

# Forcing and Combinatorics, 3.

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$\kappa \rightarrow (\lambda)_\sigma^n$  iff for all  $f : [\kappa]^n \rightarrow \sigma$  there is  $A \in [\kappa]^\lambda$  s.t.  $|f''[A]^n| = 1$

## Theorem (Erdős-Rado)

$$(2^\kappa)^+ \rightarrow (\kappa^+)_\kappa^2 \quad (2^{2^\kappa})^+ \rightarrow (\kappa^+)_\kappa^3 \quad (\exp_n(\kappa))^+ \rightarrow (\kappa^+)_\kappa^{n+1}$$

# Almost disjoint families

- there is an almost disjoint family  $\mathcal{A} \subset [\omega]^\omega$  of cardinality  $2^\omega$
- Is there an almost disjoint family  $\mathcal{B} \subset [\omega_1]^{\omega_1}$  of size  $2^{\omega_1}$ ?
- almost disjoint:  $B \cap B' \leq \omega$  for  $B \neq B' \in \mathcal{B}$ .
- there is an almost disjoint family  $\mathcal{B} \subset [\omega_1]^{\omega_1}$  of cardinality  $\omega_2$
- $2^{\omega_1} \geq \omega_3$ .
- $2^\omega = \omega_1 \implies \exists$  almost disjoint family  $\mathcal{B} \subset [\omega_1]^{\omega_1}$  of size  $2^{\omega_1}$
- $2^\omega > \omega_1$ .
- **Con(ZFC +  $2^\omega = 2^{\omega_1} = \omega_3$  +  
there is no almost disjoint family  $\mathcal{B} \subset [\omega_1]^{\omega_1}$  of size  $\omega_3$ )**

$2^{\omega_1} = \omega_3 + \bar{\aleph}$  a.d. family  $\mathcal{B} \subset [\omega_1]^{\omega_1}$  of size  $\omega_3$

- $\mathcal{M}$  c.t.m.  $\mathcal{M} \models 2^{\omega_1} = \omega_2$ .
- $\mathbb{P} = \text{FIN}(\omega_3 \times \omega)$   $|\mathbb{P}| = \omega_3$ , c.c.c.
- $\mathcal{M}[\mathcal{G}] \models 2^{\omega_1} \geq 2^\omega \geq \omega_3$ .
- $\mathcal{G} \ni p \Vdash \langle A_\alpha : \alpha < \check{\omega}_3 \rangle \subset [\check{\omega}_1]^{\check{\omega}_1}$  is almost disjoint.
- Let  $D_{\alpha,\beta} = \{\xi < \omega_1 : \exists p_\xi \in \mathbb{P} p_\xi \Vdash \xi = \sup(A_\alpha \cap A_\beta)\}$ .
- $\{p_\xi : \xi \in D_{\alpha,\beta}\}$  is an antichain:  $|D_{\alpha,\beta}| \leq \omega$ .  $g(\alpha, \beta) = \sup D_{\alpha,\beta}$
- $g : [\omega_3]^2 \rightarrow \omega_1$ . Erdős-Rado:  $(2^{\omega_1})^+ \rightarrow (\omega_2)_{\omega_1}^2$
- $\exists X \in [\omega_3]^{\omega_2} \exists \eta < \omega_1 \forall \alpha, \beta \in X g(\alpha, \beta) = \eta$ .
- $\mathcal{M}[\mathcal{G}] \models \{A_\alpha \setminus (\eta + 2) : \alpha \in X\}$  is a disjoint family

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- $\exists X \in [\omega_3]^{\omega_2} \exists \eta < \omega_1 \forall \alpha, \beta \in X \ g(\alpha, \beta) = \eta$ .
- $\mathcal{M}[\mathcal{G}] \models \{A_\alpha \setminus (\eta + 2) : \alpha \in X\}$  is a disjoint family
- Assume  $\zeta \geq \eta + 1$ ,  $\alpha, \beta \in X$  and  $\mathcal{M}[\mathcal{G}] \models \zeta \in A_\alpha \cap A_\beta$ .
- $\mathcal{M}[\mathcal{G}] \models \eta + 1 \leq \zeta \leq \xi = \sup(A_\alpha \cap A_\beta)$
- $\eta + 1 \leq \zeta \leq \xi$  and  $\exists q \in \mathcal{G} \ (q \leq p) \ q \Vdash \xi = \sup(A_{\check{\alpha}} \cap A_{\check{\beta}})$ .
- $\xi \in D_{\alpha,\beta}$  so  $\xi \leq g(\alpha, \beta) = \eta < \eta + 1 \leq \zeta \leq \xi$

# How to construct a topological space?

- $X = \langle X, \tau \rangle$  is a topological space
- $\mathcal{N} \subset \mathcal{P}(X)$  is a **network** iff for each  $x \in U \in \tau$  there is  $N \in \mathcal{N}$  with  $x \in N \subset U$ .
- $nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network}\}$ .
- $nw(X) \leq |X|, w(X)$
- $Y \subset X$  is **weakly separated** iff there is a neighbourhood assignment  $f : Y \rightarrow \tau$  s.t. for each  $x \neq y \in Y$  then either  $x \notin f(y)$  or  $y \notin f(x)$ .
- $R(X) = \sup\{|Y| : Y \subset X \text{ is weakly separated}\}$

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- $R(X) = \sup\{|Y| : Y \subset X \text{ is weakly separated}\}$
- $R(X) \leq nw(X)$
- There are spaces with  $nw(X) > R(X)$  in many models.
- **OPEN: Is it consistent that  $nw(X) \leq R(X)$  for each regular space?**

# How to approximate a topological space?

$\mathcal{N}$  is a **network** iff for each  $x \in U \in \tau$  there is  $N \in \mathcal{N}$  with  $x \in N \subset U$ .

$Y$  is **weakly separated** iff  $\exists f : Y \rightarrow \tau$  s.t.  $x \notin f(y)$  or  $y \notin f(x)$  for  $x \neq y \in Y$ .

- **Con( there is a first countable, 0-dimensional topological space  $X = \langle \omega_1, \tau \rangle$  such that  $nw(X) = \omega_1$  and  $R(X) = \omega$ ).**
- If  $\mathcal{M}$  is a **ctm**, then there is a **c.c.c poset**  $P$  in  $\mathcal{M}$  such that if  $\mathcal{G}$  is an  $\mathcal{M}$ -generic filter in  $P$  then in  $\mathcal{M}[\mathcal{G}]$  there is a first countable, 0-dimensional topological space  $X = \langle \omega_1, \tau \rangle$  such that  $nw(X) = \omega_1$  and  $R(X) = \omega$ .

$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle$ ,  $\chi(X) = \omega$ ,  $nw(X) = \omega_1$ ,  $R(X) = \omega$

$\mathcal{N}$  is a **network** iff for each  $x \in U \in \tau$  there is  $N \in \mathcal{N}$  with  $x \in N \subset U$ .

$Y$  is **weakly separated** iff  $\exists f : Y \rightarrow \tau$  s.t.  $x \notin f(y)$  or  $y \notin f(x)$  for  $x \neq y \in Y$ .

- $p = \langle A_p, n_p, \{U_p(\alpha, i) : \alpha \in A, i \in n\} \rangle \in P$  iff
  - $A_p \in [\omega_1]^{<\omega}$ ,  $n_p \in \omega$ ,  $\alpha \in U_p(\alpha, i) \subset A$
- Idea:  $U_{\mathcal{G}}(\alpha, i) = \cup \{U_p(\alpha, i) : p \in \mathcal{G}\}$ .
- $p \leq q$  iff
  - $A_p \supset A_q$ ,  $n_p \geq n_q$ ,  $U_p(\alpha, i) \cap A_q = U_q(\alpha, i)$  for  $\langle \alpha, i \rangle \in A_q \times n_q$
  - if  $U_q(\alpha, i) \subset U_q(\beta, j)$  then  $U_p(\alpha, i) \subset U_p(\beta, j)$
  - if  $U_q(\alpha, i) \cap U_q(\beta, j) = \emptyset$  then  $U_p(\alpha, i) \cap U_p(\beta, j) = \emptyset$
- $\mathbb{P} = \langle P, \leq \rangle$  satisfies c.c.c.
- $X = \langle \omega_1, \tau \rangle$  is **first countable, 0-dimensional**.
- $nw(X) = \omega_1?$      $R(X) = \omega?$

$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle, \chi(X) = \omega, nw(X) = \omega_1, R(X) = \omega$

$\mathcal{N}$  is a **network** iff for each  $x \in U \in \tau$  there is  $N \in \mathcal{N}$  with  $x \in N \subset U$ .

$Y$  is **weakly separated** iff  $\exists f : Y \rightarrow \tau$  s.t.  $x \notin f(y)$  or  $y \notin f(x)$  for  $x \neq y \in Y$ .

- $X = \langle \omega_1, \tau \rangle, \{U(\alpha, i) : \alpha < \omega_1, i < \omega\}$  base
- Need to prove:  **$nw(X) = \omega_1$  and  $R(X) = \omega$**
- **If  $X$  is  $\sigma$ -discrete then  $nw(X) = \omega_1$**
- **If  $X$  is  $\sigma$ -second-countable then  $R(X) = \omega$**
- It is impossible!
- Wait a minute:
  - If  $\mathcal{M}[\mathcal{G}][\mathcal{H}_0] \models X$  is  **$\sigma$ -discrete** then  $\mathcal{M}[\mathcal{G}] \models nw(X) = \omega_1$ .
  - If  $\mathcal{M}[\mathcal{G}][\mathcal{H}_1] \models X$  is  **$\sigma$ -second-countable** then  $\mathcal{M}[\mathcal{G}] \models R(X) = \omega$ .

$$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle, \chi(X) = \omega, nw(X) = \omega_1, R(X) = \omega$$

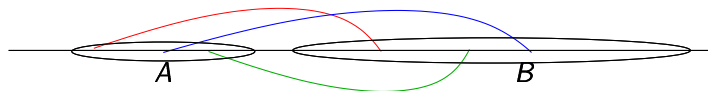
- $X = \langle \omega_1, \tau \rangle, \{U(\alpha, i) : \alpha < \omega_1, i < \omega\}$  base
- Need to prove:  $nw(X) = \omega_1$  and  $R(X) = \omega$  in  $\mathcal{M}[\mathcal{G}]$
- Find c.c.c posets  $Q_0$  and  $Q_1$  in  $\mathcal{M}[\mathcal{G}]$  such that
  - if  $\mathcal{H}_0$  is  $\mathcal{M}[\mathcal{G}]$ -generic in  $Q_0$  then  $\mathcal{M}[\mathcal{G}][\mathcal{H}_0] \models X$  is  $\sigma$ -discrete
  - if  $\mathcal{H}_1$  is  $\mathcal{M}[\mathcal{G}]$ -generic in  $Q_1$  then  $\mathcal{M}[\mathcal{G}][\mathcal{H}_1] \models X$  is  $\sigma$ -second-countable
- Let  $\langle B, f, g \rangle \in Q_0$  iff
  - $B \in [\omega_1]^{<\omega}, f, g : B \rightarrow \omega$
  - if  $f(\beta) = n$  then  $f(\beta') \neq n$  for each  $\beta' \in U(\beta, g(\beta)) \setminus \{\beta\}$
- Ordering: extension
- It works.

$$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle, \chi(X) = \omega, nw(X) = \omega_1, R(X) = \omega$$

- $X = \langle \omega_1, \tau \rangle, \{U(\alpha, i) : \alpha < \omega_1, i < \omega\}$  base
- Need to prove:  $nw(X) = \omega_1$  and  $R(X) = \omega$
- Find c.c.c posets  $Q_0$  and  $Q_1$  in  $\mathcal{M}[\mathcal{G}]$  such that
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  - if  $\mathcal{H}_1$  is  $\mathcal{M}[\mathcal{G}]$ -generic in  $Q_1$  then  $\mathcal{M}[\mathcal{G}][\mathcal{H}_1] \models X$  is  $\sigma$ -second-countable
- Let  $\langle B, f, n, g \rangle \in Q_1$  iff
  - $B \in [\omega_1]^{<\omega}, f : B \rightarrow \omega, g : B \times n \rightarrow \omega$
  - if  $f(\beta) = f(\beta') = n$  and  $g(\beta, i) = g(\beta', j)$  then  $U(\beta, i) \cap f^{-1}\{n\} = U(\beta', j) \cap f^{-1}\{n\}$
- Ordering: extension
- It works.

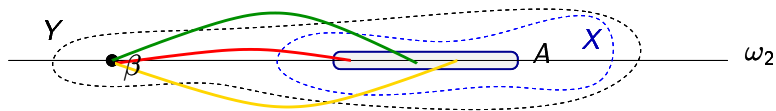
# Negative partition relations

$f$  establishes  $\omega_2 \not\rightarrow [(\omega_1 : \omega)]_{\omega_1}^2$  iff  $\forall A \in [\omega_2]^{\omega_1} \forall B \in [\omega_2]^\omega$  if  $\sup(A) < \min(B)$  then  $f''[A, B] = \omega_1$ .



Con ( GCH +  $\exists f \vdash \omega_2 \not\rightarrow [(\omega_1; \omega)]_{\omega_1}^2$  ).

- Assume CH. Define  $\mathcal{P} = \langle P, \preceq \rangle$   $\sigma$ -complete,  $\omega_2$ -c.c.
- Underlying set :  $\langle X, c, \mathcal{A}, \xi \rangle$ 
  - $X \in [\omega_2]^\omega$ ,  $c : [X]^2 \rightarrow \omega_1$
  - $\mathcal{A} \subset P(X)$ ,  $|\mathcal{A}| \leq \omega$ ,
  - $\xi \in \omega_1$
- $\langle Y, d, \mathcal{B}, \zeta \rangle \preceq \langle X, c, \mathcal{A}, \xi \rangle$  iff
  - $Y \supset X, d \supset c, \mathcal{B} \supset \mathcal{A}, \zeta \geq \xi$
  - (1)  $\forall A \in \mathcal{A} \quad \forall \beta \in (Y \setminus X) \cap \min A \quad \xi \subset d''[\{\beta\}, A]$ .



# A ZFC theorem

- $\omega \rightarrow (\omega)_2^2$
- $\omega_1 \not\rightarrow (\omega_1)_2^2$
- (Baumgartner, Hajnal)  $\omega_1 \rightarrow (\alpha)_2^2$  for  $\alpha < \omega_1$
- Step 1: If  $MA_{\aleph_1}$  holds then  $\omega_1 \rightarrow (\alpha)_2^2$  for  $\alpha < \omega_1$ .
- Fact: In any ctm  $\mathcal{M}$  there is a c.c.c poset  $P$  s.t. if  $\mathcal{G}$  is  $\mathcal{M}$ -generic filter then  $\mathcal{M}[\mathcal{G}] \models MA_{\aleph_1}$
- Step 2:  $\mathcal{M} \models f : [\omega_1]^2 \rightarrow 2$ ;

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- Fact: In any ctm  $\mathcal{M}$  there is a c.c.c poset  $P$  s.t. if  $\mathcal{G}$  is  $\mathcal{M}$ -generic filter then  $\mathcal{M}[\mathcal{G}] \models MA_{\aleph_1}$
- Step 2:  $\mathcal{M} \models f : [\omega_1]^2 \rightarrow 2$ ;
- $\alpha = \{\alpha_n : n < \omega\}$ .
- Define  $Q = \langle Q, \leq \rangle$  as follows.  $\sigma \in Q$  iff
  - $\sigma$  is an injective function,  $\text{dom}(\sigma) = \{\alpha_0, \dots, \alpha_n\}$  for some  $n < \omega$
  - $\text{ran}(\sigma) \subset \omega_1$ ,  $\sigma$  is order preserving
  - $\text{ran}(\sigma)$  is  $f$ -homogeneous
- $Q$  is ill founded iff  $\exists A \subset \omega_1$   $f$ -homogeneous,  $\text{tp}(A) = \alpha$
- $Q$  is well-founded in  $\mathcal{M}$  iff it is well-founded in  $\mathcal{M}[\mathcal{G}]$
- $\exists A$  in  $\mathcal{M}[\mathcal{G}]$ , so  $Q$  is ill-founded in  $\mathcal{M}[\mathcal{G}]$  so  $Q$  is ill-founded in  $\mathcal{M}$ , so  $\exists A$  in  $\mathcal{M}$