Ninth Summer School on Potential Theory
Nyíregyháza, Hungary, July 26-31, 2010

# Asymptotically minimal polynomials and tuples

Béla Nagy
University of Szeged

## 1 Notations

$K \subset \mathbf{C}$ compact set.
Denote $Pc(K)$ the polynomially convex hull of $K$.
The set $K$ is polynomially convex, if $Pc(K) = K$ , in other words, $K$ "has no holes".
Outer domain of $K$ is denoted by $\Omega = \Omega(K)$ and $\Omega := \mathbf{C} \setminus Pc(K)$ or $\Omega := \mathbf{C}_\infty \setminus Pc(K)$, unambiguous from context.
Logarithmic energy of a measure $\mu$ is $I(\mu) = \int\int -\log|z-w| d\mu(z) d\mu(w)$.
Equilibrium measure of $K$ is denoted by $\nu_K$. $cap(K) := \exp(-I(\nu_K))$.

Green's function of $K$ : $g_K(z) := g_{\mathbf{C}_\infty \setminus K}(z, \infty)$. Defining properties:

- $g_K(z)$ is subharmonic on the whole $\mathbf{C}$,
- $g_K(z)$ is harmonic on $\Omega(K)$,
- $g_K(z) \to 0$ as $z \to z_0$ for (quasi) every $z_0 \in \partial Pc(K)$,
- $g_K(z) - \log|z|$ is bounded as $z \to \infty$.

Or, $g_K(z) = \int \log|z-w| d\nu_K(z) + I(\nu_K)$.

Sweeping out or balayage of a measure $\mu$ from $K$:
$K$ compact set (consisting of finitely many components), $\mu$ Borel measure, $\mu \geq 0$, $supp(\mu) \subset K$. Then there exists a unique measure $\hat{\mu}$ such that their potential, $U^\mu(z), U^{\hat{\mu}}(z)$ agree on $\Omega(K)$. Some properties: $supp(\hat{\mu}) \subset \partial\Omega$, $\hat{\mu}(intK) = 0$.

## 2 Chebyshev polynomials

**Definition.** *Let $T_n = T_n(K; z)$ be the unique polynomial with degree $n$ such that its leading coefficient is 1 (monic) and its sup norm over $K$, $||T_n||_{\infty,K}$ is smallest. $T_n$ is called Chebyshev polynomial of $K$ with degree $n$.*

**Proposition** ([11], Ch. III, Th. 3.1)**.**

$$\lim_{n\to\infty} ||T_n(K;.)||_{\infty,K}^{1/n} = cap(K).$$

**Proposition** ([11], Ch. III, Th. 3.4)**.** *Let $K$ be a compact set, $T_n$ denote its Chebyshev polynomial with degree $n$. Then the zeros of $T_n$ lie in the (linear) convex hull of $K$.*

**Example.** *Zeros can be outside $K$: $K := \{|z| = 1\}$. Then $T_n(z) = z^n$. In this case, zeros remain in $Pc(K) = \{|z| \leq 1\}$.*

**Example.** *Zeros may stay outside $Pc(K)$: Fix $0 < \alpha < \pi/2$ and let $K := \{\exp(it) : \pi/2 - \alpha \leq t \leq \pi/2 + \alpha$ or $3\pi/2 - \alpha \leq t \leq 3\pi/2 + \alpha\}$. Then $K$ is polynomially convex, $K = \overline{K}$ where now $\overline{K} = \{\overline{z} : z \in K\}$. So $T_n$ must have real coefficients and must have a real zero.*

**Theorem** ([11], Ch. III, Th. 3.4)**.** *Let $K$ be a compact set, $T_n$ denote its Chebyshev polynomial with degree $n$. Then, for every $K_1 \subset \mathbf{C}$ compact set, $K_1 \cap K = \emptyset$, there exists $m \in \mathbf{N}$ suct that for every $n \in \mathbf{N}$, we know that $|\{z \in K_1 : T_n(z) = 0\}| \leq m$. In other words, on every compact set disjoint from $K$ there can be only boundedly many zeros.*

# 3 Fekete polynomials

**Definition.** *Let $K \subset \mathbf{C}$ compact. Consider for all $n \in \mathbf{N}$ the quantity*

$$(w_1, \ldots, w_n) \mapsto \Big( \prod_{1 \le j < k \le n} |w_j - w_k| \Big)^{\frac{1}{n(n-1)}}.$$

*It has a finite maximum on $K \times \ldots \times K = K^n$, denote it by $\delta_n(K)$. If it attains its maximum at $z_1, \ldots, z_n \in K$, then $z_1, \ldots, z_n$ is called a Fekete $n$-tuple or Fekete points of $K$.*

**Definition.** *$K \subset \mathbf{C}$ compact, $z_1, \ldots, z_n$ Fekete points. Then $p(z) = (z - z_1) \ldots (z - z_n)$ is called a Fekete polynomial.*

**Proposition** (Fekete-Szegő, [10], Th. 5.5.2 or [11], Ch. III, Th. 1.3)**.**

$$\lim_{n \to \infty} \delta_n(K) = cap(K).$$

**Proposition.** *If $z_1, \ldots, z_n$ are Fekete points for $K$, then $z_1, \ldots, z_n \in \partial Pc(K)$.*

**Theorem** (Bernstein-Walsh, [10], Th. 5.5.7.a))**.** *$K \subset \mathbf{C}$ compact, $cap(K) > 0$, $\Omega$ is its outer domain, $P$ is a polynomial of degree $n$. Then*

$$\Big( \frac{|P(z)|}{||P||_K} \Big)^{1/n} \le \exp\big(g_\Omega(z)\big).$$

This is sharp.

**Theorem** ([10], Th. 5.5.7.b))**.** *$K \subset \mathbf{C}$ compact, $cap(K) > 0$, $\Omega$ is its outer domain. Let $Q(z)$ be a Fekete polynomial for $K$ with degree $n$. Then for all $z \in \Omega$*

$$\exp\big(g_\Omega(z)\big) \Big( \frac{cap(K)}{\delta_n(K)} \Big)^{\tau_\Omega(z,\infty)} \le \Big( \frac{|Q(z)|}{||Q||_K} \Big)^{1/n}$$

*where $\tau_\Omega(z, \infty)$ denotes the Harnack distance for $\Omega$.*

# 4 Asymptotically minimal polynomials

**Lemma** ([11], Ch. III, Th. 3.1, or [10], Th. 5.5.4)**.** *$K \subset \mathbf{C}$ compact. Then for any monic polynomial $p_n(z) = z^n + \ldots$ with degree $n$, we have $||p_n||_{\infty,K}^{1/n} \ge cap(K)$.*

**Definition.** *Let $p_n(z)$, $n \in N \subset \mathbf{N}$ be a sequence of monic polynomials. We say that $\{p_n\}$, $n \in N$ is asymptotically minimal sequence on $K$, or AMIN for short, if $\limsup_{n \to \infty} ||p_n||_{\infty,K}^{1/n} = cap(K)$.*

In other words, they have as small norms (asymptotically) as possible.

**Example.** *Chebyshev or Fekete polynomials are AMIN.*

**Example.** *We can put zeros anywhere: Fix $a \in \mathbf{C}$. If $p_n(z)$ is AMIN on $K$, then $p_n(z)(z - a)$ is again AMIN on $K$.*

Comment: If $K \subset \mathbf{C}$ compact, $\mu$ is a measure, then let $\widehat{\mu} := \widehat{\mu|_K} + \mu|_{\mathbf{C} \setminus K}$ where the first term is the balayage of $\mu|_K$ out of $int(K)$ onto $\partial Pc(K)$.

**Theorem** ([11], Ch. III Th. 4.7 (ii))**.** *Let $\{p_n(z)\}$, $n \in \mathbf{N}$ be AMIN on $K$. Denote $\mu_n$ the normalized zero counting measure of $p_n$, that is, if $p_n(z) = (z - z_1) \ldots (z - z_n)$, then $\mu_n = 1/n \sum \delta_{z_j}$. Then $\widehat{\mu_n} \to \nu_K$ in the weak star sense as $n \to \infty$ where $\widehat{\mu_n}$ is defined as in the comment.*

**Theorem.** *Let $p_n(z)$ be monic polynomials, and consider their normalized zero counting measures. That is, if $p_n(z) = (z - z_1)\dots(z - z_n)$, then $\mu_n = 1/n \sum \delta_{z_j}$. If $\widehat{\mu_n} \to \nu_K$ in the weak star sense as $n \to \infty$ where $\widehat{\mu_n}$ is defined as in the comment, then $p_n$ AMIN on $K$.*

**Theorem** ([11], Ch. III, Th. 4.7 (iv))**.** *$K \subset \mathbf{C}$ compact, $\{p_n\}$, $n \in N \subset \mathbf{N}$ AMIN on $K$. If $U \subset \Omega(K)$ is open, $U^- \cap K = \emptyset$, and $p_n$ have no zeros in $U$ at all, then $|p_n(z)|^{1/n} \to \exp\big(g_\Omega(z)\big)$ locally uniformly on $U$ as $n \to \infty$.*

**Proposition.** *$K \subset \mathbf{C}$ compact, polynomially convex, $z_0 \in \partial K$, $U$ open, $z_0 \in U$. $\{p_n\}$, $n \in N \subset \mathbf{N}$ AMIN on $K$. If $\{p_n\}$ have no zeros in $U$ at all, then $\partial K \cap U$ is an analytic arc.*

So if the boundary is not so smooth, then there must be a zero somewhere close.

**Proposition** (Pommerenke)**.** *$K \subset \mathbf{C}$ compact, connected, $\partial K$ is $C^1$ smooth. Then there exists $\delta > 0$ such that for all Fekete points $z_1, \dots, z_n$, $|z_j - z_k| \le \delta/n$.*

**Proposition** (Korevaar, Monterie 2001, [3], Prop. 3.3)**.** *$K \subset \mathbf{C}$ compact, connected, $z_0 \in \partial K$, $U$ open, $z_0 \in U$. If $g_\Omega$ is Lip$\lambda$, $1/2 \le \lambda \le 1$ on $U$, then there exists $\delta > 0$ such that for all $n$ and for all Fekete $n$-points $z_1, \dots, z_n$, if $z_j, z_k \in U$, then $|z_j - z_k| \ge \delta/n^{1/\lambda}$.*

**Proposition.** *$K \subset \mathbf{C}$ compact, $z_0 \in \partial Pc(K)$, $U$ open, $z_0 \in U$. Assume that $g_\Omega$ is Lip$\lambda$, $1/2 \le \lambda \le 1$ on $U$. Let $p_n$ be AMIN on $K$ such that all zeros lie in $K$. Then, there exists $\delta > 0$ such that for all zeros in $U$, $z_j, z_k \in U$ $|z_j - z_k| \ge \delta/n^{1/\lambda}$.*

# 5 Lemniscates

**Theorem** (Hilbert's lemniscate theorem, [10] Th. 5.5.8 or [11], p. 79)**.** *Let $K \subset \mathbf{C}$ compact, polynomially convex set and $U$ open set, $K \subset U$. Then there exists a polynomial $r$ such that its lemniscate is in between $K$ and $U$. More precisely, $K \subset r^{-1}[D]$ and $r^{-1}[D^-] \subset U$, where $r^{-1}[D] = \{z \in \mathbf{C} : |r(z)| < 1\}$ and $r^{-1}[D^-] = \{z \in \mathbf{C} : |r(z)| \le 1\}$.*

**Definition.** *For $s > 0$ let*

$$\begin{aligned} L_s = L_s(K) &:= \{z \in \mathbf{C} : g_K(z) \le \log(1+s)\} \\ s_n = s_n(K) &:= \inf\{s > 0 : \exists r, \deg(r) = nK \subset r^{-1}[D] \subset L_s(K)\} \\ M_n(K) &:= \inf\{\max\{g(z) : |P(z)| = r\} : deg(P) = n, r > 0, K \subset P^{-1}[|z| < r]\}. \end{aligned}$$

With this notation, Hilbert's lemniscate theorem states that $s_n(K) \to 0$ as $n \to 0$.
Assume $K$ is connected for the rest of this section. So we can use Riemann mapping theorem and so on.

**Theorem** (Andrievskii, 2000, [1], Th. 2)**.** *If $\partial K$ is a Jordan curve of bounded secant variation (e.g. $C^1$ smooth), then for some $c > 0$ and for all $n$, $s_n(K) \le c/n$.*

**Theorem** (Kosukhin, 2005, [4], Th. 1)**.** *Assume that $\partial K = \Gamma$ is a closed Jordan curve. Denote $\psi : \Omega(K) \to \{|z| > 1\}$ the Riemann conformal map ($\psi(\infty) = \infty$ and $\psi'(\infty) > 0$) and $\varphi := \psi^{-1}$ be its inverse. Let $\omega(\varphi, C; \pi/n)$ be the modulus of continuity of $\varphi$ on $C = \{|z| = 1\}$, $\omega(g; d)$ and $\omega(\psi; d)$ be the modulus of continuity on $\Omega$ of $g = g_\Omega(.)$ and $\psi$ respectively. If $d \ge \omega(\varphi, C; \pi/n)$, then*

$$M_n(K) \le \omega(g; d) + \omega\Big(\varphi, C; \frac{\pi}{n}\Big) \int_d^\infty \frac{\min\big(\omega(\psi; t), 2\big)}{t^2} dt.$$

# References


[1] Vladimir Andrievskii, *On the approximation of a continuum by lemniscates*, J. Approx. Theory **105** (2000), no. 2, 292–304. MR MR1775150 (2001g:41027)

[2] J. Korevaar, *Fekete extreme points and related problems*, Approximation theory and function series (Budapest, 1995), Bolyai Soc. Math. Stud., vol. 5, János Bolyai Math. Soc., Budapest, 1996, pp. 35–62. MR MR1432660 (98c:31004)

[3] J. Korevaar and M. A. Monterie, *Fekete potentials and polynomials for continua*, J. Approx. Theory **109** (2001), no. 1, 110–125. MR MR1818072 (2002c:41006)

[4] O. N. Kosukhin, *On the rate of approximation of closed Jordan curves by lemniscates*, Mat. Zametki **77** (2005), no. 6, 861–876. MR MR2246962 (2007j:30028)

[5] T. Kövari and Ch. Pommerenke, *On the distribution of Fekete points*, Mathematika **15** (1968), 70–75. MR MR0231990 (38 #316)

[6] Béla Nagy and Vilmos Totik, *Sharpening of Hilbert's lemniscate theorem*, J. Anal. Math. **96** (2005), 191–223. MR MR2177185 (2006g:30008)

[7] Ch. Pommerenke, *Über die Verteilung der Fekete-Punkte*, Math. Ann. **168** (1967), 111–127. MR MR0206238 (34 #6057)

[8] ———, *Über die Verteilung der Fekete-Punkte. II*, Math. Ann. **179** (1969), 212–218. MR MR0247108 (40 #377)

[9] Christian Pommerenke, *Konforme Abbildung und Fekete-Punkte*, Math. Z. **89** (1965), 422–438. MR MR0181740 (31 #5967)

[10] Thomas Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995. MR MR1334766 (96e:31001)

[11] Edward B. Saff and Vilmos Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 316, Springer-Verlag, Berlin, 1997, Appendix B by Thomas Bloom. MR MR1485778 (99h:31001)

[12] Vilmos Totik, *Polynomial inverse images and polynomial inequalities*, Acta Math. **187** (2001), no. 1, 139–160. MR MR1864632 (2002h:41017)