 3.1

1. For k a field, the underlying topological space of $\operatorname{Spec} k$ is a single point. The stalk of the structure sheaf at this point is k.

2. The generic stalk of $\operatorname{Spec} \mathbf{Z}$, i.e. the stalk of $\mathcal{O}_{\operatorname{Spec} \mathbf{Z}}$ at the generic point (0) is $\mathbf{Q}$. The inclusion $\mathbf{Z} \to \mathbf{Q}$ corresponds to a morphism $\operatorname{Spec} \mathbf{Q} \to \operatorname{Spec} \mathbf{Z}$, identifiable as the inclusion of the generic point into $\operatorname{Spec} \mathbf{Z}$. At a closed point corresponding to the prime ideal (p) the stalk $\mathcal{O}_{\operatorname{Spec} \mathbf{Z},(p)}$ is isomorphic to the subring of $\mathbf{Q}$ formed by fractions whose denominator is not divisible by p. The maximal ideal of this ring is generated by p; we have $\mathcal{O}_{\operatorname{Spec} \mathbf{Z},(p)}/p\mathcal{O}_{\operatorname{Spec} \mathbf{Z},(p)} \cong \mathbf{F}_p$. The natural projection $\mathbf{Z} \to \mathbf{F}_p$

corresponds to a map $\operatorname{Spec} \mathbf{F}_p \to \operatorname{Spec} \mathbf{Z}$, the inclusion of the closed point (p).

3. The generic stalk of $\operatorname{Spec} \mathbf{C}[x]$ is the rational function field $\mathbf{C}(x)$. At the closed point $(x-a)$ the stalk consists of those elements of $\mathbf{C}[x]$ whose denominator does not vanish at a; the maximal ideal of $\mathcal{O}_{\operatorname{Spec} \mathbf{C}[x],(x-a)}$ is generated by functions vanishing at a. The quotient by this maximal ideal is isomorphic to $\mathbf{C}$; the image of a function by the projection $\mathcal{O}_{\operatorname{Spec} \mathbf{C}[x],(x-a)} \to \mathbf{C}$ is its value at a. Here again a map $\mathbf{C}[x] \to \mathbf{C}[x]/(x-a) \cong \mathbf{C}$ corresponds to the inclusion of the point $a \in \mathbf{C}$.

Thus by comparing the two last examples, we may think of elements of $\mathbf{Q}$ as functions on the space $\operatorname{Spec} \mathbf{Z}$. If the denominator of $f \in \mathbf{Q}$ is not divisible by a prime p, then f is "defined" in a neighbourhood of (p); its image in $\mathbf{F}_p$ is its "value" at p. This is the coarsest analogy one may observe; it will be considerably refined later.

Example 3.2 *Affine spaces.* For a field k we define *affine n-space over k* as the affine scheme $\mathbf{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ (with $k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k). An explanation for this name is provided by a form of a classical theorem of Hilbert's called the *Nullstellensatz* (see e.g. Lang [1], Chapter IX.1): this says that if k is algebraically closed, then any maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with some $a_i \in k$. Thus in this case we may identify the set of closed points of $\mathbf{A}^n_k$ with elements of k^n. Note that the above statement is false even for $n = 1$ if k is not algebraically closed: for instance, the polynomial $x^2 + 1$ generates a maximal ideal of $\mathbf{R}[x]$ not of the above form.

We next give the basic example of a non-affine scheme. Before discussing it, an easy definition.

Definition 3.3 An *open subscheme* of a scheme X is the ringed space consisting of an open subset U and the restriction $O_X|_U$ of the structure sheaf of X to U.

Indeed, one checks that U admits an open covering by affine schemes (use basic open sets, for instance).

Example 3.4 It is possible to define projective spaces $\mathbf{P}^n_A$ over any commutative ring A (not to mention any scheme...) using a patching construction (see Construction 5.3 for details). We may patch together the affine schemes

$$D_+(x_i) \cong \operatorname{Spec} A[(x_0/x_i), \ldots, (x_{i-1}/x_i), (x_{i+1}/x_i), \ldots, (x_n/x_i)]$$

over the isomorphic open subschemes

$$D(x_j/x_i) \cong \operatorname{Spec} A[(x_0/x_i), \dots, (x_{i-1}/x_i), (x_{i+1}/x_i), \dots, (x_n/x_i)]_{x_j/x_i}$$

of $D_+(x_i)$ and

$$D_+(x_i/x_j) \cong \operatorname{Spec} A[(x_0/x_j), \dots, (x_{j-1}/x_j), (x_{j+1}/x_j), \dots, (x_n/x_j)]_{x_i/x_j}$$

of $D_+(x_j)$ by remarking that

$$A[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}]_{\frac{x_j}{x_i}} = A[\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}]_{\frac{x_i}{x_j}}$$

as subrings of $A[x_0, \dots, x_n, x_0^{-1}, \dots, x_n^{-1}]$, so we may use the identity maps as patching isomorphisms.

The next definition enables us to define the basic objects of study in algebraic geometry, namely loci of zeros of systems of polynomials in affine or projective space.

Definition 3.5 A morphism $\phi : Y \to X$ of schemes is a *closed immersion* if the underlying continuous map is the inclusion of a closed subset of X and moreover there is a covering of X by affine open subschemes $U_i = \operatorname{Spec} A_i$ such that for all i the open subscheme of Y defined by $\phi^{-1}(U_i)$ is isomorphic to an affine scheme $\operatorname{Spec} B_i$ with the induced maps $A_i \to B_i$ surjections. We say that Y is a *closed subscheme* of X if there is a closed immersion of Y into X.

Remark 3.6 It can be shown that any closed subscheme of an affine scheme $X = \operatorname{Spec} A$ is of the form $\operatorname{Spec} A/I$ with some ideal I. However, the reader should be warned that in general it is possible to give several different closed subscheme structures on a given closed subset of the underlying topological space of a scheme.

Example 3.7 An (irreducible) *affine hypersurface* of dimension $n-1$ over a field k is the closed subscheme of $\mathbf{A}_k^n$ given by the quotient of the polynomial ring $k[x_1, \dots, x_n]$ by the principal ideal generated by an irreducible polynomial f (here the covering of $\mathbf{A}_k^n$ is just the one-element covering by the whole space). Affine hypersurfaces of dimension 1 are also called *plane curves*. For instance, the quotient of $k[x_1, x_2]$ modulo the principal ideal generated by the polynomial $x_1x_2 - 1$ defines an affine plane curve: the conic of equation $x_1x_2 = 1$.

A *projective hypersurface* is a closed subscheme Y of some $\mathbf{P}_k^n$ which restricts to an affine hypersurface on each canonical open subset $D_+(x_i)$ via the isomorphisms $D_+(x_i) \cong \mathbf{A}_k^n$. As above, in dimension 1 we get *projective plane curves.* For instance, the locus of zeros of the homogenous polynomial $X_0X_1 - X_2^2$ in $\mathbf{P}_k^2$ defines a projective plane curve given on $D_+(X_0)$ by the affine plane curve of equation $x_1 = x_2^2$, on $D_+(X_1)$ that of equation $x_0 = x_2^2$ and on $D_+(X_2)$ that of equation $x_0x_1 = 1$ (notice that different types of affine conics arise from the same projective conic).

4. Quasi-coherent Sheaves

In Section 2 we saw that any ring A defines an affine scheme $X = \operatorname{Spec} A$. Here we study how to associate sheaves on X to modules over the ring A, a construction that will be very useful in the next chapter.

As the construction of affine schemes makes one expect, sheaves associated to A-modules should be, in some sense, modules over the structure sheaf $\mathcal{O}_X$. The following definition makes this precise.

Definition 4.1 Let X be any scheme. A *sheaf of $\mathcal{O}_X$-modules* or an *$\mathcal{O}_X$-module* for short is a sheaf of abelian groups $\mathcal{F}$ on X such that for each open $U \subset X$ the group $\mathcal{F}(U)$ is equipped with an $\mathcal{O}_X(U)$-module structure $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ making the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commute for each inclusion of open sets $V \subset U$. In the special case when $\mathcal{F}(U)$ is an ideal in $\mathcal{O}_X(U)$ for all U we speak of a *sheaf of ideals* on X.

Examples 4.2 Here are two natural situations where O_X-modules arise.

1. Let $\phi : X \to Y$ be a morphism of schemes. We know that on the level of structure sheaves ϕ is given by a morphism $\phi^\sharp : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$, whence an O_Y-module structure on $\phi_*\mathcal{O}_X$.

2. In the previous situation the kernel $\mathcal{I}$ of the morphism $\phi^\sharp : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ (defined by $\mathcal{I}(U) = \ker(\mathcal{O}_Y(U) \to \phi_*\mathcal{O}_X(U))$) is a sheaf of ideals on Y. This is particularly interesting when X is a closed subscheme of Y and ϕ is the natural inclusion. In this case we call $\mathcal{I}$ the *sheaf of ideals defining X*.

3. More generally, given any $\mathcal{O}_X$-module $\mathcal{F}$ one can define its *annihilator* as the ideal sheaf whose sections over an open set U consist of those $f \in \mathcal{O}_X(U)$ for which $fs = 0$ for all $s \in \mathcal{F}(U)$. For instance, the annihilator of the $\mathcal{O}_X$-module 0 is $\mathcal{O}_X$.

We may now proceed to construct $\mathcal{O}_X$-modules over affine schemes from modules over rings. For this we first need an algebraic concept.

Definition 4.3 Let A be a ring, $S \subset A$ a multiplicatively closed subset and M an A-module. The *localisation of M by S* is the A_S-module M_S given by $M \otimes_A A_S$.

As in the case of rings, given an element $f \in A$ or a prime ideal P, we shall use the notations M_f for $M \otimes_A A_f$ and M_P for $M \otimes_A A_P$.

Construction 4.4 Let A be a ring and M an A-module. For any multiplicatively closed $S \subset A$ there is a natural map $M \to M_S$ obtained by tensoring the natural map $A \to A_S$ by M and similarly there is a natural map $M_g \to M_f$ for any inclusion $D(f) \subset D(g)$. The sheaf axioms for $\mathcal{O}_X$ imply that the M_f satisfy the sheaf axioms over basic open sets, so that Lemma 2.7 may be applied to get a sheaf $\tilde{M}$ over X which is an $\mathcal{O}_X$-module by construction.

The rule $M \to \tilde{M}$ is naturally a functor from the category of A-modules to the category of $\mathcal{O}_X$-modules and it is easy to check that it is fully faithful.

One cannot expect, however, that in this way an equivalence of categories arises, as the following simple counter-example shows.

Example 4.5 Let A be the local ring of the affine line $\mathbf{A}^1_k$ in 0, i.e. the localisation of the polynomial ring $k[x]$ by the ideal (x). Then $X = \operatorname{Spec} A$ consists only of two points: a closed point coming from (x) and a so-called generic point η coming from the ideal (0). The stalks of $\mathcal{O}_X$ are A in the closed point and $k(x)$ in the generic point. Now define an $\mathcal{O}_X$-module $\mathcal{F}$ on $X = \operatorname{Spec} A$ by putting $\mathcal{F}(X) = A$ and $\mathcal{F}(\eta) = 0$, the restriction $\mathcal{F}(X) \to \mathcal{F}(\eta)$ being the zero map. As the only nonempty open subsets of X are η and X itself, these data indeed define an $\mathcal{O}_X$-module whose A-module of global sections is A. But this $\mathcal{O}_X$-module is not isomorphic to $\tilde{A}$ as the stalks at η are different.

Definition 4.6 Let X be a scheme. A *quasi-coherent sheaf* on X is an $\mathcal{O}_X$-module $\mathcal{F}$ for which there is an open affine cover $\{U_i : i \in I\}$ of X such that the restriction of $\mathcal{F}$ to each $U_i = \operatorname{Spec} A_i$ is isomorphic to an $\mathcal{O}_{U_i}$-module of the form $\tilde{M}_i$ with some A_i-module M_i. If moreover each M_i is finitely generated over A_i, then $\mathcal{F}$ is called a *coherent sheaf.*

Remark 4.7 It can be shown that for an affine scheme $X = \operatorname{Spec} A$ the functor $M \to \tilde{M}$ establishes an equivalence between the category of A-modules and that of quasi-coherent sheaves; since we shall not need this, we omit the proof and refer the interested reader to Hartshorne [1], Corollary II.5.5.

We now return to the first example in 4.2 and investigate the question of determining whether a morphism $\phi : X \to Y$ yields a quasi-coherent sheaf $\phi_* \mathcal{O}_X$ in Y. Unfortunately, this is not true in general but Section II.5 of Hartshorne [1] contains several sufficient conditions. For our purposes the following easy condition on ϕ will suffice.

Definition 4.8 A morphism $\phi : X \to Y$ of schemes is *affine* if Y has an covering by affine open subsets $U_i = \operatorname{Spec} A_i$ such that for each i the open subscheme $\phi^{-1}(U_i)$ of X is affine as well.

Any morphism of affine schemes is obviously affine. We shall see other examples of affine morphisms in the next chapter.

Lemma 4.9 *If $\phi : X \to Y$ is an affine morphism, then $\phi_* \mathcal{O}_X$ and the ideal sheaf defined by the kernel of $\phi^\sharp$ are quasi-coherent sheaves on Y.*

Proof: Assume first $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine schemes. Then $\phi_* \mathcal{O}_X$ is just $\tilde{B}$ with B regarded as an A-module via the map $\lambda : A \to B$ inducing ϕ. Indeed, it is enough to check this over basic open sets $D(f)$ for which we may argue in the same way as in the second half of the proof of Theorem 2.14. Moreover, a similar reasong shows that the ideal sheaf on Y defined by the kernel of $\phi^\sharp$ is just $\tilde{I}$ with $I = \ker(\lambda)$. Once we have these results at hand, the general case of the lemma follows from the definition of affine morphisms and quasi-coherent sheaves. □

The lemma applies in particular to a closed immersion $i : X \to Y$ of schemes which is affine by definition. Thus to any closed subscheme of Y we may associate a quasi-coherent sheaf of ideals. We conclude this section by proving the converse.

Proposition 4.10 *The above construction gives a bijection between closed subschemes $X \subset Y$ and quasi-coherent sheaves of ideals on Y.*

Proof: Given a quasi-coherent sheaf of ideals $\mathcal{I}$ on Y, we may take a covering of Y by affine open subschemes $U_j = \operatorname{Spec} A_j$ as in the definition of quasi-coherence. Define for each j a closed immersion $i_j : X_j \to U_j$ as the map induced by the projection $A_j \to A_j/I_j$, where I_j is the ideal for which

$\mathcal{I}|_{U_j} \cong \tilde{I}_j$. To see that $X = \bigcup X_j$ is closed in X, note first that $X \cap U_j = X_j$ for all j (look at the restriction of $\mathcal{I}$ to basic open sets contained in the intersections $U_i \cap U_j$). But then any point of $U_j \cap (Y \setminus X)$ has an open neighbourhood contained in $U_j \setminus (X \cap U_j)$, whence the claim. Finally, the i_j endow X with the structure of a closed subscheme. It is manifest that the two constructions are inverse to each other. □

5. Fibres of a Morphism

We next define the fibre of a morphism of schemes *as a scheme* and not just a point set. Motivated by the situation in the topological setting, we introduce more generally the notion of a *fibre product* of schemes and get the definition of fibres as a special case.

Given topological spaces $Y \to X$, $Z \to X$ over the same space X, their fibre product can be defined as the space representing the functor.

$$S \mapsto \{(\phi, \psi) \in \mathrm{Hom}(S, Y) \times \mathrm{Hom}(S, Z) : p \circ \phi = q \circ \psi\}$$

on the category of topological spaces over X. We can adopt the same definition in the context of schemes if we show that the similarly defined functor on the category of schemes equipped with morphisms to a fixed base scheme X is representable. We first prove representability in the category of affine schemes.

Proposition 5.1 *Assume given affine schemes* $Y = \mathrm{Spec} A$ *and* $Z = \mathrm{Spec} B$ *equipped with morphisms* $p : Y \to X$, $q : Z \to X$ *into an affine scheme* $X = \mathrm{Spec}\, R$. *Then the contravariant functor*

$$S \mapsto \{(\phi, \psi) \in \mathrm{Hom}(S, Y) \times \mathrm{Hom}(S, Z) : p \circ \phi = q \circ \psi\}$$

on the category of affine schemes is representable by $Y \times_X Z := \mathrm{Spec}\,(A \otimes_R B)$.

Proof: Indeed, by Theorem 2.14 the statement of the proposition is equivalent to saying that given ring homomorphisms $\mu : R \to A$ and $\nu : R \to B$, the ring $A \otimes_R B$ represents the functor

$$C \mapsto \{(\kappa, \lambda) \in \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C) : \kappa \circ \mu = \lambda \circ \nu\}$$

on the category of commutative rings with unit. But this is precisely the defining property of the tensor product of R-algebras. Indeed, for $(a, b) \in A \times B$ the map $(a, b) \mapsto \kappa(a)\lambda(b)$ is R-bilinear, hence induces an R-module homomorphism $A \otimes_R B \to C$. When $A \otimes_R B$ is equipped with its ring structure, it is moreover an R-algebra homomorphism. □

Remark 5.2 It is not true that the underlying topological space of a fibre product of affine schemes is the topological fibre product of the underlying topological spaces of the schemes. As an easy example, take $X = \operatorname{Spec} k$ with some field k, $Y = \operatorname{Spec} L$, with $L|k$ a finite separable extension of k, $Z = \operatorname{Spec} \bar{k}$. Then the topological fibre product of Y and Z over X is a one-element set, whereas $L \otimes_k \bar{k}$ is a direct sum of $[L : k]$ copies of $\bar{k}$ and hence its prime spectrum consists of $[L : k]$ points.

To extend this construction to arbitrary schemes, the idea is of course to cover them with open affine subschemes and then to "patch" the fibre products of these affine schemes together. How this can be done precisely is explained next.

Construction 5.3 Assume given a family of schemes $\{X_i : i \in I\}$ and for each $(i,j) \in I^2$ an open subscheme $U_{ij} \subset X_i$ such that $U_{ii} = X_i$. Assume moreover we are given isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$ satisfying the *cocycle condition*

$$\phi_{jk} \circ \phi_{ij} = \phi_{ik}$$

on $U_{ij} \cap U_{ik}$ for all $i, j.k$ (here we tacitly assume that $\phi_{ij}(U_{ij} \cap U_{ik}) \subset U_{jk}$). Note that the cocycle condition for $i = j = k$ implies that ϕ_{ii} is the identity and then for $i = k$ that $\phi_{ji} = \phi_{ij}^{-1}$.

We now construct a scheme X having an open covering $\{U_i : i \in I\}$ such that each U_i is isomorphic to X_i as a scheme and via these isomorphisms the U_{ij} correspond to the intersections $U_i \cap U_j$. The above compatibility relations for the ϕ_{ij} are thus necessary conditions for such a scheme X to exist.

Define the underlying set of X to be the disjoint union of those of the X_i modulo the equivalence relation which identifies points of U_{ij} with those of U_{ji} via ϕ_{ij}. The compatibility conditions for the ϕ_{ij} ensure that this is indeed an equivalence relation; we endow X with the quotient topology. The composite maps $p_i : X_i \to \coprod X_i \to X$ map each X_i homeomorphically onto an open subset $U_i \subset X$. Now to define the structure sheaf $\mathcal{O}_X$ of X it suffices by Lemma 2.7 to define its sections over a basis of open sets in X in a compatible fashion. The open sets U which are contained in one of the U_i clearly form a basis. For such a U one is tempted to define $\mathcal{O}_X(U)$ as $\mathcal{O}_{X_i}(p_i^{-1}(U))$, but the problem is that U may be contained in the intersection of several U_i. However, the rings obtained for each choice of U_i are all isomorphic via the ϕ_{ij}, so to remedy this one defines $\mathcal{O}_X(U)$ to be the subring of $\prod_{U \subset U_i} \mathcal{O}_{X_i}(p_i^{-1}(U))$ consisting of sequences of sections mapped to each other by the ϕ_{ij}. More precisely, we take those sequences (s_i) with $\phi_{ij}^\sharp(s_j) = s_i$ for all (i,j) (here s_i is viewed as a section in $\phi_{ij*}(\mathcal{O}_{X_i}|_{U_{ij}})(p_j^{-1}(U))$). One defines restriction maps for subsets $V \subset U$ as induced by the product of the restriction maps of the $\mathcal{O}_{X_i}$; in fact, any element of $\mathcal{O}_X(V)$ is determined by its components indexed by the sets U_i containing U. It is now straightforward to check the sheaf axioms over U as well as the fact that the ringed space thus obtained is a scheme.

Armed with this patching construction, we may now construct fibre products of arbitrary schemes. Just as in topology, let us refer to a morphism $p : Y \to X$ of schemes as a *scheme over* X.

Proposition 5.4 *Given two schemes* $p : Y \to X$, $q : Z \to X$ *over the same scheme* X, *the contravariant functor*

$$S \mapsto F_{YZ}(S) := \{(\phi, \psi) \in \mathrm{Hom}(S, Y) \times \mathrm{Hom}(S, Z) : p \circ \phi = q \circ \psi\}$$

the category of schemes is representable by a scheme $Y \times_X Z$.

The scheme $Y \times_X Z$ is called the *fibre product of* Y *and* Z *over* X. It is equipped with two canonical morphisms into Y and Z making the diagram

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{p} & X \end{array}$$

commute (they correspond to the identity morphism of $Y \times_X Z$).

Proof: We show first that if Y, Z and X are all affine, then the scheme $Y \times_X Z$ defined in Proposition 5.1 is indeed a fibre product *in the category of schemes*. For this we have to see that for an arbitrary scheme S any element of $F_{YZ}(S)$ factors as a composite $(\pi_1, \pi_2) \circ \phi$ with some morphism $\phi : S \to Y \times_S Z$. Choosing an affine open cover $\{S_i : i \in I\}$ of S, for each i we dispose of a morphism $\phi_i : S_i \to Y \times_X Z$ with the above property according to Proposition 5.1. Since by definition for any affine open subset $U \subset S_i \cap S_j$ the elements of $F_{YZ}(U)$ are in bijection with $\mathrm{Hom}(U, Y \times_X Z)$, we see that the restrictions of ϕ_i and ϕ_j to the open subschemes $S_i \cap S_j$ are the same for all (i, j). Hence there is a unique morphism ϕ agreeing with ϕ_i over S_i (the existence of the underlying continuous map is straightforward; for the existence of $\phi^\sharp$ use Lemma 2.7 (3)).

Still assuming X affine, choose affine open coverings $\{Y_i : i \in I\}$ and $\{Z_j : j \in J\}$ of Y and Z, respectively. First fix some $l \in J$. We then dispose of affine shemes $Y_i \times_X Y_l$ for each $i \in I$. Now note that quite generally if $Y \times_X Z$ represents the functor F_{YZ} and $U \subset Y$ is an open subscheme, then the open subscheme $\pi_1^{-1}(U) \subset Y \times_X Z$ represents the functor F_{UZ} and as such is unique up to unique isomorphism by the Yoneda lemma. Applying this remark with Z_l in place of Z, Y_i (resp. Y_j) in place of Y and $Y_i \cap Y_j$ in place of U we see that there exist unique isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$, where U_{ij} (resp. U_{ji}) is the inverse image of $Y_i \cap Y_j$ by the projection $Y_i \times_X Z_l \to Y_i$ (resp. $Y_j \times_X Z_l \to Y_j$). The uniqueness of the ϕ_{ij} implies that the compatibility conditions in Construction 5.3 are satisfied, so we may patch the $Y_i \times_X Z_l$ together along the U_{ij} to obtain a scheme $Y \times_X Y_l$. The projections $Y \times_X Z_l \to Y$ and $Y \times_X Z_l \to Z_l$ are defined by patching the projections from the elements of the open covering $\{Y_i \times_X Z_l : i \in I\}$ of $Y \times_X Z_l$ as in the previous paragraph. To show that $Y \times_X Z_l$ represents F_{YZ_l} one considers for a pair $(\phi, \psi) \in F_{YZ_l}(S)$ the restrictions $(\phi|_{\phi^{-1}(Y_i)}, \psi) \in F_{Y_iZ_l}$ and patches the corresponding morphisms $S \to Y_i \times_X Z_l$ together, again arguing as in the previous paragraph.

Now by exactly the same method one shows that the schemes $Y \times_X Z_l$ patch together to give a scheme $Y \times_X Z$ representing F_{YZ}. Finally one extends the construction to arbitary X by choosing a covering of X by affine open subschemes X_k and noting that the open subschemes $Y_k = p^{-1}(X_k)$ form an open covering of Y such that the fibre products $Y_k \times_{X_k} q^{-1}(X_k)$ represent the functors F_{Y_kZ} where the Y_k are viewed as schemes *over* X (indeed, given $(\phi, \psi) \in F_{Y_kZ}(S)$ we must have $\psi(S) \subset q^{-1}(X_k)$), so one may repeat the previous procedure to patch the schemes $Y_k \times_X Z = Y_k \times_{X_k} q^{-1}(X_k)$ together.

□

Now if we imitate the situation in topology, to define the fibre of a morphism $Y \to X$ at some point P of X we first need to define the inclusion morphism $\{P\} \to X$. This is achieved as follows. Take an affine open neighbourhood $U = \operatorname{Spec} A$. Then P is identified with a prime ideal of A and we dispose of a morphism $A \to A_P$ which we may compose with the natural projection $A_P \to A_P/PA_P =: \kappa(P)$. We get a morphism $\operatorname{Spec} \kappa(P) \to U$, whence by composition with the inclusion map $U \to X$ a morphism $i_P : \operatorname{Spec} \kappa(P) \to X$.

Lemma 5.5 *The morphism $i_P : \operatorname{Spec} \kappa(P) \to X$ just defined does not depend on the choice of U.*

Proof: If $V = \operatorname{Spec} B$ is another affine open neighbourhood of P, then there is some affine open subscheme $W \subset V \cap U$. We may assume that W as a subscheme of U is of the form $D(f)$ with some $f \in A \setminus P$. But the localisation map $A \to A_P$ factors through A_f (since f is a unit in A_P), which means that the map $\operatorname{Spec} A_P \to U$ factors through W. By symmetry, we get the same conclusion for V. □

Definition 5.6 Given a morphism $\phi : Y \to X$ and a point P of X, the *fibre of ϕ at P* is the scheme $Y_P := Y \times_X \operatorname{Spec} \kappa(P)$, the fibre product being taken with respect to the maps ϕ and i_P.

We saw in Remark 5.2 that the underlying topological space of a fibre product is not a topological fibre product in general. However, the good news is:

Proposition 5.7 *Given a morphism $\phi : Y \to X$ and a point P of X, the underlying topological space of the fibre Y_P is homeomorphic to the subspace $\phi^{-1}(P)$ of the underlying space of Y.*

Proof: We may assume we are dealing with affine schemes $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. We first show there is a bijection between $\phi^{-1}(P)$ and $\operatorname{Spec} B \otimes_A \kappa(P)$ as sets, the homomorphism $\lambda : A \to B$ defining the A-module structure of B being the one corresponding to ϕ by Theorem 2.14. Now the above λ induces a map $\overline{\lambda} : A/P \to B/\lambda(P)B$ and a point $Q \in \operatorname{Spec} B$ is in $\phi^{-1}(P)$ if and only if its image $\overline{Q}$ in $B/\lambda(P)B$ satisfies $\overline{\lambda}^{-1}(\overline{Q}) = (0)$. This is the same as saying that $\overline{\lambda}(A/P) \cap \overline{Q} = \{0\}$, or else, putting $S = \overline{\lambda}(A/P) \setminus \{0\}$, that $\overline{Q}$ defines a prime ideal of the localisation $(B/\lambda(P)B)_S$. But the latter ring is none but $B \otimes_A \kappa(P)$. To see this, note first the isomorphism $B/\lambda(P)B \cong B \otimes_A (A/P)$ coming from the exact sequence

$$B \otimes_A P \to B \otimes_A A \to B \otimes_A (A/P) \to 0$$

coming from tensoring with B the short exact sequence

$$0 \to P \to A \to A/P \to 0$$

of A-modules. Here we have $B \otimes_A A \cong B$ coming from the multiplication map $b \otimes a \mapsto ba$ and so the image of $B \otimes_A P$ in B is exactly $\lambda(P)B$ (since B is an A-module via λ). Now the natural map $A/P \to B \otimes_A (A/P) = B/\lambda(P)B$ is given by $a \mapsto 1 \otimes a$ and the localisation of $B \otimes_A (A/P)$ by the subset $\{1 \otimes a : a \in (A/P) \setminus \{0\}\}$ is exactly $B \otimes_A \kappa(P)$.

In the above procedure we identified $Y_P = \operatorname{Spec} B \otimes_A \kappa(P)$ with a subset of $\operatorname{Spec} B$; by looking at basic open sets $D(f)$ one sees easily that the topology of Y_P corresponds to the subspace topology. □

6. Special Properties of Schemes

In this section we have assembled some technical notions concerning schemes that are to be used in the sequel. The reader is advised to take a brief glance at it and to come back later if necessary. The first definition is:

Definition 6.1 A scheme X is called *integral* if for all open subsets $U \subset X$ the ring $\mathcal{O}_X(U)$ is an integral domain.

This algebraic notion has a strong consequence for the underlying topological space of the scheme. Namely, call a topological space X *irreducible* if it cannot be written as a union of two closed subsets properly contained in X, or, equivalently, if any two open subsets have a nonempty intersection. Now the basic fact is:

Lemma 6.2 *The underlying topological space of an integral scheme is irreducible.*

Proof: Indeed, if U_1 and U_2 are nonempty disjoint open subsets of a scheme X, then the sheaf axioms imply that $\mathcal{O}_X(U_1 \cup U_2)$ is isomorphic to the direct sum $\mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2)$,which is not an integral domain. □

Remark 6.3 In fact, a scheme is integral if and only if its underlying space is irreducible and if the rings $\mathcal{O}_X(U)$ contain no nilpotent elements. See Hartshorne [1], Proposition II.3.1.

Proposition 6.4 *Let X be an integral scheme.*

1. *There is a unique point $\eta \in X$ whose closure is the underlying space of X.*
2. *The stalk $\mathcal{O}_{X,\eta}$ is a field K which is naturally isomorphic to the fraction field of any local ring of X.*

Proof: We begin with the first statement. For uniqueness, assume η_1, η_2 both have the required property. Then any affine open subset $U = \operatorname{Spec} A$ contains both η_1 and η_2: they correspond to prime ideals P and Q of A with the property $V(P) = V(Q) = U$. Since A is an integral domain, this is only possible for $P = Q = (0)$. This argument also shows the existence of η: indeed, define it as the point corresponding to the ideal (0) of A. Its closure in X contains U, hence it must be the whole of X by the previous lemma. The second statement is obvious from this construction: $\mathcal{O}_{X,\eta}$ is none but the fraction field of A which is the common fraction field of all local rings of U; for the points of $X \setminus U$ we work with other affine open subsets which all have a non-empty intersection with U by irreducibility of X and hence have a local ring in common. □

Definition 6.5 The point η of the proposition is called the *generic point* of X and the field K the *function field* of X.

For an integral scheme X we say that $f \in K$ is *regular* at a point P if $f \in \mathcal{O}_{X,P}$; it *has a zero at* P if it is contained in the maximal ideal of $\mathcal{O}_{X,P}$.

Lemma 6.6 ?? *Let X be an integral scheme The sets of points*

$$D_f = \{P \in X : f \text{ is regular at } P\}$$

and

$$D_f^0 = \{P \in X : f \text{ is regular and does not have a zero at } P\}$$

are open in X.

Proof: Let f be regular at P and take an affine open neighbourhood $U = \operatorname{Spec} A$ of P. We may write $f = x/y$ with $x, y \in A$; thus f is regular in the open neighbourhood $D(y)$ of P, which proves openness of D_f. As for D_f^0, it is the intersection $D_f \cap D_{f^{-1}}$. □

Definition 6.7 An integral closed subscheme of some affine (resp. projective) n-space over a field k is called an *affine (resp. projective)* variety over k.

Remark 6.8 In the literature one often finds a stronger condition imposed on affine and projective varieties X, namely that they should also be *geometrically integral*, which means integrality of the scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ for an algebraic closure $\bar{k}$ of k. But in some texts no integrality condition is required at all.

Next we state a finiteness condition which is always satisfied for affine and projective varieties. Recall that a ring is noetherian if all of its ideals are finitely generated.

Definition 6.9 A scheme X is noetherian if it is compact and has a covering by affine open subschemes of the form Spec A with A a noetherian ring.

Remark 6.10 It can be shown that *any* affine open subset of a noetherian scheme is of the form form Spec A with A a noetherian ring. See Hartshorne [1], Proposition II.3.2.

We next introduce the notion of dimension for schemes. Of course, we would like affine and projective n-space to be n-dimensional, a point to be 0-dimensional, a plane curve 1-dimensional, a surface 2-dimensional, etc. One heuristic approach is the following inductive "argument": a curve should be of dimension 1 because its irreducible proper closed subsets are only points, a surface should have dimension 2 as it contains only curves and points as proper closed subsets etc. This approach is summarised in the following definition.

Definition 6.11 The *dimension* of a scheme X is the supremum of the integers n for which there exists a strictly increasing chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subsets properly contained in X.

Remark 6.12 The dimension is either a positive integer or infinite. It is mainly interesting for noetherian schemes because noetherian rings have no infinite ascending chains of prime ideals. However, there exist noetherian rings whose associated affine scheme has infinite dimension; see Atiyah-Macdonald [1], Exercise 11.4 for an example due to Nagata.

In order to be able to give examples in the affine case, we first prove an easy lemma.

Lemma 6.13 *Let $X = \operatorname{Spec} A$ be an affine scheme. Then any irreducible closed subset of X is of the form $V(P)$, with P a prime ideal of A.*

Proof: Let $Z = V(I)$ be a closed subset of X. We may and do assume that I is the intersection of the prime ideals corresponding to the points of Z. Assume $fg \in I$ for some $f, g \in A$. Then any prime ideal containing I must contain f or g, hence the union of the closed subsets $V(I + (f))$ and $V(I + (g))$ is Z. Therefore Z is irreducible if and only if one of them, say $V(I + (f))$ equals Z. By our assumption on I this is equivalent to $f \in I$, whence the claim. □

By the lemma, the dimension of $\operatorname{Spec} A$ is the supremum of the lengths of chains of prime ideals in A. In ring theory this is called the *Krull dimension of A* and is usually denoted by $\dim A$.

Thus for instance, the Krull dimension of a field is 0, that of $\mathbf{Z}$ is 1. But in general with the above definition the dimension is hard to determine in practice. It is not even clear that affine or projective spaces have the dimension we expect. Fortunately, this can be remedied by means of a criterion for wich we need to recall a definition first.

Definition 6.14 The transcendence degree of a field extension $K|k$ is the maximal number of elements of K algebraically independent over k; the transcendence degree of an integral domain A containing k is defined as the transcendence degree of its fraction field over k and is denoted by $tr.deg_k\, A$.

Now comes the criterion which we only quote from the literature.

Proposition 6.15 *Let k be a field and A an integral domain which is a finitely generated k-algebra. Then the Krull dimension of A is equal to its transcendence degree over k.*

For a proof, see e.g. Matsumura [1], Theorem 5.6.

Example 6.16 As immediate applications of the proposition, we get that $\mathbf{A}_k^n$ and $\mathbf{P}_k^n$ both have dimension n as expected, and that affine and projective plane curves have dimension 1. In general, affine or projective varieties of dimension 1 are called *curves*, those of dimension 1 *surfaces.*