**COMMUTATIVE ALGEBRA**
**Homework 4**

1. Let $A$ be a ring, $I = (x_1, \ldots, x_n)$ and ideal of $A$, and $K(\underline{x})$ the associated Koszul complex. For an $A$-module $M$ show that there is a chain of isomorphisms

$$H_n(K(\underline{x}) \otimes_A M) \cong \{m \in M \mid Im = 0\} \cong \mathrm{Hom}_A(A/I, M).$$

2. Let $A$, $I$ and $K(\underline{x})$ be as in the previous exercise.

*a*) Given an exact sequence of $A$-modules

$$0 \to M' \to M \to M'' \to 0,$$

show that there is a long exact homology sequence

$$\ldots H_i(K(\underline{x}) \otimes_A M') \to H_i(K(\underline{x}) \otimes_A M) \to H_i(K(\underline{x}) \otimes_A M'') \to H_{i-1}(K(\underline{x}) \otimes_A M') \to \ldots$$

*b*) If $M$ is an $A$-module and $y_1, \ldots, y_m$ an $M$-regular sequence contained in $I$, show that $H_{n-m}(K(\underline{x}) \otimes_A M) \cong \mathrm{Hom}_A(A/I, M/(y_1, \ldots, y_m)M)$ and moreover $H_i(K(\underline{x}) \otimes_A M) = 0$ for $n - m < i \leq n$.

[*Hint*: Proceed by induction on $m$ using Exercise 1 and part *a*) applied to $M'' = M/y_1M$.]

3. Let $A$ be a Noetherian local ring, $x_1, \ldots, x_n$ a sequence of elements contained in the maximal ideal $P$, and $M$ a finitely generated $A$-module.

*a*) Show that $H_i(K(x_1, \ldots, x_n) \otimes_A M) = 0$ implies $H_i(K(x_1, \ldots, x_{n-1}) \otimes_A M) = 0$ for all $i \geq 0$.

*b*) Write $d$ for the depth of $M$. Verify that $H_i(K(x_1, \ldots, x_n) \otimes_A M) \neq 0$ for $i \leq n - d$.

[*Remark:* Note that $H_i(K(x_1, \ldots, x_n) \otimes_A M) = 0$ for $i > n - d$ by the previous exercise; this gives a characterization of $d$ via the Koszul complex.]

4. Let $A$ be a regular local ring of dimension $n$ and residue field $k$. Establish isomorphisms $\mathrm{Tor}_i^A(k, k) \cong \Lambda^i k^n$ for all $i \geq 1$.

5. This exercise gives another proof of the Hilbert syzygy theorem.
Let $k$ be a field, $A = k[x_1, \ldots, x_n]$, $B = A[y_1, \ldots, y_n]$ polynomial rings. Set $t_i := y_i - x_i$ and view $A$ as a $B$-module via $A \cong B/(t_1, \ldots, t_n)$. Finally, set $A' := k[y_1, \ldots, y_n]$; it is a subring of $B$.

*a*) Show that the Koszul complex $K(t_1, \ldots, t_n)$ gives a free $B$-module resolution of $A$.

*b*) Given an $A$-module $M$, show that $K(t_1, \ldots, t_n) \otimes_B M$ gives a free $A'$-module resolution of $M$.

*c*) Verify that for an $A$-module $N$ the $A$-module structure of $N$ may be identified with its $A'$-module structure inherited from the $B$-structure.

*d*) Conclude that the global dimension of $A = k[x_1, \ldots, x_n]$ is at most $n$.