A PIECEWISE TORONTO SPACE

L. SOUKUP

ABSTRACT. We show that it is consistent that there is a hereditarily separable, 0-dimensional T_2 space X of cardinality ω_1 such that for each uncountable subspace Y of X there is a continuous bijection $\varphi : Y \longrightarrow X$ and there is a partition $(Y_i)_{i<n}$ of Y into finitely many pieces such that $\varphi \restriction Y_i$ is homeomorphism for each $i < n$.

1. INTRODUCTION

Two topological spaces X and Y are called *piecewise homeomorphic* iff for some natural number n there are partitions $(X_i)_{i<n}$ of X and $(Y_i)_{i<n}$ of Y such that X_i and Y_i are homeomorphic. An uncountable topological space is called *piecewise Toronto* if its every two uncountable subspaces are piecewise homeomorphic.

There are spaces which are piecewise Toronto in a trivial way: they are piecewise homeomorphic to $\mathbf{D}_\kappa$, where $\mathbf{D}_\kappa$ denotes the discrete topological space of size κ. It is easy to see that such spaces are just the scattered spaces of finite height. To exclude these trivial examples let us observe that an uncountable hereditarily separable space can not be a scattered space with finite height and so it is not piecewise homeomorphic to $\mathbf{D}_\kappa$ for any $\kappa > \omega$. In section 2 we show that the existence of a hereditarily separable piecewise Toronto space is consistent with ZFC. In fact, the space Z we construct in corollary 2.2 will have a stronger property: for each uncountable subspace T of Z there is a continuous bijection $\varphi : T \longrightarrow Z$ and there is a partition $(T_i)_{i<n}$ of T into finitely many pieces such that $\varphi \restriction T_i$ is homeomorphism for each $i < n$.

Our notation is standard, see e.g. [2]. We will also use the following pieces of notion and notation.

For $c \in \mathrm{Fn}(\omega_1, 2; \omega)$ write $[c] = \{f \in 2^{\omega_1} : f \supset c\}$.

Let $F : \omega_1 \times \omega_1 \longrightarrow 2$ be a function. F is *nice* iff for each $\{\alpha, \beta\} \in \left[\omega_1\right]^2$ the set $\Delta_F(\alpha, \beta) \overset{def}{=} \{\nu < \omega_1 : F(\alpha, \nu) \neq F(\beta, \nu)\}$ is uncountable. For $A, B \subset \omega_1$ we write $F \restriction A \times B \equiv 0$ ($F \restriction A \times B \equiv 1$) iff $F(\alpha, \beta) = 0$ ($F(\alpha, \beta) = 1$) for each $\alpha \in A$ and $\beta \in B$.

We say that F is a *HFD_w-function* iff

$$\forall f : \omega_1 \xrightarrow{1-1} \omega_1 \ \forall m < \omega \ \forall g : \omega_1 \times m \xrightarrow{1-1} \omega_1 \ \forall H : m \longrightarrow 2$$
$$\exists \alpha < \beta < \omega_1 \ \forall j < m \ F\big(f(\alpha), g(\beta, j)\big) = H(j).$$

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Let $\mathrm{Z}(F) = \{F(\alpha, \cdot) : \alpha < \omega_1\} \subset 2^{\omega_1}$, where $F(\alpha, \cdot)$ denotes the function defined by the formula $F(\alpha, \cdot)(\xi) = F(\alpha, \xi)$.

Two functions $F_0 : A_0 \times B_0 \longrightarrow 2$ and $F_1 : A_1 \times B_1 \longrightarrow 2$ are *isomorphic*, denoted by $F_0 \cong F_1$, iff there are bijections $g : A_0 \longrightarrow A_1$ and $h : B_0 \longrightarrow B_1$ such that $F_1\big(g(\alpha), h(\beta)\big) = F_0\big(\alpha, \beta\big)$ for each $\alpha \in A_0$ and $\beta \in B_0$.

2. Generic construction

Theorem 2.1. *If $2^{\omega_1} = \omega_2$, then there is a c.c.c poset P of size ω_2 and in the generic extension V^P there is a nice HFD$_w$-function function $F : \omega_1 \times \omega_1 \longrightarrow 2$ such that*

$$(\dagger) \qquad \forall X \in \big[\omega_1\big]^{\omega_1}\ \exists Y \in \big[\omega_1\big]^{<\omega}\ F \cong F \restriction \big(X \times (\omega_1 \setminus Y)\big).$$

Before proving this theorem we give a corollary.

Corollary 2.2. *If ZF is consistent then so is ZFC +* there is a 0-dimensional, hereditarily separable space Z such that

(‡) for each uncountable subspace $T \subset Z$ there are a continuous bijection $\varphi : T \longrightarrow Z$ and a partition $(T_i)_{i<n}$ of T into finitely many pieces such that $\varphi \restriction T_i$ is homeomorphism for each $i < n$. .

Proof of corollary 2.2. Assume that $2^{\omega_1} = \omega_2$ in the ground model, consider the model V^P obtained by applying theorem 2.1 and fix the nice HFD$_w$-function function $F : \omega_1 \times \omega_1 \longrightarrow 2$ satisfying (†). Let $Z = \mathrm{Z}(F)$. Since F is a an HFD$_w$-function, it follows that $Z(F)$ is hereditarily separable.

To check (‡) let $T = \{F(\alpha, \cdot) : \alpha \in X\}$ be an arbitrary uncountable subspace of X. By (†) there is $Y \in \big[\omega_1\big]^{<\omega}$ such that $F \restriction \big(X \times (\omega_1 \setminus Y)\big) \cong F$ witnessed by bijections $g : X \longrightarrow \omega_1$ and $h : \omega_1 \setminus Y \longrightarrow \omega_1$.

Define the bijection $\varphi : T \longrightarrow Z$ by the formula $\varphi(F(\alpha, \cdot)) = F(g(\alpha), \cdot)$.

By the choice of g and h for each $\alpha \in \omega_1$ and $c \in \mathrm{Fn}(\omega_1, 2; \omega)$ we have

$$F(\alpha, \cdot) \supset c \iff F(g^{-1}(\alpha), \cdot) \supset c \circ h,$$

and so

$$\varphi^{-1}[c] = [c \circ h] \cap T,$$

which implies that φ is continuous.

Fix an enumeration $\{\varepsilon_i : i < n\}$ of ${}^{Y}2$, let $T_i = \{f_\alpha \in T : \varepsilon_i \subset f_\alpha\}$ and consider the partition $T = \bigcup_{i<n} T_i$.

To show that the map $\varphi \restriction T_i$ is a homeomorphism between T_i and $Z_i = \varphi'' T_i$ we need to show that $\varphi''([d] \cap T_i)$ is open in Z_i for each $d \in \mathrm{Fn}(\omega_1, 2; \omega)$. We can assume that $d \cup \varepsilon_i \in \mathrm{Fn}(\omega_1, 2; \omega)$ otherwise $[d] \cap T_i = \emptyset$. Let $d' = d \restriction \omega_1 \setminus Y$ and $c' = d' \circ h^{-1}$. Since $d' = c' \circ h$ we have

$$\varphi''([d] \cap T_i) = \varphi''([d'] \cap T_i) = \varphi''([c' \circ h]) \cap \varphi'' T_i = [c'] \cap Z_i.$$

The corollary is proved. □

Proof of theorem 2.1. We construct $P = \mathcal{C} * P'$ in two steps: in the first step, forcing with $\mathcal{C} = \mathrm{Fn}(\omega_1 \times \omega_1, 2; \omega)$, we introduce our desired function F, which will be a nice HFD$_w$-function but (†) will fail. Then, in the second step, we add many bijections between certain subsets of ω_1 to $V^{\mathcal{C}}$ to guarantee (†) in such a way that F remains a nice HFD$_w$-function during the iteration.

In $V^{\mathcal{C}}$ let $F = \bigcup \mathcal{G}$, where $\mathcal{G}$ is the $\mathcal{C}$-generic filter over V.

If $c \in \mathcal{C}$ let $\operatorname{supp} c = (\operatorname{dom}\operatorname{dom} c) \cup (\operatorname{ran}\operatorname{dom} c)$.

To obtain $P' = P_{\omega_2}$ we carry out a finite support iteration of c.c.c posets

$$\langle P_\alpha : \alpha \le \omega_2, Q_\alpha : \alpha < \omega_2 \rangle$$

in the following way: in the α^{th} step, we pick an uncountable set X_α of ω_1 in the intermediate model $V^{\mathcal{C}*P_\alpha}$ and then we try to find a finite set Y_α and c.c.c poset Q_α such that forcing with Q_α preserves (‡) and

$$V^{\mathcal{C}*P_\alpha*Q_\alpha} \models \text{“}F \cong F \restriction \big(X_\alpha \times (\omega_1 \setminus Y_\alpha)\big) \textit{ witnessed by } \textit{bijections } f_\alpha : \omega_1 \longrightarrow X_\alpha \textit{ and } g_\alpha : \omega_1 \longrightarrow \omega_1 \setminus Y_\alpha.\text{”}$$

Now assume that P_α is constructed and let us see the induction step.

First, using a bookkeeping function, we pick the set $X_\alpha \in \big[\omega_1\big]^{\omega_1} \cap V^{\mathcal{C}*P_\alpha}$ in such a way that

$$(*) \qquad \{X_\alpha : \alpha < \omega_2\} = \big[\omega_1\big]^{\omega_1} \cap V^{\mathcal{C}*P_{\omega_2}}.$$

To construct the poset Q_α we need the following induction hypotheses. To formulate it we introduce two notions. A function $F : \omega_1 \times \omega_1 \longrightarrow 2$ is *strongly non-trivial* provided that each uncountable family of pairwise disjoint, finite subsets of ω_1 contains four distinct elements, a, b, c, d such that $F \restriction a \times b \equiv 0$ and $F \restriction b \times a \equiv 0$ and $F \restriction c \times d \equiv 1$ and $F \restriction d \times c \equiv 1$.

Given a set $I \subset \omega_1$ we say that $A \subset \omega_1$ is *up-dense* (*down-dense*) *for F in I* iff for each $b \in \operatorname{Fn}(I, 2; \omega)$ there is $\alpha \in A$ such that $F(\alpha, \beta) = b(\beta)$ $\big(F(\beta, \alpha) = b(\beta)\big)$ for each $\beta \in \operatorname{dom}(b)$.

Induction Hypothesis .

(I) $V^{\mathcal{C}*P_\alpha} \models$ “F is strongly non-trivial”,

(II) $V^{\mathcal{C}*P_\alpha} \models$ “ $\forall X \in \big[\omega_1\big]^{\omega_1}\ \exists Y \in \big[\omega_1\big]^{<\omega}\ \forall \delta < \omega_1\ \exists A \in \big[X \setminus \delta\big]^{\omega}$ A is up-dense for F in $\omega_1 \setminus Y$”,

The preservation of the induction hypotheses (I) and (II) during the iteration will be verified later in lemmas 2.7 and 2.11. Let us observe that we will not have to check (‡) in our final model because the following lemma clearly holds:

Lemma 2.3. *If* (II) *holds, then F is an HFD_w-function.*

We continue the construction of the poset Q_α. Using (II) fix $Y_\alpha \in \big[\omega_1\big]^{<\omega}$ and pairwise disjoint countable subsets $\{D_\xi : \xi < \omega_1\}$ of X_α which are up-dense for F in $\omega_1 \setminus Y_\alpha$.

Let us recall that for each $\beta < \alpha$ in the β^{th} step we already constructed bijections $f_\beta : \omega_1 \longrightarrow X_\beta$ and $g_\beta : \omega_1 \longrightarrow \omega_1 \setminus Y_\beta$ witnessing $F \cong F \restriction \big(X_\beta \times (\omega_1 \setminus Y_\beta)\big)$. For each $\beta < \alpha$ the set $C_\beta = \{\nu < \omega_1 : (f''_\beta\nu \cup g''_\beta\nu) \subset \nu\}$ is clearly club and C_β belongs to $V^{\mathcal{C}*P_\beta*Q_\beta} \subset V^{\mathcal{C}*P_\alpha}$. Since P_α satisfies c.c.c and $|\alpha| < 2^{\omega_1} = \omega_2$ it follows that there is a club set $C \subset \omega_1$ even in V such that $|C \setminus C_\beta| \le \omega$ for each $\beta < \alpha$.

The club set $C = \{\gamma_\nu : \nu < \omega_1\}$ gives a natural partition $\mathcal{A}_\alpha = \{A^\alpha_\nu : \nu < \omega_1\}$ of ω_1 into countable pieces: let $A^\alpha_\nu = [\gamma_\nu, \gamma_{\nu+1})$ for $\nu < \omega_1$. We can thin out C to contain only limit ordinals and in this case every A^α_ν is infinite. Define the map $\operatorname{rk}_\alpha : \omega_1 \longrightarrow \omega_1$ by the formula $\xi \in A^\alpha_{\operatorname{rk}_\alpha(\xi)}$.

If $\beta < \alpha$ then $|C \setminus C_\beta| \le \omega$ and so all but countably many A^α_ν's are f_β-closed. By shrinking C we can assume every A^α_η contains some D_ξ and so

(i) $A^\alpha_\eta \cap X_\alpha$ is up-dense for F in $\omega_1 \setminus Y_\alpha$.

Since $A^\alpha_\eta \in V$ and infinite, it follows

(ii) A^α_η is down-dense for F in ω_1.

For $\eta < \omega_1$ let $O_\eta = [\omega\eta, \omega\eta + \omega)$ and $B^\alpha_\eta = \bigcup\{A^\alpha_\eta : \nu \in O_\eta\}$. Put $\mathcal{B}_\alpha = \left\langle B^\alpha_\eta : \eta < \omega_1 \right\rangle$.

Given two sets Z and W denote by $\mathrm{Bij}_\mathrm{p}(Z, W)$ the family of bijections between finite subsets Z and W.

If $p \in \mathrm{Bij}_\mathrm{p}(\omega_1, \omega_1)$ a sequence $\vec{x} = \langle x_0, x_1, \dots, x_n\rangle$ of countable ordinals is a *p-loop* iff $n \geq 1$, $x_0 = x_n$ and there is a sequence $\langle k_0, \dots, k_{n-1}\rangle \in {}^n\{-1, +1\}$ such that

(iii) $\mathrm{rk}_\alpha(x_{i+1}) = \mathrm{rk}_\alpha(p^{k_i}(x_i))$ for each $i < n$,

(iv) there is no $i < n$ such that $\{k_i, k_{i+1}\} = \{-1, +1\}$, $x_{i+1} = p^{k_i}(x_i)$ and $x_{i+2} = p^{k_{i+1}}(x_{i+1})$.

We say that p is *loop-free* if there is no p-loop.

If $A, B, C, D \subset \omega_1$ let

$$\mathrm{Iso}_\mathrm{p}(A, B, C, D) = \Big\{ \langle p, q\rangle : p \in \mathrm{Bij}_\mathrm{p}(A, B),\ q \in \mathrm{Bij}_\mathrm{p}(C, D) \wedge \forall \alpha \in \mathrm{dom}(p)\ \forall \nu \in \mathrm{dom}(q)\ F(\alpha, \nu) = F\big(p(\alpha), q(\nu)\big)\Big\}.$$

Now we are in the position to define the poset Q_α. We put a pair of finite functions $\langle p, q\rangle \in \mathrm{Iso}_\mathrm{p}(\omega_1, X_\alpha, \omega_1, \omega_1 \setminus Y_\alpha)$ into Q_α iff

(v) $p''B_\eta \cup q''B_\eta \subset B_\eta$ for each $\eta < \omega_1$,

(vi) p and q are loop-free.

As promised, Q_α is ordered by the reverse inclusion: $\langle p', q'\rangle \leq \langle p, q\rangle$ iff $p' \supset p$ and $q' \supset q$.

Let $\mathrm{supp}\,\langle p, q\rangle = \mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \mathrm{dom}\, q \cup \mathrm{ran}\, q$ for $\langle p, q\rangle \in Q_\alpha$.

We need to show that Q_α satisfies c.c.c and a Q_α-generic filter gives bijections $f_\alpha : \omega_1 \longrightarrow X_\alpha$ and $g_\alpha : \omega_1 \longrightarrow \omega_1 \setminus Y_\alpha$ witnessing $F \cong F \restriction \big(X_\alpha \times (\omega_1 \setminus Y_\alpha)\big)$. First we prove an auxiliary lemma.

Lemma 2.4. *If $p, q \in \mathrm{Bij}_\mathrm{p}(\omega_1, \omega_1)$, $\mathrm{rk}_\alpha{}''\,\mathrm{supp}\, p \cap \mathrm{rk}_\alpha{}''\,\mathrm{supp}\, q = \emptyset$ and $\vec{x} = \langle x_0, \dots, x_n\rangle$ is a $(p \cup q)$-loop, then $\vec{x}$ is either a p-loop or a q-loop.*

Proof. Assume that $x_0 \in \mathrm{supp}\, p$. Then $x_0 \notin \mathrm{supp}\, q$, so $\mathrm{rk}_\alpha(x_1) = \mathrm{rk}_\alpha(p^{k_0}(x_0))$ for some $k_0 \in \{-1, +1\}$. Since $p^{k_0}(x_0) \in \mathrm{supp}\, p$ we have $\mathrm{rk}_\alpha(x_1) = \mathrm{rk}_\alpha(p^{k_0}(x_0)) \notin \mathrm{rk}_\alpha{}''\,\mathrm{supp}\, q$ and so $x_1 \notin \mathrm{supp}\, q$. Repeating this argument we yield $\{x_0, \dots, x_n\} \subset \mathrm{supp}\, p \setminus \mathrm{supp}\, q$ and so $\vec{x}$ is a p-loop. □

Lemma 2.5. *Q_α satisfies c.c.c.*

Proof. We work in $V^{\mathcal{C} * P_\alpha}$. Assume that $\{q_\xi : \xi < \omega_1\} \subset Q_\alpha$, $q_\xi = \langle q_{\xi,0}, q_{\xi,1}\rangle$ $c_\xi = \mathrm{supp}\, q_\xi$ and $r_\xi = \mathrm{rk}_\alpha{}'' c_\xi$. Applying standard Δ-system and counting arguments we can find $I \in \left[\omega_1\right]^{\omega_1}$ such that

(1) $\{c_\xi : \xi \in I\}$ forms a Δ-system with kernel c,

(2) $\{r_\xi : \xi \in I\}$ forms a Δ-system with kernel r,

(3) $\mathrm{rk}_\alpha{}'' c = r$,

(4) $\mathrm{rk}_\alpha{}''(c_\xi \setminus c) = r_\xi \setminus r$ for each $\xi \in I$,

(5) $q_{\xi,i} \restriction c = q'_i$ for each $\xi \in I$ and $i < 2$.

Since F is strongly non-trivial in $V^{\mathcal{C}*P_\alpha}$ there is $\{\xi,\zeta\}\in\big[I\big]^2$ such that $F\restriction(c_\xi\setminus c)\times(c_\zeta\setminus c)\equiv 0$ and $F\restriction(c_\zeta\setminus c)\times(c_\xi\setminus c)\equiv 0$. We show that taking $q_i=q_{\xi,i}\cup q_{\zeta,i}$ for $i<2$ we have $q=\langle q_0,q_1\rangle\in Q_\alpha$. Clearly $q\in\mathrm{Iso_p}(\omega_1,X_\alpha,\omega_1,\omega_1\setminus Y_\alpha)$ and q satisfies (v). Since $q_i=q'_i\cup(q_{\xi,i}\setminus q'_i)\cup(q_{\zeta,i}\setminus q'_i)$ and the sets $\mathrm{rk}_\alpha{}''q'_i$, $\mathrm{rk}_\alpha{}''(q_{\xi,i}\setminus q'_i)$ and $\mathrm{rk}_\alpha{}''(q_{\zeta,i}\setminus q'_i)$ are pairwise disjoint we have that q satisfies (vi) as well by lemma 2.4. □

If $\mathcal{G}^{Q_\alpha}$ is the Q_α-generic filter over $V^{\mathcal{C}*P_\alpha}$ let $f_\alpha=\cup\{q:\langle q,q'\rangle\in\mathcal{G}^{Q_\alpha}\}$ and $g_\alpha=\cup\{q':\langle q,q'\rangle\in\mathcal{G}^{Q_\alpha}\}$.

Lemma 2.6. $V^{\mathcal{C}*P_\alpha*Q_\alpha}\models$ *"$F\cong F\restriction\big(X_\alpha\times(\omega_1\setminus Y_\alpha)\big)$ is witnessed by f_α and g_α."*

Proof. We need to prove that $\mathrm{dom}(f_\alpha)=\omega_1$, $\mathrm{ran}(f_\alpha)=X_\alpha$, $\mathrm{dom}(g_\alpha)=\omega_1$ and $\mathrm{ran}\,g_\alpha=\omega_1\setminus Y_\alpha$ which follows if for each $\nu\in\omega_1$, $\mu\in X_\alpha$, $\rho\in\omega_1$ and $\sigma\in\omega_1\setminus Y_\alpha$ the families

$$\begin{aligned}D^{up}_\nu &= \{\langle q_0,q_1\rangle\in Q_\alpha:\nu\in\mathrm{dom}(q_0)\},\\ R^{up}_\mu &= \{\langle q_0,q_1\rangle\in Q_\alpha:\mu\in\mathrm{ran}(q_0)\},\\ D^{down}_\rho &= \{\langle q_0,q_1\rangle\in Q_\alpha:\rho\in\mathrm{dom}(q_1)\},\\ R^{down}_\sigma &= \{\langle q_0,q_1\rangle\in Q_\alpha:\sigma\in\mathrm{ran}(q_1)\}\end{aligned}$$

are all dense in Q_α. Fix $q=\langle q_0,q_1\rangle\in Q_\alpha$. Write $\mathrm{rk}_\alpha(\nu)=\omega\eta+n$. Pick $\omega\eta\le\zeta<\omega\eta+\omega$ such that $\mathrm{supp}(q)\cap A^\alpha_\zeta=\emptyset$. Since $A^\alpha_\zeta\cap X_\alpha$ is up-dense for F in $\omega_1\setminus Y_\alpha$ we can find $\nu'\in A^\alpha_\zeta\cap X_\alpha$ such that $F(\nu',q_1(\xi))=F(\nu,\xi)$ for each $\xi\in\mathrm{dom}\,q_1$. Let $q'=\langle q_0\cup\{\langle\nu,\nu'\rangle\},q_1\rangle$. By the choice of ζ', $\mathrm{rk}_\alpha(\nu')=\zeta\notin\mathrm{rk}_\alpha{}''(\mathrm{supp}(q))$, so this extension of q can not introduce a $q_0\cup\{\langle\nu,\nu'\rangle\}$-loop, i.e. $q'\in Q_\alpha$. Thus $q'\in D^{up}_\nu$ and $q'\le q$ which was to be proved. The density of R^{up}_μ can be verified by a similar argument using that A^α_ζ is up-dense for F in ω_1.

To check the density of D^{down}_ρ and R^{down}_σuse that $A^\alpha_\eta\setminus Y_\alpha$ is down-dense for F in $\omega_1\setminus Y_\alpha$. □

The induction step is complete so the theorem is proved provided we can verify the induction hypotheses (I) and (II) in every $V^{\mathcal{C}*P_\gamma}$. First we deal with (I) because it is fairly easy. Checking (II) is the crux of our proof.

Lemma 2.7. *The induction hypothesis* (I) *holds, i.e. F is strongly non-trivial in every $V^{\mathcal{C}*P_\alpha}$.*

Proof. First remark that F is clearly strongly non-trivial in $V^{\mathcal{C}}$. By [1, lemma 4.10] we can assume that $\alpha=\gamma+1$ and F is strongly non-trivial in $V^{\mathcal{C}*P_\gamma}$. Working in $V^{\mathcal{C}*P_\alpha}$ assume that $q\Vdash$ "$\{\dot x_\xi:\xi<\omega_1\}$ *are pairwise disjoint, finite subsets of* ω_1." For each $\xi<\omega_1$ pick a condition $q_\xi\le q$ and a finite subset x_ξ of ω_1 such that $q_\xi\Vdash$ "$\dot x_\xi=x_\xi$". Write $q_\xi=\langle q_{\xi,0},q_{\xi,1}\rangle$. Since Q_γ satisfies c.c.c, we can assume that the sets x_ξ are pairwise disjoint.

We can assume that $x_\xi\subset\mathrm{dom}\,q_{\xi,0}$ because in lemma 2.6 we showed that the sets D^{up}_ν are dense in Q_γ.

From now on we can argue as in lemma 2.5. Let $c_\xi=\mathrm{supp}\,q_\xi$ and $r_\xi=\mathrm{rk}_\gamma{}''c_\xi$. We can find $I\in\big[\omega_1\big]^{\omega_1}$ such that $\{c_\xi:\xi\in I\}$ forms a Δ-system with kernel c and $\{r_\xi:\xi\in I\}$ forms a Δ-system with kernel r, moreover $\mathrm{rk}_\gamma{}''c=r$, $\mathrm{rk}_\gamma{}''(c_\xi\setminus c)=r_\xi\setminus r$, $q_{\xi,i}\restriction c$ is independent from ξ and $x_\xi\subset c_\xi\setminus c$ for each $\xi\in I$. Write $c'_{\xi,i}=c_{\xi,i}\setminus c$, $q'_{\xi,i}=q_{\xi,i}\restriction c'_{\xi,i}$, $r'_{\xi,i}=r_{\xi,i}\setminus r$ and $q'_i=q_{\xi,i}\restriction c$.

Since F is strongly non-trivial in $V^{\mathcal{C}*P_\xi}$ there are pairwise different ordinals $\xi_0,\xi_1,\zeta_0,\zeta_1\in I$ $F\restriction c'_{\xi_0}\times c'_{\zeta_0}\equiv 0$ and $F\restriction c'_{\zeta_0}\times c'_{\xi_0}\equiv 0$ and $F\restriction c'_{\xi_1}\times c'_{\zeta_1}\equiv 1$ and $F\restriction c'_{\zeta_1}\times c'_{\xi_1}\equiv 1$. .

For $j<2$ let $q^j_i=q_{\xi_j,i}\cup q_{\zeta_j,i}$ for $i<2$ and $q^j=\left\langle g^j_0,q^j_1\right\rangle$. Then $q^j\in \mathrm{Iso_p}(\omega_1,X_\alpha,\omega_1,\omega_1\setminus Y_\alpha)$ and q^j clearly satisfies (v). Since $q^j_i=q'_i\cup q'_{\xi_j,i}\cup q'_{\zeta_j,i}$ and the sets $\mathrm{rk}_\gamma{}''q'_i$, $\mathrm{rk}_\gamma{}''q'_{\xi_j,i}$ and $\mathrm{rk}_\gamma{}''q'_{\zeta_j,i}$ are pairwise disjoint we have that q^j satisfies (vi) as well by lemma 2.4. Thus

$$q^0\Vdash \text{“}F\restriction \dot{x}_{\xi_0}\times\dot{x}_{\zeta_0}\equiv 0\wedge F\restriction\dot{x}_{\zeta_0}\times\dot{x}_{\xi_0}\equiv 0\text{”}$$

and

$$q^0\Vdash \text{“}F\restriction \dot{x}_{\xi_1}\times\dot{x}_{\zeta_1}\equiv 1\wedge F\restriction\dot{x}_{\zeta_1}\times\dot{x}_{\xi_1}\equiv 1\text{”}.$$

□

Now we start to work on (II).

Definition 2.8. Assume that $\mathcal{H}$ is a family of functions, $\mathrm{dom}(h)\cup\mathrm{ran}(h)\subset\omega_1$ for each $h\in\mathcal{H}$. A sequence $\vec{x}=\langle x_0,x_1,\dots,x_n\rangle\in{}^n\omega_1$ is called *$\mathcal{H}$-loop* if $n\ge 1$, $x_0=x_n$, and there are sequences $\langle h_0,\dots,h_{n-1}\rangle\in{}^n\mathcal{H}$ and $\langle k_0,\dots,k_{n-1}\rangle\in{}^n\{-1,+1\}$ such that

(vii) $h_i^{k_i}(x_i)=x_{i+1}$ for each $i<n$,
(viii) there is no $i<n-1$ such that $h_i=h_{i+1}$ and $\{k_i,k_{i+1}\}=\{-1,+1\}$.

Let $Z\subset\omega_1$. We say that $\mathcal{H}$ *acts loop-free on* Z if

(ix) Z is h-closed for each $h\in\mathcal{H}$,
(x) Z does not contain any $\mathcal{H}$-loop.

Definition 2.9. A condition $p=\langle c,q\rangle\in\mathcal{C}*P_\alpha$ is called *determined* iff

(1) q is a function, $\mathrm{dom}(q)\in\left[\omega_1\right]^{<\omega}$,
(2) $q(\eta)=\langle q(\eta,0),q(\eta,1)\rangle$, and $q(\eta,i)$ is a function, for each $i<2$ and $\eta\in\mathrm{dom}(q)$,
(3) $\bigcup\{\mathrm{supp}\,q(\eta,i):i<2,\eta\in\mathrm{dom}(q)\}\subset\mathrm{supp}\,c$,
(4) $\mathrm{dom}(c)=\mathrm{supp}\,c\times\mathrm{supp}\,c$.

The determined conditions are dense in $\mathcal{C}*P_\alpha$.

Lemma 2.10. *In $V^{\mathcal{C}*P_\alpha}$ for each $J\in\left[\alpha\right]^{<\omega}$ there is $\mu<\omega_1$ such that both $\{f_\xi:\xi\in J\}$ and $\{g_\xi:\xi\in J\}$ act loop-free on $\omega_1\setminus\mu$.*

Proof. We work in $V[\mathcal{G}]$, where $\mathcal{G}$ is the $\mathcal{C}*P_\alpha$-generic filter over V. The lemma will be proved by induction on $\max J$. Let $\zeta=\max J$ and $J'=J\setminus\{\zeta\}$. Using the inductive hypothesis fix $\mu<\omega_1$ such that

(a) $\mu=\bigcup\{B\in\mathcal{B}_\zeta:B\cap\mu\ne\emptyset\}$,
(b) if $A\in\mathcal{A}_\zeta$ and $A\subset\omega_1\setminus\mu$ then A is f_ξ-closed and g_ξ-closed for each $\xi\in J'$,
(c) $\{f_\xi:\xi\in J'\}$ and $\{g_\xi:\xi\in J'\}$ act loop-free on $\omega_1\setminus\mu$.

Assume on the contrary that $\langle x_0,\dots,x_n\rangle\in{}^n(\omega_1\setminus\mu)$ is an (e.g.) $\{f_\xi:\xi\in J\}$-loop witnessed by the sequences $\langle h_i:i<n\rangle\in{}^n\{f_\xi:\xi\in J\}$ and $\langle k_i:i<n\rangle\in{}^n\{-1,+1\}$. Let $M=\{m<n:h_m=f_\zeta\}$. By the induction hypothesis $M\ne\emptyset$. Write $M=\{m_j:j<\ell\}$, $m_0<\dots<m_{\ell-1}$. Let $y_0=x_{m_0}$, $y_1=x_{m_1}$, …, $y_{\ell-1}=x_{m_{\ell-1}}$ and $y_\ell=x_{m_0}$. Pick a determined condition $\langle c,q\rangle\in\mathcal{G}$ such that $y_j,f_\zeta^{k_{m_j}}(y_j)\in\mathrm{dom}(q(\zeta,0))\cap\mathrm{ran}(q(\zeta,0))$ for each $j<\ell$. We claim that $\langle y_j:j\le\ell\rangle$ is a $q(\zeta,0)$-loop witnessed by the sequence $\langle k_{m_j} : j < \ell\rangle$, which contradicts the choice of Q_ζ. Condition (iii) holds because $\mathrm{rk}_\zeta(y_{j+1}) = \mathrm{rk}_\zeta(f_\zeta^{k_{m_j}}(y_j))$ by (b) Assume on the contrary that (iv) fails, i.e, there is $j < \ell$ such that $\{k_{m_j}, k_{m_{j+1}}\} = \{-1,+1\}$, $y_{j+1} = f_\zeta^{k_{m_j}}(y_j)$ and $y_{j+2} = f_\zeta^{k_{m_{j+1}}}(y_{j+1})$. Since $f_\zeta^{k_{m_j}}(y_j) = f_\zeta^{k_{m_j}}(x_{m_j}) = x_{m_j+1}$ and $y_{j+1} = x_{m_{j+1}}$, and so $x_{m_j+1} = x_{m_{j+1}}$, by (c) it follows that $m_j + 1 = m_{j+1}$. Similarly, $m_{j+1} + 1 = m_{j+2}$. Thus $x_{m_j} = y_j$, $x_{m_j+1} = y_{j+1}$ and $x_{m_j+2} = y_{j+2}$. So $h_{m_j} = h_{m_j+1} = f_\zeta$ and $\{k_{m_j}, k_{m_j+1}\} = \{-1,+1\}$ which contradicts our assumption that $\langle h_i : i < n\rangle$ and $\langle k_i : i < n\rangle$ satisfied 2.8.(ii). □

Lemma 2.11. *The induction hypothesis* (II) *holds in* $V^{\mathcal{C}*P_\alpha}$, *i.e.*
$V^{\mathcal{C}*P_\alpha} \models$ "$\forall X \in [\omega_1]^{\omega_1}\ \exists Y \in [\omega_1]^{<\omega}\ \forall \delta < \omega_1\ \exists A \in [X \setminus \delta]^{\omega}$ *A is up-dense for F in* $\omega_1 \setminus Y$",

Proof. Assume that

$$1_{\mathcal{C}*P_\alpha} \Vdash X = \{\dot{x}_\xi : \xi < \omega_1\} \in [\omega_1]^{\omega_1}.$$

Pick determined conditions $p_\xi = \langle c_\xi, q_\xi\rangle \in \mathcal{C} * P_\alpha$ and $x_\xi \in \omega_1$ such that $p_\xi \Vdash$ "$\dot{x}_\xi = x_\xi$". We can assume that $x_\xi \in \operatorname{supp} c_\xi$. Write $J_\xi = \operatorname{dom} q_\xi$, $q_\xi(\eta) = \langle q_\xi(\eta,0), q_\xi(\eta,1)\rangle$ for $\eta \in J_\xi$ and $Z_\xi = \operatorname{supp}(c_\xi)$.

Now there is $K \in [\omega_1]^{\omega_1}$ such that the conditions $\{p_\xi : \xi \in K\}$ are "*pairwise twins*", i.e.

(1) $\{Z_\xi : \xi \in K\}$ forms a Δ-system with kernel Z,
(2) $\{J_\xi : \xi \in K\}$ forms a Δ-system with kernel J,
(3) $\max Z < \min(Z_\xi \setminus Z) < \max(Z_\xi \setminus Z) < \min(Z_{\xi'} \setminus Z)$ for $\xi < \xi' \in K$,
(4) $|Z_\xi| = |Z_{\xi'}|$ for $\{\xi,\xi'\} \in [K]^2$. Denote by $\varphi_{\xi,\xi'}$ the natural bijection between Z_ξ and $Z_{\xi'}$.
(5) $c_{\xi'}\big(\varphi_{\xi,\xi'}(\nu), \varphi_{\xi,\xi'}(\nu')\big) = c_\xi(\nu,\nu')$ for $\langle \nu,\nu'\rangle \in Z_\xi \times Z_\xi$ and $\{\xi,\xi'\} \in [K]^2$,
(6) $q_{\xi'}(\eta,i) = \{\langle \varphi_{\xi,\xi'}(\nu), \varphi_{\xi,\xi'}(\nu')\rangle : \langle \nu,\nu'\rangle \in q_\xi(\eta,i)\}$ for $\eta \in J$, $i < 2$ and $\{\xi,\xi'\} \in [K]^2$.

Since $\mathcal{B}_\eta$ is a partition of ω_1 into countable pieces for $\eta \in J$, there is a club set $C = \{\gamma_\nu : \nu < \omega_1\} \subset \omega_1$ in $V^{\mathcal{C}*\mathcal{P}_\alpha}$ such that for each $\eta \in J$ and $\nu < \omega_1$ we have

$$[\gamma_\nu, \gamma_{\nu+1}) = \bigcup\{B \in \mathcal{B}_\eta : B \cap [\gamma_\nu, \gamma_{\nu+1}) \ne \emptyset\}.$$

Since $\mathcal{C} * P_\alpha$ is c.c.c we can assume that $C \in V$.

By thinning out K we can assume that if $\xi < \xi' \in K$ then there is $\gamma \in C$ such that $\max(Z_\xi \setminus Z) < \gamma < \min(Z_{\xi'} \setminus Z)$, moreover $\max Z < \min C$.

By lemma 2.10 fix $\mu \in C$ such that $\delta \le \mu$ and $1_{\mathcal{C}*P_\alpha} \Vdash$ "$\{f_\eta : \eta \in J\}$ *and* $\{g_\eta : \eta \in J\}$ *act loop-free on* $\omega_1 \setminus \mu$".

A pair $\langle \vec{\eta}, \vec{k}\rangle$ is called *relevant* iff $\vec{\eta} = \langle \eta_0, \ldots, \eta_{n-1}\rangle \in {}^nJ$ and $\vec{k} = \langle k_0, \ldots, k_{n-1}\rangle \in {}^n\{-1,+1\}$ for some $n \in \omega$. For each relevant pair $\langle \vec{\eta}, \vec{k}\rangle$ let

$$f_{\langle \vec{\eta}, \vec{k}\rangle} = f_{\eta_{n-1}}^{k_{n-1}} \circ \cdots \circ f_{\eta_0}^{k_0}$$

and

$$g_{\langle \vec{\eta}, \vec{k}\rangle} = g_{\eta_{n-1}}^{k_{n-1}} \circ \cdots \circ g_{\eta_0}^{k_0}$$

If $p = \langle c, q\rangle$ is determined and $J \subset \operatorname{dom}(q)$ we define the *q-approximation of* $f_{\langle\vec\eta,\vec k\rangle}$, $f^q_{\langle\vec\eta,\vec k\rangle}$, in the natural way:

$$f^q_{\langle\vec\eta,\vec k\rangle} = q(\eta_{n-1},0)^{k_{n-1}} \circ \cdots \circ q(\eta_0,0)^{k_0}.$$

Similarly,

$$g^q_{\langle\vec\eta,\vec k\rangle} = q(\eta_{n-1},1)^{k_{n-1}} \circ \cdots \circ q(\eta_0,1)^{k_0}.$$

We say that $\left\langle\vec\eta,\vec k\right\rangle$ is *irreducible* if there is no $i < n-1$ such that $\eta_i = \eta_{i+1}$ and $\{k_i, k_{i+1}\} = \{-1,+1\}$.

Let $\xi \in K$ be arbitrary. An irreducible $\left\langle\vec\eta,\vec k\right\rangle$ is *active* iff $\operatorname{dom} f^q_{\langle\vec\eta,\vec k\rangle} \cap (Z_\xi \setminus Z) \neq \emptyset$ or $\operatorname{dom} g^q_{\langle\vec\eta,\vec k\rangle} \cap (Z_\xi \setminus Z) \neq \emptyset$, i.e., there is a sequence $\vec x = \langle x_0,\dots,x_{n-1}\rangle \in {}^n(Z_\xi \setminus Z)$ such that $x_{i+1} = q_\xi(\eta_i,0)^{k_i}(x_i)$ for $i<n$ or $x_{i+1} = q_\xi(\eta_i,1)^{k_i}(x_i)$ for $i<n$. Observe that the definition of activeness above does not depend on the choice ξ because the conditions $\{\langle c_\xi, q_\xi\rangle : \xi \in K\}$ are pairwise twins.

We say that $\vec x$ witnesses that $\left\langle\vec\eta,\vec k\right\rangle$ is active.

Let $K' \in \left[K\right]^\omega$, $\dot A = \{\langle p_\xi, x_\xi\rangle : \xi \in K'\}$ and $\zeta \in K \setminus K'$. Let $r^* = \langle c^*, q^*\rangle \le p_\zeta$ be a determined condition such that for each active $\left\langle\vec\eta,\vec k\right\rangle$ and $w \in Z$ the values $f^{q^*}_{\langle\vec\eta,\vec k\rangle}(w)$ and $g^{q^*}_{\langle\vec\eta,\vec k\rangle}(w)$ are defined.

Let

$$Y = \{g^{r^*}_{\langle\vec\eta,\vec k\rangle}(w) : f_{\langle\vec\eta,\vec k\rangle} \text{ is active and } w \in Z\}.$$

Claim . Y is finite.

Proof of the claim. Since $\{f_\eta : \eta \in J\}$ and $\{g_\eta : \eta \in J\}$ act loop-free on $Z_\zeta \setminus Z$, the elements of a witnessing sequence are pairwise different, so there are only finitely many of them and a witnessing sequence works only for one active $\left\langle\vec\eta,\vec k\right\rangle$. So there is only finitely many active $\left\langle\vec\eta,\vec k\right\rangle$. □

We show that

$$(\bullet) \qquad r^* \Vdash \dot A \text{ is up-dense in } \omega_1 \setminus Y \text{ for } F.$$

Indeed, assume that $r' \le r^*$, $r' = \langle c', q'\rangle$ is determined, $B \in \left[\omega_1 \setminus Y\right]^{<\omega}$ and $b \in {}^B 2$.

Pick $\xi \in K$ such that $\operatorname{supp}(c') \cap \operatorname{supp}(c_\xi) = Z$ and $\operatorname{dom}(q') \cap \operatorname{dom}(q_\xi) = J$. To complete the proof of the lemma it is enough to construct a common extension $p = \langle c, q\rangle$ of $r' = \langle c', q'\rangle$ and $p_\xi = \langle c_\xi, q_\xi\rangle$ such that $c(x_\xi, \beta) = b(\beta)$ for each $\beta \in B$.

Let $\operatorname{supp}(c) = \operatorname{supp}(c') \cup \operatorname{supp}(c_\xi)$. Put $\operatorname{dom} q = \operatorname{dom}(q') \cup \operatorname{dom}(q_\xi)$ and for $i < 2$ let

$$q(\eta, i) = \begin{cases} q'(\eta,i) \cup q_\xi(\eta,i) & \text{if } \eta \in J,\\ q'(\eta,i) & \text{if } \eta \in \operatorname{dom} q' \setminus J,\\ q_\xi(\eta,i) & \text{if } \eta \in \operatorname{dom} q_\xi \setminus J.\end{cases}$$

Put $c^- = c' \cup c_\xi$. Let $H = \operatorname{supp}(c')$ and $Z'_\xi = Z_\xi \setminus Z = Z_\xi \setminus H$.

Now $\operatorname{dom}(c) = \operatorname{supp}(c) \times \operatorname{supp}(c) = (H \cup Z'_\xi) \times (H \cup Z'_\xi) = (H \times H) \cup (H \cup Z'_\xi) \cup (Z'_\xi \times H) \cup (Z'_\xi \times Z'_\xi)$.

Definition 2.12. For $\langle a,b\rangle, \langle a',b'\rangle \in \operatorname{dom}(c)$ put $\langle a,b\rangle \equiv \langle a',b'\rangle$ iff there is $\left\langle \vec{\eta},\vec{k}\right\rangle$ such that $f^q_{\langle\vec{\eta},\vec{k}\rangle}(a)=a'$ and $g^q_{\langle\vec{\eta},\vec{k}\rangle}(b)=b'$.

We should define $c \supset c^-$ such that

$$(\star) \qquad \text{if } \langle a,b\rangle \equiv \langle a',b'\rangle \text{ then } c(a,b)=c(a',b'),$$

$$(\star\star) \qquad c(x_\xi,\beta)=b(\beta)$$

The first two claims are straightforward.

Claim 2.12.1. *The sets $(H\times H)$, $(H\cup Z'_\xi)$, $(Z'_\xi\times H)$ and $(Z'_\xi\times Z'_\xi)$ are $\equiv$-closed.*

Claim 2.12.2. *Assume that $\langle a,b\rangle \equiv \langle a',b'\rangle$. If $\langle a,b\rangle\in Z'_\xi\times H$ then there is an active $\left\langle \vec{\eta},\vec{k}\right\rangle$ such that $a'=f^{q_\xi}_{\langle\vec{\eta},\vec{k}\rangle}(a)$ and $b'=g^{q'}_{\langle\vec{\eta},\vec{k}\rangle}(b)$. If $\langle a,b\rangle\in H\times Z'_\xi$ then there is an active $\left\langle \vec{\eta},\vec{k}\right\rangle$ such that $a'=f^{q'}_{\langle\vec{\eta},\vec{k}\rangle}(a)$ and $b'=g^{q_\xi}_{\langle\vec{\eta},\vec{k}\rangle}(b)$.*

Claim 2.12.3. *Assume that $\langle a,b\rangle \equiv \langle a',b'\rangle$. If $\langle a,b\rangle, \langle a',b'\rangle\in Z'_\xi\times H$ and $a=a'$ then $\langle a,b\rangle = \langle a',b'\rangle$. If $\langle a,b\rangle, \langle a',b'\rangle\in H\times Z'_\xi$ and $b=b'$ then $\langle a,b\rangle = \langle a',b'\rangle$.*

Proof of the claim 2.12.3. Assume first that $\langle a,b\rangle, \langle a',b'\rangle\in Z'_\xi\times H$ and $b\neq b'$. By 2.12.2 there is an active $\left\langle \vec{\eta},\vec{k}\right\rangle$ such that $a'=f^{q_\xi}_{\langle\vec{\eta},\vec{k}\rangle}(a_e)$ and $b'=g^{q'}_{\langle\vec{\eta},\vec{k}\rangle}(b)$. Since $b\neq b'$ it follows that $\left\langle \vec{\eta},\vec{k}\right\rangle \neq \langle\emptyset,\emptyset\rangle$. Since $1\Vdash$ "$\{f_\eta:\eta\in J\}$ acts loop-free on $\omega_1\setminus\mu$" it follows that $a\neq f^{q'}_{\langle\vec{\eta},\vec{k}\rangle}(a)$ and so $a\neq a'$.

If $\langle a,b\rangle, \langle a',b'\rangle\in H\times Z'_\xi$ then the same arguments work using that $1\Vdash$ "$\{g_\eta: \eta\in J\}$ acts loop-free on $\omega_1\setminus\mu$". □

Claim 2.12.4. *If $\langle a,b\rangle, \langle a',b'\rangle\in \big((Z'_\xi\times H)\cup(H\times Z'_\xi)\big)\cap\operatorname{dom}(c^-)$ and $\langle a,b\rangle \equiv \langle a',b'\rangle$ then $c^-(a,b)=c^-(a',b')$.*

Proof of the claim 2.12.4. Assume first that $\langle a,b\rangle, \langle a',b'\rangle\in Z'_\xi\times H$. Fix an active $\left\langle \vec{\eta},\vec{k}\right\rangle$ such that $a'=f^{q_\xi}_{\langle\vec{\eta},\vec{k}\rangle}(a)$ and $b'=g^{q'}_{\langle\vec{\eta},\vec{k}\rangle}(b)$. Since $\langle a,b\rangle, \langle a',b'\rangle\in\operatorname{dom}(c^-)$ it follows that $\langle a,b\rangle, \langle a',b'\rangle\in\operatorname{dom}(c_\xi)$ and so $b,b'\in Z$. If $b\in\operatorname{dom} g^{q_\xi}_{\langle\vec{\eta},\vec{k}\rangle}$, then we are done because in this case $b'=g^{q_\xi}_{\langle\vec{\eta},\vec{k}\rangle}(b)$ and so $c^-(a,b)=c_\xi(a,b)=c_\xi(a',b')=c^-(a',b')$ for $\langle c_\xi,q_\xi\rangle\in\mathcal{C}\times\mathcal{P}_\alpha$. Unfortunately, $b_e\in\operatorname{dom} g^{q_\xi}_{\langle\vec{\eta},\vec{k}\rangle}$ can not be guaranteed, so we need an additional argument here.

Let $\varphi=\varphi_{\xi,\zeta}$ be the function witnessing that p_ξ and p_ζ are twins.

Since $\left\langle \vec{\eta},\vec{k}\right\rangle$ is active and $b\in Z$ it follows that $g^{q^*}_{\langle\vec{\eta},\vec{k}\rangle}(b)$ is defined and so $g^{q^*}_{\langle\vec{\eta},\vec{k}\rangle}(b)=b'$. Put $\underline{b}=\varphi(b)$, $\underline{b}'=\varphi(b)$ $\underline{a}=\varphi(a)$ and $\underline{a}'=\varphi(a')$. Since $c^-(a,b)=c_\xi(a,b)=c_\zeta(\underline{a},\underline{b})=c^*(\underline{a},\underline{b})$ and $c^-(a',b')=c_\xi(a',b')=c_\zeta(\underline{a}',\underline{b}')=c^*(\underline{a}',\underline{b}')$ it is enough to show that $c^*(\underline{a},\underline{b})=c^*(\underline{a}',\underline{b}')$.

First observe that $\underline{b}=\varphi(b)=b$, $\underline{b}'=\varphi(b')=b'$ and so $\underline{b}'=f^{q^*}_{\langle\vec{\eta},\vec{k}\rangle}(\underline{b})$. Moreover $\underline{a}'=\varphi(a')=\varphi(f^{q_\zeta}_{\langle\vec{\eta},\vec{k}\rangle}(a))=f^{q_\zeta}_{\langle\vec{\eta},\vec{k}\rangle}(\varphi(a))=f^{q_\zeta}_{\langle\vec{\eta},\vec{k}\rangle}(\underline{a})=f^{q^*}_{\langle\vec{\eta},\vec{k}\rangle}(\underline{a})$. Thus using

$r^* = \langle c^*, q^* \rangle \leq \langle c_\zeta, q_\zeta \rangle$ we have

$$c^*(\underline{a}', \underline{b}') = c^*(f^{q^*}_{\langle \vec{\eta}, \vec{k} \rangle}(\underline{a}), f^{q^*}_{\langle \vec{\eta}, \vec{k} \rangle}(\underline{b})) = c^*(\underline{a}, \underline{b}),$$

which completes the proof of the claim. □

Claim 2.12.5. *If $\langle a, b \rangle \in (Z'_\xi \times H) \cap \operatorname{dom}(c^-)$ and $\langle a, b \rangle \equiv \langle a', b' \rangle$ then $b' \in Y$.*

Proof of the claim 2.12.5. Since $\langle a, b \rangle \in (Z'_\xi \times H) \cap \operatorname{dom}(c^-)$ we have $b \in Z$. Fix an active $\left\langle \vec{\eta}, \vec{k} \right\rangle$ such that $a' = f^{q_\xi}_{\langle \vec{\eta}, \vec{k} \rangle}(a)$ and $b' = f^{q'}_{\langle \vec{\eta}, \vec{k} \rangle}(b)$. Since $\left\langle \vec{\eta}, \vec{k} \right\rangle$ is active it follows that $g^{q^*}_{\langle \vec{\eta}, \vec{k} \rangle}(b)$ is defined and $g^{q^*}_{\langle \vec{\eta}, \vec{k} \rangle}(b) \in Y$. But $g^{q'}_{\langle \vec{\eta}, \vec{k} \rangle}(b) = g^{q^*}_{\langle \vec{\eta}, \vec{k} \rangle}(b)$ so $b' \in Y$ which was to be proved. □

By claims 2.12.3–2.12.5 we can find a condition $c \in \mathcal{C}$ with $\operatorname{supp} c = \operatorname{supp} c' \cup \operatorname{supp} c_\xi$ and $\operatorname{dom} c = \operatorname{supp} c \times \operatorname{supp} c$ such that

(a) $c \supset c' \cup c_\xi$,
(b) $(\star)$ holds, i.e. $c(a, b) = c(a', b')$ whenever $\langle a, b \rangle \equiv \langle a', b' \rangle$,
(c) $c(x_\xi, \beta) = b(\beta)$ for $\beta \in B$.

Then by $(\star)$ we have $\langle c, q \rangle \in \mathcal{C} * \mathcal{P}_\alpha$ and

$$\langle c, q \rangle \Vdash (\forall \beta \in \operatorname{dom} b)\ F(x_\xi, \beta) = b(\beta).$$

Thus $(\bullet)$ holds. Hence lemma 2.11 is proved. □

So we have shown that (II) is preserved during the inductive construction, which was the last step to prove theorem 2.1 □

References

[1] A. Hajnal, Zs. Nagy, L. Soukup, *On the number of non-isomorphic subgraphs of certain graphs without large cliques and independent subsets*, in "A TRIBUTE TO PAUL ERDŐS ", e.d. A Baker, B. Bollobás, A. Hajnal, Cambridge University Press, 1990, p. 223-248
[2] T. Jech, *Set Theory*, Academic Press. New York, 1978.
[3] J. Steprans, *Steprans' Problems*, in OPEN PROBLEMS IN TOPOLOGY (J. van Mill and G.M. Reed eds.), North-Holland, Amsterdam, 1990 pp. 13-20.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, P.O. BOX 127, H-1364, BUDAPEST, HUNGARY
E-mail address: soukup@renyi.hu