Conjectures and Results on the Multivariate Bernstein Inequality on Convex Bodies

Sz. Révész

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How to estimate the derivative of a multivariate polynomial?

$P : \mathbb{R}^n \to \mathbb{R}$, $\deg P = n$, gradient at x and directional derivative at $\mathbf{x}$ in direction $\mathbf{y}$ are

$$[DP](\mathbf{x}) := DP(\mathbf{x}) := \left(\ldots, \frac{\partial P}{\partial x_j}, \ldots \right), \qquad D_{\mathbf{y}} P(\mathbf{x}) := \langle DP(\mathbf{x}); \mathbf{y} \rangle.$$

In univariate case, $|Dp(x)| \leq n \cdot \sqrt{\|p\|^2 - p(x)^2} \cdot \ \omega(x) \qquad \omega(x) := \frac{1}{\sqrt{1-x^2}}$.

Observe the nice separation of effects of the *degree*, of the *polynomial values* (at x and at other points of a set), and the *location* of the point x (in relation with the *geometry* of the set considered).

Bernstein-Szegő normalization: $B_n(P; x) := n \cdot \sqrt{\|P\|^2 - P(x)^2}$; we want to determine / estimate DP and $D_y P$ normalized, i.e. $GP(\mathbf{x}) := DP(\mathbf{x}) / B_n(P; \mathbf{x})$.

One can also consider (bounded) polynomials on (real) normed spaces X and on their complexifications $Y = X + iX$. For this talk it is safe to read "$X = \mathbb{R}^d$" and "$Y = \mathbb{C}^d$".

Two possible means of estimating $GP(\mathbf{x})$: absolute value, i.e. absolute maximum over all directions $\mathbf{y}$, that is, the norm of it as a linear functional acting on the space of directions – or in the average, say volume.

Let now $\mathcal{G}(\mathbf{x}) := \mathrm{con}\,\{GP(\mathbf{x}) : P \in \mathcal{P}\}$ be the (convex hull of the) set of all (normalized) gradient vectors. We want to find

(i) the value of $\sup_{\mathbf{v}\in\mathcal{G}(\mathbf{x})} \|\mathbf{v}\| = \sup_{P\in\mathcal{P}_n(X)} \|GP(\mathbf{x})\|$, where $\|GP(\mathbf{x})\| := \sup_{\mathbf{y}\in S_X^*} |\langle GP(\mathbf{x}); \mathbf{y}\rangle|$;

(ii) the value of $\mathrm{vol}\,\mathcal{G}(\mathbf{x}) = \mathrm{vol}\,\{\mathrm{con}\,\{GP(\mathbf{x}) : P \in \mathcal{P}_n(X)\}\}$. (This makes sense only in finite dimensional spaces, i.e. if $X = \mathbb{R}^d$.)

Pluripotential theory approach

Here the key object of study is the *Siciak-Zaharjuta extremal function* of an arbitrary (compact) set $E \subset Y$ (or: $\mathbb{C}^d$), defined as

$$V_E(z) := \sup\left\{\frac{1}{n}\log|p(z)| : 0 \neq p \in \mathcal{P}_n(Y),\ ||p||_E \leq 1,\ n \in \mathbb{N}\right\} \quad (z \in Y) \quad (1)$$

Note that $V_E(y) = 0$ for $y \in Y$. V_E is analogous to the one-dimensional Green function. If $E \subset X$ is real, it suffices to take here real polynomials, i.e. $\mathcal{P}_n(X)$.

In general, V_E may not be continuous, and then one takes the *upper regularization* V_E^* defined as $V_E^*(z) := \limsup_{w \to z} V_E(w)$.

For a nice set – surely for convex bodies – $V_E = V_E^*$.

This V_E^* constitutes a *plurisubharmonic function* (PSH function), that is an upper semicontinuous function with subharmonic restrictions on each "complex lines" (1 dimensional complex subspaces). It follows from the fact that $\log|p(z)|$ is subharmonic on complex lines.

The extension of the Laplace- and Poisson equations is the so-called complex Monge-Ampère equation:

$$(\partial\overline{\partial}u)^d := d!4^d \mathrm{det}\left[\frac{\partial^2 u}{\partial z_j \overline{\partial} z_k}(z)\right] dV(z), \tag{2}$$

where $dV(z) = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_d \wedge dy_d$ is just the usual volume element in $\mathbb{C}^d$. Recall that for a complex variable $z = x + iy$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial z}\right), \qquad \frac{\partial}{\overline{\partial} z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Note also that in one complex variable $4\frac{\partial^2 u}{\partial z \overline{\partial} z} = \Delta u$, and that $f = u + iv$ is regular iff $\frac{\partial}{\overline{\partial} z} f = 0$ (Cauchy-Riemann equations for $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$).

At first the Monge-Ampère equation (2) is applied only to smooth functions $u \in PSH \cap C^2$, say, but due to the work of Bedford and Taylor, the operator extends, in the appropriate sense, even to the whole set of locally bounded PSH functions (which covers the case of V_E^* for any non-pluripolar E).

Therefore, it makes sense to consider

$$(\partial\overline{\partial}V_E^*)^d \ , \tag{3}$$

which is then a compactly supported measure λ_E and is called the *complex equilibrium measure* of the set E.

It is shown in the theory that in fact the support lies in the polynomial convex hull $\widehat{E}$ of E: in case E is convex, $\widehat{E} = E$ and $V_E^* = V_E$: moreover, it is also shown that this measure is kind of normalized, as $\lambda|_E(\mathbb{C}^d) = \lambda|_E(\widehat{E}) = (2\pi)^d$.

For a not necessarily differentiable function F we can consider the *lower (semi)derivative*

$$D_{\mathbf{y}}^+ F(\mathbf{a}) := \liminf_{t\to 0^+} \frac{F(\mathbf{a}+t\mathbf{y}) - F(\mathbf{a})}{t}$$

Theorem 1 (Baran, 1994 & 2004). *Let $E \subset X$ be any bounded, closed set, $x \in \operatorname{int} E$ and $0 \neq y \in X$. Then for all $p \in \mathcal{P}_n(X)$ we have*

$$|\langle Dp(x), y\rangle| \leq n D_y^+ V_E(x) \sqrt{||p||_E^2 - p(x)^2}. \tag{4}$$

In other words

$$|\langle Gp(x); y\rangle| \leq D_y^+ V_E(x).$$

It is a bit disturbing that we don't have $Gp(x)$, but only its inner product with the directional vectors y. To reformulate, we use standard convex geometry: the *polar*, or *dual* of a set $E \subset X$ is defined as

$$E^\star := \{y \in X^* : |\langle u, y\rangle| \leq 1 \ (\forall u \in E)\}.$$

So the result reads as $\mathcal{G}(x)^\star \supset \{y \ : \ D_y^+ V_E(x) \leq 1\}$.

In reflexive (so surely in finite dimensional) spaces $E^{\star\star} = \operatorname{con} E$, whence we also find $\mathcal{G}(x) \subset \{y \ : \ D_y^+ V_E(x) \leq 1\}^\star$.

Theorem 2 (Baran, 1995). *Let E be a compact subset of $\mathbb{R}^d$ with non-empty interior. Then the equilibrium measure $\lambda|_E$ is absolutely continuous in the interior of E with respect to the Lebesgue measure of $\mathbb{R}^d$. Denote its density function by $\lambda(x)$ for all $x \in \operatorname{int} E$. Then we have*

$$\frac{1}{d!}\lambda(x) \geq \operatorname{vol} \mathcal{G}(x)$$

for a.a. $x \in \operatorname{int} E$.

Moreover, if E is a symmetric convex domain of $\mathbb{R}^d$, then we have $\frac{1}{d!}\lambda(x) = \operatorname{vol} \mathcal{G}(x)$ for a.a. $x \in \operatorname{int} E$.

Conjecture 3. (Baran, 1995). *We have*

$$\frac{1}{d!}\lambda(x) = \operatorname{vol} \mathcal{G}(x)$$

even if E is a non-symmetric convex body in $\mathbb{R}^d$.

Proposition 4 (Baran, 1995). *The extremal function of the standard simplex Δ in $\mathbb{R}^d$ is*

$$V_\Delta(z) = \log |h(|z_1| + |z_2| + \cdots + |z_n| + |1 - (z_1 + z_2 + \cdots + z_n)|)| \,.$$

Here $h(z) := z + \sqrt{z^2 - 1}$ is inverse to the Joukowski map $\zeta \to (1/2)(\zeta + 1/\zeta)$, with that choice of the square-root that it is positive for positive z exceeding 1, so that h maps to the exterior of the unit disk.

From this we can calculate

Proposition 5. *For the standard simplex Δ of $\mathbb{R}^d$ and with any unit directional vector $y = (y_1, \ldots, y_n)$ and any point $x = (x_1, \ldots, x_n) \in \operatorname{int} \Delta$ we have the formula*

$$D_y^+ V_\Delta(x) = \sqrt{\frac{y_1^2}{x_1} + \cdots + \frac{y_n^2}{x_n} + \frac{(y_1 + \ldots y_n)^2}{(1 - (x_1 + \cdots + x_n))}} \,. \tag{5}$$

The inscribed ellipse method of Sarantopoulos

The *method of inscribed ellipses* was introduced into the subject by Y. Sarantopoulos. This works for arbitrary interior points of any sets (e.g. a possibly nonsymmetric convex body). The key of all is the next

Lemma 6 (**Inscribed Ellipse Lemma, Sarantopoulos, 1991).** *Let K be any subset in a vector space X. Suppose that $x \in K$ and the ellipse*

$$\mathbf{r}(t) := \mathbf{r}_{\mathbf{a},\mathbf{x},\mathbf{y};b}(t) := \cos t\ \mathbf{a} + b \sin t\ \mathbf{y} + \mathbf{x} - \mathbf{a} \qquad (t \in [-\pi, \pi))\,. \tag{6}$$

lies inside K. Then we have for any polynomial $p \in \mathcal{P}_n(X)$ the Bernstein type inequality

$$|\langle Dp(x), y\rangle| \le \frac{n}{b}\sqrt{||p||^2_{C(K)} - p^2(x)}. \tag{7}$$

Proof. Let $T(t) := p(\mathbf{r}(t))$.

Observe $T \in \mathcal{T}_n([-\pi, \pi))$. Indeed, we must substitute in

$$p(X) = \sum_{\substack{\mathbf{k}=k_1,\ldots,k_d \\ k_1+\ldots k_d \le n}} a_{\mathbf{k}} X_1^{k_1} \cdots X_d^{k_d}$$

the values $X_j = r_j(t) = a_j \cos t + by_j \sin t + x_j - a_j \quad (j = 1, \ldots d)$.

Apply the classical Bernstein-Szegő inequality: $\quad |T'(t)| \le n\sqrt{\|T\|^2 - T^2(t)}$.

Obviously $\|T\| \le \|p\|_{C(K)}$ whenever the ellipse (6) fully lies in K.

Now the chain rule yields

$$T'(t) = \langle Dp(\mathbf{r}(t)), \dot{\mathbf{r}}(t) \rangle = \langle Dp(\mathbf{r}(t)), -\sin t\mathbf{a} + b \cos t\mathbf{y} \rangle.$$

Substituting $t = 0$ and noting $T(0) = p(\mathbf{r}(0)) = p(x)$ leads to (7). $\square$

Theorem 7 (Sarantopoulos, 1991). *Let p be any polynomial of degree at most n over the normed space X. Denote $S := S_X$ the unit sphere of X. Then we have for any unit vector $y \in S_X$ the Bernstein type inequality*

$$|\langle Dp(x), y\rangle| \leq \frac{n\sqrt{||p||^2 - p^2(x)}}{\sqrt{1 - \|x\|^2}}. \tag{8}$$

Equivalently,

$$\|Dp(x)\| \leq \frac{n\sqrt{||p||^2 - p^2(x)}}{\sqrt{1 - \|x\|^2}}.$$

What are these many norms? It's confusing ... Well,

$$\|Dp(x)\|_{X^\star} \leq \frac{n\sqrt{||p||^2_{C(B_X)} - p^2(x)}}{\sqrt{1 - \|x\|^2_X}}.$$

Proof. Take $\mathbf{r}(t) := \cos t\, x + \sqrt{1 - \|x\|^2} \sin t\, y$. To check $\mathbf{r} \subset B_X$ use Cauchy-Schwarz ($\sim$ -Chebyshev-Bunyakovskii). (This ellipse is optimal...) $\square$

Theorem 8 (**Sarantopoulos, 1991).** *Let K be a symmetric convex body and y a unit vector in the normed space X. Let p be any polynomial of degree at most n. We have*

$$|\langle Dp(x), y\rangle| \leq \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{\tau(K,y)\cdot\sqrt{1-\varphi^2(K,x)}} \ ,$$

where $\varphi(K,x)$ is the Minkowski functional (norm) with respect to K, and $\tau(K,y)$ is the maximal chord in direction of y. *In particular, we have*

$$\|Dp(x)\| \leq \frac{2n\sqrt{\|p\,\|^2_{C(K)} - p^2(x)}}{w(K)\cdot\sqrt{1-\varphi^2(K,x)}} \ ,$$

where $w(K)$ stands for the minimal width *of K.*

This is quite satisfactory as regards symmetric convex bodies. What to do with non-symmetric convex bodies?

Generalized Minkowski functional.

First let

$$\gamma(K,x) := \inf\left\{2\frac{\sqrt{||x-a||\ ||x-b||}}{||a-b||} : a,b \in \partial K, \text{s.t.} \quad x \in [a,b]\right\}. \tag{9}$$

Then we can set

$$\alpha(K,x) := \sqrt{1-\gamma^2(K,x)}. \tag{10}$$

When $K = -K$, we have $\alpha(K,x) = \varphi(K,x)$, so it is indeed a generalization of the usual Minkowski gauge – norm.

Also, $\alpha(K,x)$ can be introduced the following manner. We define $\alpha(K,x) := \min\{\lambda \ : \ x \in K_\lambda\}$, where $K_\lambda := \cap_{y \in S_{X^*}} S(K,y,\lambda)$ with $S(K,y,\lambda)$ being the λ-homothetic copy of the minimal strip, orthogonal to $y \in S_{X^*}$ and covering K. That is, $S(K,y,\lambda)$ is defined by

$$S(K,y,\lambda) := \{z \in X \ : \ (1+\lambda) \inf_{x \in K} \langle y,x \rangle + (1-\lambda) \sup_{x \in K} \langle y,x \rangle \\ \leq 2\langle y,z \rangle \leq (1-\lambda) \inf_{x \in K} \langle y,x \rangle + (1+\lambda) \sup_{x \in K} \langle y,x \rangle\}.$$

There are many equivalent definitions of the generalized Minkowski functional, (some of them due to Minkowski himself). For more see Sz. Révész and Y. Sarantopoulos, A generalized Minkowski functional with applications in approximation theory, *J. Convex Analysis* **11** (2004), no. 2, 303-334.

Theorem 9 (Kroó–Révész, [9], 1998). *Let K be an arbitrary convex body, $x \in \mathrm{int}K$ and $\|y\| = 1$, where X can be an arbitrary normed space. Then we have*

$$|\langle Dp(x), y\rangle| \leq \frac{2n\sqrt{||p||^2_{C(K)} - p^2(x)}}{\tau(K,y)\sqrt{1-\alpha(K,x)}}, \tag{11}$$

for any polynomial p of degree at most n. Moreover, we also have

$$||Dp(x)|| \leq \frac{2n\sqrt{||p||^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha(K,x)}} \leq \frac{2\sqrt{2}\cdot n\sqrt{||p||^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2\,(K,x)}}. \tag{12}$$

Note that in [9] the best ellipse (the one with maximal possible b parameter) is not found; for most cases, the construction there gives only a good estimate, but not an exact value.

On the other hand the estimate is within a constant $\sqrt{2}$ factor of the truth.

Note that the generalized Minkowski functional was found to be the exact thing to consider for the *Chebyshev problem of polynomial growth* in the multivariate setting. So it was natural to pose

Conjecture 10. (Révész–Sarantopoulos, 2001). *Let X be a topological vector space, and K be a convex body in X. For every point $x \in \mathrm{int}K$ and every (bounded) polynomial p of degree at most n over X we have*

$$\|Dp(x)\| \leq 2\frac{n\sqrt{||p||^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2(K,x)}},$$

where $w(K)$ stands for the minimal width of K.

But a few years later, calculating a certain concrete case, I found the following surprising fact.

Proposition 11 (2005)**.** *Either Conjecture 3, or Conjecture 10 must fail.*

So I became suspicious about my own conjecture and proposed the counter-conjecture to Nikola Naidenov.

Theorem 12 (Nikola Naidenov, 2006)**.** *Conjecture 10 is false!*

More precisely, N. Naidenov exhibited counterexamples with e.g.

$$|Dp(x)| \approx 2.015... \frac{n\sqrt{||p||^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2(K,x)}}.$$

How to find them?

Naidenov wrote: "The counterexample was found by a computer...".

Remark 13. *Somehow, my computer did not find it...*

Well, whatever is the situation with the generalized Minkowski functional, but there has to be a best inscribed ellipse (with largest b parameter), and that gives an estimate, which is perhaps sharp. But what is this best b? We have no good ways to compute this value, which may be called *the exact yield of the inscribed ellipse method*.

We tried to do this with an instructive, important example – the standard simplex Δ of $\mathbb{R}^d$.

So let $E(\Delta, x, y)$ be defined as the maximal possible b parameter of any inscribed ellipse in Δ, passing through x and tangent to y at x.

Theorem 14 (Milev-Révész, 2003). $E(\Delta, y, x)$ *has the precise value*

$$E(\Delta, x, y) = \left\{ \frac{y_1^2}{x_1} + \cdots + \frac{y_d^2}{x_d} + \frac{(y_1 + \ldots + y_d)^2}{1 - x_1 - \ldots - x_d} \right\}^{-1/2}. \tag{13}$$

To get this we wrote out the explicit linear conditions of the ellipse being to the same side of each side polygon then the remaining vertex; the conditions regarding x, y are also linear restraint, furthermore, under these conditions the b-parameter of the ellipse is a linear goal function and via the Kuhn-Tucker Theorem one can solve the maximization problem.

As an immediate corollary, we obtain the best possible estimate for Δ, available via the inscribed ellipse method:

Theorem 15 (Milev-Révész, 2003). *Let $p \in \mathcal{P}_n^d$. Then for every $x \in \mathrm{int}\Delta$ and $y \in \mathbb{S}^{d-1}$ we have*

$$|D_y p(x)| \leq \frac{n\sqrt{||p||^2_{C(\Delta)} - p^2(x)}}{E(\Delta, x, y)}, \tag{14}$$

Hence we are led to the following surprising corollary.

Corollary 16. *The pluripotential theoretical estimate* (4) *of Baran, calculated for the standard simplex of $\mathbb{R}^d$ in* (5)*, gives the result exactly identical to* (14)*, obtained from the inscribed ellipse method.*

Methods translated to the search of extremal ellipses

After some rounds of analysis in several complex variables, Baran's pluripotential theoretic estimates boil down to the search of certain extremal ellipses. We have thus two extremal problems for ellipses.

1.) Given the directions of the major and minor axis, as well as the eccentricity (ratio of axes), find the ellipse with maximal area. We call the corresponding extremal ellipse an a-*maximal ellipse*. This plays a key role in the pluripotential theoretic approach.

2.) Fix $x \in K^o$ and a non-zero vector $y \in \mathbb{R}^n$ and find the b-maximal inscribed ellipse through x, tangent to y at x: $y \in T_x\mathcal{E}$. That is, $\mathcal{E} = \mathcal{E}_b = \mathcal{E}_b(x, y)$ is given by a parametrization

$$\theta \to r(\theta) := a \cos\theta + by \sin\theta + (x - a) \tag{15}$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^+$ are such that $r(\theta) \in K$ for all θ. Note that $r(0) = x$ and $r'(0) = by$; thus one is allowed to vary a and b in (15). Denote the maximum as $b^\star(x, y)$. We will call an ellipse with $b = b^\star$ a *maximal inscribed ellipse* (for x, y) or simply $b-$*maximal*.

Proposition 17. *For any convex body K, a $b-$maximal ellipse $\mathcal{E}$ is also an $a-$maximal ellipse.*

Proof. First observe that an $a-$maximal ellipse $\mathcal{E}$ is characterized by the property that *no translate of $\mathcal{E}$ lies entirely in the interior K^o of K*. For if $\mathcal{E} + v \subset K^o$ for some $v \neq 0$, one can dilate $\mathcal{E} + v$ to get an ellipse with the same orientation and eccentricity as $\mathcal{E}$ which lies in K but has larger area.

Conversely, if $\mathcal{E}$ is not an $a-$maximal ellipse, then one can find an ellipse $\mathcal{E}'$ with the same orientation and eccentricity as $\mathcal{E}$ which lies in K but has larger area. The convex hull H of $\mathcal{E} \cup \mathcal{E}'$ lies in K and we can translate $\mathcal{E}$ within H to an ellipse $\mathcal{E}''$ lying in the two-dimensional surface $S(\mathcal{E}')$ determined by $\mathcal{E}'$; if $\mathcal{E}''$ does not lie in the "interior" of $S(\mathcal{E}')$, we simply translate it within this surface (since the area of $\mathcal{E}'$! is greater than that of $\mathcal{E}''$) until it does.

Moreover: $\mathcal{E}$ is *not* an $a-$maximal ellipse if and only if there is a unit vector v and $\delta > 0$ such that $\mathcal{E} + sv \subset K^o$ for $0 < s < \delta$.

This follows, since if K is a convex body, $u \in K$ and $u + v \in K^o$, then the entire half-open segment $(u, u + v]$ lies in K^o.

Suppose that $\mathcal{E}$ given by $\theta \to a\cos\theta + by\sin\theta + (x - a)$ is a $b-$maximal ellipse for x, y. For the sake of obtaining a contradiction, we assume that $\mathcal{E}$ is *not* an $a-$maximal ellipse. So, $\exists v \neq 0$ and $\delta > 0$ so that $\mathcal{E}_s := \mathcal{E} + sv \subset K^o$ for $0 < s < \delta$.

For $0 < \epsilon < \delta/2$, consider the ellipse $\tilde{\mathcal{E}}(\epsilon)$ given by

$$\begin{aligned} r_\epsilon(\theta) &= (a - \epsilon v)\cos\theta + by\sin\theta + x - (a - \epsilon v) \\ &= a\cos\theta + by\sin\theta + (x - a) + \epsilon v(1 - \cos\theta). \end{aligned} \tag{16}$$

Observe that the point $r_\epsilon(\theta) \in \tilde{\mathcal{E}}(\epsilon)$ lies on the ellipse $\mathcal{E}_{s_\theta} := \mathcal{E} + \epsilon(1 - \cos\theta)v$ where $s_\theta = \epsilon(1 - \cos\theta) \leq 2\epsilon < \delta$. Thus $\tilde{\mathcal{E}}(\epsilon) \subset K^o$.

Note that $r_\epsilon(0) = x \in \tilde{\mathcal{E}}(\epsilon)$ and $r'_\epsilon(0) = by$; in particular, the "b" for $\tilde{\mathcal{E}}(\epsilon)$ is the same as the "b" for $\mathcal{E}$.

Since $\tilde{\mathcal{E}}(\epsilon) \subset K^o$, dilation at x yields a larger b, so $\mathcal{E}$ was not a $b-$maximal ellipse for x, y. Contradiction: whence $\mathcal{E}$ must have been a-maximal, too. $\square$

A unifying result.

For any compact set $K \subset \mathbb{R}^n$ with non-empty interior, take $x \in K^o$ and $y \in \mathbb{R}^n \setminus \{0\}$. According to the above, we have the pointwise inequalities

$$\delta_M(x,y) := \sup_{\mathbf{0} \neq P \in \mathcal{P}_n} \langle GP(x), y \rangle \leq \begin{cases} \delta_B(x,y) := \liminf\limits_{t \to 0^+} \frac{V_K(x+ity)}{t} \\ \delta_S(x,y) := \frac{1}{b^*(x,y)} \end{cases} . \qquad (17)$$

These quantities are so-called *metrics* /Finsler metrics/ in the differential geometry sense. The assertion that e.g. the inscribed ellipse method is optimal, that is it gives sharp estimates, would be to say that $\delta_M(x,y) = \delta_S(x,y)$. The above corollary says that $\delta_S(x,y) = \delta_B(x,y)$ in case of Δ. Realizing this in 2005 I started to conjecture that it may hold in general.

Theorem 18 (Burns, Levenberg, Ma'u, R, 2007)**.** *Let K be a convex body in $\mathbb{R}^n$. Then the limit in the definition of the directional derivative (in the imaginary direction) of V_K exists and equals $\frac{1}{b^*(x,y)}$:*

$$\delta_B(x,y) := \lim_{t\to 0^+} \frac{V_K(x+ity)}{t} = \frac{1}{b^*(x,y)} =: \delta_S(x,y). \tag{18}$$

So, a posteriori, we find even

Corollary 19. *For any convex body K, an ellipse $\mathcal{E} \subset K$ is $a-$maximal if and only if it is $b-$maximal for all $x \in \mathcal{E} \cap K^o$ and $y \in T_x\mathcal{E}$.*

Now the natural question is if these equivalent methods lead to the best possible result, i.e. if $\delta_M(x,y) = \delta_S(x,y) = \delta_B(x,y)$?

I still don't know of any direct, geometrical proof of this fact, although the assertion is purely geometrical.

What is missing, is the lower estimation. We look for polynomials with large gradient vectors. To find the largest, we should consider all polynomials of a given degree n on $\mathbb{R}^d$, i.e. of d variables, and compute $DP(x)$ and $GP(x)$.

Work in progress with Nikola Naidenov and Lozko Milev from Sofia.

Idea: Perhaps, $\|DP(x)\|$, or even $|\langle DP(x), y\rangle|$ are convex functionals on the space of polynomials $\mathcal{P}_n(\mathbb{R}^d)$. Then a Krein-Milman approach – computing extreme points of the unit ball of the space of polynomials equipped with the maximum norm on our convex body K – can reduce the necessary calculations to find the exact extrema.

Difficulties:

(i) The normalization by the degree is fine, but $\sqrt{\|P\|^2 - P(x)^2}$ is ugly. It seems to spoil the convexity of the functional to be maximized.

Answer: yes, but we may as well consider $\|DP(x)\|^2$ and thus we find a convex formulation.

(ii) Normalizing with respect to the maximum norm on Δ determines a rather complicated type of norm, expressed in function of the coefficients as coordinates. E.g.: Muñoz-Seouane-R, 2009 – description of *homogeneous* degree 2 polynomials with this norm (description of the unit ball) is already quite complicated.

(iii) The number of parameters blow up with the degree. $\mathcal{P}_n^d$ is a $\binom{n+d-1}{d}$-dimensional vector space in the coefficients as variables. Degree 1 is linear – nothing can be seen as DP is a constant. Degree is, even for $d = 2$, a space of dimension 6. degree 3 is already of dimension 10, seems impossible to deal with.

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