Bergman function

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1. Definitions and basic properties
   - Basic definitions
   - Orthonormal basis
   - Examples
   - Proper mappings
   - Connection between the Bergman kernels and proper mappings
2. Applications
3. Open problems

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Let D be a domain in $\mathbb{C}^n$. By $L^2_h(D)$ we denote the set of square-integrable holomorphic functions on D, i.e

$$L^2_h(D) := \mathcal{O}(D) \cap L^2(D).$$

$L^2_h(D)$ equipped with the inner product (induced from $L^2(D)$) $\langle f, g \rangle = \int_D f\overline{g} d\mathcal{L}^{2n}$) is a Hilbert space.

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Note that for fixed $z \in D$ the evaluation functional

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is continuous (exercise). Therefore, it follows from the Riesz representation theorem that there is $k_D(\cdot, z) \in L^2_h(D)$ such that

$$f(z) = \langle f, k_D(\cdot, z)\rangle, \quad f \in L^2_h(D).$$

$$(\dagger) \quad f(z) = \langle f, k_D(\cdot, z)\rangle = \int_D f(w)\overline{k_D(w,z)}d\mathcal{L}^{2n}(w).$$

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It suffices to apply the formula $(\dagger)$ to $f = k_D(\cdot, z)$ to get that $k_D(z,z) = \langle k_D(\cdot, z), k_D(\cdot, z)\rangle$.

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Actually $k_D(w,z) = \langle k_D(\cdot,z), k_D(\cdot,w)\rangle = \overline{\langle k_D(\cdot,w), k_D(\cdot,z)\rangle} = \overline{k_D(z,w)}$.

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Thus the Bergman function $k_D(z,w)$ is holomorphic with respect to the first variable and antiholomorphic with respect to the second one. In particular, k_D is a real-analytic positive function on D.

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It is clear that $|f(z)|^2 = |\langle f, k_D(\cdot, z)\rangle|^2 \leq ||f||^2 ||k_D(\cdot, z)||^2 = ||f||^2 k_D(z,z) = ||f||^2 k_D(z)$.

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To show the opposite inequality observe that if $k_D(z) = 0$ then $f(z) = 0$ for any $f \in L^2_h(D)$.

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In the case when $k_D(z) > 0$, a function $f = \frac{k_D(\cdot,z)}{\sqrt{k_D(z)}}$ satisfies the relations $|f(z)|^2 = k_D(z)$ and $||f|| = 1$.

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Theorem

$$k_D(z, w) = \sum_{j\in J} \varphi_j(z)\overline{\varphi_j(w)}.$$

In particular

$$k_D(z) = \sum_{j\in J} |\varphi_j(z)|^2.$$

- If D is a bounded domain then for any $z \in D$ there is a $j \in J$ such that $\varphi_j(z) \neq 0$. Therefore $k_D(z) > 0$. Hence $\log k_D$ is a well defined real-analytic function on D.

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- Note also that if $(\varphi_j)_j$ and $(\psi_k)_k$ are orthonormal bases of $L^2_h(D_1)$ and $L^2_h(D_2)$ respectively, then $(\varphi_j \times \psi_k)_{j,k}$ forms an orthonormal basis for $L^2_h(D_1 \times D_2)$ (here $(f \times g)(z, w) := f(z)g(w)$).

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  Using this simple observation one may obtain the following property

$$k_{D_1 \times D_2}((z_1, z_2), (w_1, w_2)) = k_{D_1}(z_1, w_1) k_{D_2}(z_2, w_2).$$

The formula $k_D(z,w) = \sum_{j \in J} \varphi_j(z)\overline{\varphi_j(w)}$ allows us to compute the Bergman kernels for few examples of domains.

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- Let $\mathbb{B}_n := \{|z_1|^2 + \ldots + |z_n|^2 < 1\}$. It is easy to see that z^α, $\alpha \in (\mathbb{Z}_+)^n$ form an orthogonal basis of $\mathbb{B}_n$. A direct computation shows that $||z^\alpha|| = \sqrt{\frac{\alpha!\pi^n}{(n+|\alpha|)!}}$.

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$$k_{\mathbb{B}_n}(z,w) = \frac{n!}{\pi^n}(1 - \langle z, w\rangle)^{-n+1}, \quad z, w \in \mathbb{B}_n.$$

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- Similarly as before, one can show that

$$k_{\mathbb{D}^n}(z,w) = \prod_{j=1}^{n} k_{\mathbb{D}}(z_j, w_j) = \frac{1}{\pi^n}\frac{1}{(1 - z_j\overline{w_j})^2}.$$

Bergman projection

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$$P_D(f)(z) = \int_D f(w)\overline{k_D(w,z)}d\mathcal{L}^{2n}(w).$$

P_D is called the *Bergman projection*.

Now we describe the behavior of the Bergman kernel under biholomorphic mappings:

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Proposition

Let $F : D \to G$ be a biholomorphic mapping between domains D, $G \subset \mathbb{C}^n$. Then we have

$$k_G(F(z), F(w)) \det F'(z) \overline{\det F'(w)} = k_D(z, w), \quad z, w \in D.$$

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$$0 = \int_D f(w) (\overline{k_D(w,z)} - \overline{k_G(F(w), F(z))} \cdot \overline{\det F'(w)} \cdot \det F'(z)) d\mathcal{L}^{2n}(w).$$

Proper maps

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- f is non-degenerated, i.e. $\det f' \not\equiv 0$;
- (**Remmert's Proper Mapping Theorem**) for any analytic subset A of D, the image $f(A)$ is analytic in G;
- in particular $f(\mathcal{V}_f)$ is a proper analytic subset of G, where $\mathcal{V}_f = \{x \in D : \det[f'(x)] = 0\}$.

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Theorem

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Such an m is called the *order* of f.

Examples

1. $\mathrm{Prop}(\mathbb{D},\mathbb{D})$ consists of the finite Blaschke products (i.e. mappings $B(z) = e^{i\theta}\prod_{j=1}^{m}\frac{z-\lambda_j}{1-z\overline{\lambda_j}}$, where $\lambda_1,\ldots,\lambda_m \in \mathbb{D}$, $\theta \in \mathbb{R}$, $m \in \mathbb{N}$).

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2. $\mathrm{Prop}(\mathbb{D}^n,\mathbb{D}^n) = \{\mathbb{D}^n \ni z \to (B_1(z_{\sigma(1)}),\ldots,B_n(z_{\sigma(n)})) \in \mathbb{D}^n :$ B_i – finite Blaschke product, $\sigma \in \Sigma_n\}$.

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3. (**Alexander theorem** 1980) $\mathrm{Prop}(\mathbb{B}_n,\mathbb{B}_n) = \mathrm{Aut}(\mathbb{B}_n)$, $n \geq 2$.

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Theorem

Suppose D and G are bounded domains in $\mathbb{C}^n$ and that $F : D \to G$ is a proper holomorphic mapping of order m. Let $\Phi_1, \ldots, \Phi_m$ denote the m local inverses to F (defined locally on $G \setminus F(\mathcal{V}_F)$). Then the Bergman kernels transform in the following way:

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$$\sum_{k=1}^{m} k_D(z, \Phi_k(w))\overline{\det \Phi_k'(w)} = \det F'(z) k_G(F(z), w)$$

for all $z \in D$ and $w \in G \setminus F(\mathcal{V}_F)$.

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Theorem

Let $F : D \to G$ be a proper holomorphic mapping between domains D, G in $\mathbb{C}^n$. Then

$$P_D(\det F' \cdot h \circ F) = \det F' \cdot (P_G(h)) \circ F, \quad h \in L^2(D).$$

$$(\dagger) \quad P_D(\det F' \cdot h \circ F) = \det F' \cdot (P_G(h)) \circ F, \quad h \in L^2(G).$$

Sketch of proof: Since P is a projection, $P(\varphi) = \varphi$ for a holomorphic φ. Therefore $(\dagger)$ holds for any $h \in L^2_h(D)$.
Set $\mathcal{V} := F(\mathcal{V}_F) = F(\{\det f' = 0\})$. Integration by parts gives $P_D(\det F' \cdot \frac{\partial g}{\partial w_j} \circ F) = 0$, $1 \leq j \leq n$, for any $g \in \mathcal{C}_0^\infty(G \setminus \mathcal{V})$. So it remains to verify that

$$H := \{\frac{\partial g}{\partial w_j} : \ g \in \mathcal{C}_0^\infty(G \setminus \mathcal{V}), \ 1 \leq j \leq n\}$$

is contained in $(L^2_h(G))^\perp$ and is dense in $(L^2_h(G))^\perp$.
In other words it suffices to show that $(L^2_h(G))^\perp \cap H^\perp = \{0\}$. So take any $h \in (L^2_h(G))^\perp \cap H^\perp$. Then $\int_G h \overline{\frac{\partial g}{\partial w_j}} d\mathcal{L}^{2n} = 0$, $g \in \mathcal{C}_0^\infty(G \setminus \mathcal{V})$. Whence h satisfies the Cauchy-Riemann equations on $G \setminus \mathcal{V}$, so h is holomorphic on $G \setminus \mathcal{V}$. Using this we conclude that h extends holomorphically to the whole G (h is L^2 !), which gives $h = 0$ (as $h \in L^2_h(G)^\perp$).

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Let $m_1, \ldots, m_n \in \mathbb{N}$. A domain $D \subset \mathbb{C}^n$ is said to be a $(m_1, \ldots, m_n)$-circled (shortly *quasi-circled*) if

$$(\lambda^{m_1} x_1, \ldots, \lambda^{m_n} x_n) \in D \quad \text{for any} \quad |\lambda| = 1,\ x = (x_1, \ldots, x_n) \in D. \quad (2)$$

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According to the above definition, circular (balanced) domains are just $(1, \ldots, 1)-$ circular (balanced).
For an $m = (m_1, \ldots, m_n)$-balanced domain D one may define its Minkowski functional in the following way

$$\mu_{D,m}(x) := \inf\{\lambda > 0 :\ (\lambda^{-m_1} x_1, \ldots, \lambda^{-m_n} x_n) \in D\}, \quad x \in \mathbb{C}^n.$$

Properties of the function $\mu_{D,m}$ are similar to the ones of the standard Minkowski functional. Let us recall some of them:

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Example

$$\text{Set} \quad \Phi_j : \mathbb{C}^n \ni (\lambda_1, \ldots, \lambda_n) \mapsto \sum_{1 \le k_1 < \ldots < k_j \le n} \lambda_{k_1} \ldots \lambda_{k_j} \in \mathbb{C} \tag{3}$$

$$\text{and} \quad \Phi := (\Phi_1, \ldots, \Phi_n). \tag{4}$$

Put $\mathbb{G}_n := \Phi(\mathbb{D}^n)$. A domain $\mathbb{G}_n$ is called the **symmetrized polydisc**. It is quite easy to show (exercise) that the symmetrized polydisc is $(1, 2, \ldots, n)$ balanced but it is not even circular.

We say that a domain D has a property $(\star)$ if for any compact K of D there is a neighborhood U of $\overline{D}$ such that the Bergman kernel function $(z, w) \to k_D(z, \overline{w})$ extends holomorphically to $U \times \tilde{K}$, where $\tilde{K} = \{\bar{x} : x \in K\}$.

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Any quasi-balanced domain with the continuous Minkowski functional satisfies $(\star)$.

Actually, fix $\theta \in \mathbb{R}$ and put $\Phi_\theta : D \ni z \to (e^{i\theta m_1} z_1, \ldots, e^{i\theta m_n} z_n) \in D$. It follows from the transformation rule for biholomorphic mappings that

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$k_D((\lambda^{m_1} z_1, \ldots, \lambda^{m_n} z_n), w) = k_D(z, (\overline{\lambda}^{m_1} w_1, \ldots, \overline{\lambda}^{m_n} w_n)), \quad |\lambda| = 1.$

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Putting $D_r = \{\mu_{D,m} < r\}$ we see that the Bergman function extends holomorphically to $D_r \times \tilde{D_{1/r}}$. It remains to observe that the continuity of $\mu_{D,m}$ implies that for any $K \subset\subset D$ there is $r < 1$ such that $K \subset D_r$.

We have the following theorem:

Bell, Kos.

Let D, G be bounded domains in $\mathbb{C}^n$. Suppose that G is $(m_1, \ldots, m_n)$-circled, $0 \in G$ and that D satisfies $(\star)$.

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Corollary

Any proper holomorphic mappings between quasi-balanced domains whose Minkowski functionals are continuous extends holomorphically to a neighborhood of $\overline{D}$.

Theorem

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Crucial step:

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Crucial step:
Using properties of the Bergman function we show that that there are smooth functions $\Phi_{i,k}$, $\operatorname{supp} \Phi_{i,k} \subset\subset G$ such that

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for $i = 1, \ldots, n$, and $k \in \mathbb{N}$.

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As already mentioned $L^2_h(D)$ denote the Bergman space of a domain D in $\mathbb{C}^n$, i.e. $L^2_h(D) = L^2(D) \cap \mathcal{O}(D)$.
It is clear that $\dim L^2_h(\mathbb{C}^2) = 0$. It is also clear that $\dim L^2_h(\mathbb{D}) = \infty$. It is very natural to ask the following question:

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Question

Is there a pseudoconvex domain with finite dimensional and non-trivial Bergman space?

This problem has been lastly studied by Jucha in the case of Hartogs domains, i.e. domains of the form

$$D = D_\varphi = D_\varphi(G) = \{(z, w) \in G \times \mathbb{C}^N : \ ||w|| < e^{-\varphi(z)}\},$$

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The results obtained by Jucha suggest that the answer to a simpler conjecture should be positive:

Question

Is $L^2_h(D)$ either trivial or infinite dimensional for a Hartogs domain D?

Let us recall the famous

Feferman's theorem

Assume that D, G are strictly pseudoconvex domains with the smooth boundaries. Then any biholomorphic mapping $f : D \to G$ extends to a diffeomorphism $\bar{D} \to \bar{G}$.

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Bell and Ligocka found a simply proof of this theorem - their methods relied upon investigations of Bergman kernels.
There is a famous conjecture that the assumption of the strict-pseudoconvexity is not necessary:

Question

Does a biholomorphic mapping $f : D \to G$ between smooth domains extend to a diffeomorphic mapping between $\bar{D}$ and $\bar{G}$?

It is believed that a very subtle investigation of Bergman kernels should give an affirmative answer to this question.