THE D-PROPERTY ON UNIONS OF SCATTERED SPACES

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Abstract. We show in a direct way that a space is D if it is a finite union of subparacompact scattered spaces. This result can not be extended to countable unions, since it is known that there is a regular space which is a countable union of paracompact scattered spaces and which is not D. Nevertheless, we show that every space which is the union of countably many regular Lindelöf $\mathcal{C}$-scattered spaces has the D-property. Also, we prove that a space is D if it is a locally finite union of regular Lindelöf $\mathcal{C}$-scattered spaces.

1. Introduction

All spaces under consideration are Hausdorff. An *open neighbourhood assignment* (ONA) for a space X is a function η from X to the topology of X such that $x \in \eta(x)$ for every $x \in X$. If Y is a subset of X, we write $\eta[Y] = \bigcup\{\eta(y) : y \in Y\}$. Then, we say that X is a *D-space*, if for every open neighbourhood assignment η for X there is a closed discrete subset D of X such that $\eta[D] = X$.

It is obvious that every compact space is a D-space. However, it is not known whether every Lindelöf space is D, and it is also unknown whether the D-property is implied by paracompactness, subparacompactness or metacompactness. Nevertheless, it was shown in [5] that on the class of generalized ordered spaces paracompactness is equivalent to the D-property, and it was proved in [7] that for subspaces of finite produts of ordinals property D is equivalent to metacompactness.

On the other hand, it is known that some (finite unions of) generalized metric spaces are D (see [1], [2], [4], [7] and [9]).

The relationship between D-spaces and topological games was studied by Peng (see [8] and [9]). Recall that a space X is *scattered*, if every nonempty closed subspace of X has an isolated point. And we say that a space X is *$\mathcal{C}$-scattered*, if every nonempty closed subspace Y of X has a point with a compact neighbourhood in Y. By means of stationary strategies on the topological game $G(\mathbf{DC}, X)$ defined in [10], it was shown by Peng in [9] that if a space X is the union of finitely many regular subparacompact $\mathcal{C}$-scattered spaces, then X has the D-property. In this paper, we shall prove in a direct way (without using topological games) that a space is D if it is a finite union of subparacompact scattered spaces. Also, the space constructed by van Douwen and Wicke in [6] provides an example of a regular space which is a union of countably many paracompact scattered spaces but which is not D. However, by using the compact-open game, we shall prove that a space

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is D if it is a countable union of regular Lindelöf $\mathcal{C}$-scattered spaces. Also, we shall prove that a space is D if it is a locally finite union of regular Lindelöf $\mathcal{C}$-scattered spaces.

2. Proofs of the results for countable unions

First, we consider the Cantor-Bendixson process for topological spaces. For any space X and any ordinal α, we define the *α-derivative of* X as follows: $X^0 = X$; if $\alpha = \beta + 1$, $X^\alpha = \{x \in X : x$ is an accumulation point of $X^\beta\}$; and if α is a limit, $X^\alpha = \bigcap\{X^\beta : \beta < \alpha\}$. It is well-known that a space X is scattered iff there is an ordinal α such that $X^\alpha = \emptyset$. Then, we define the *height of* a scattered space X by $\mathrm{ht}(X) =$ the least ordinal α such that $X^\alpha = \emptyset$.

If X is a scattered space and α is an ordinal, we write $I_\alpha(X) = X^\alpha \setminus X^{\alpha+1}$. Intuitively, $I_\alpha(X)$ denotes the set of points of X which are at level α. Now, assume that x is a point of a scattered space X and U is a neighbourhood of x. Let β be the ordinal such that $x \in I_\beta(X)$. Then, we say that U is a *cone on x* if $U \cap X^\beta = \{x\}$. Clearly, every point x of a scattered space has a local base whose elements are cones on x.

By a *basic ONA* for a scattered space X we mean a function η that assigns to every point x of X an open cone $\eta(x)$ on x. The following lemma is easy to prove.

Lemma 2.1. *Let X be a scattered space. Assume that for every basic ONA η for X there is a closed discrete subset D of X such that $X = \eta[D]$. Then, X is a D-space.*

We shall use without explicit mention Lemma 2.1 and also the well-known fact that the D-property is hereditary with respect to closed subspaces.

Now, our aim is to give a direct proof for the following result, in which we use a modification of the argument given in [1, Theorem 1.4].

Theorem 2.1. *If a space X is the union of a finite collection of subparacompact scattered spaces, then X is D.*

Proof. It is easy to check that any space which is the union of finitely many scattered spaces is also scattered. Then, suppose that $X = X_1 \cup \cdots \cup X_k$ where $X_1, \ldots, X_k$ are subparacompact scattered spaces. We proceed by induction on k. If $k = 0$, we have $X = \emptyset$, and so we are done. Now assume that the statement holds for $k = l$ for some $l \geq 0$. Then, in order to show that the statement holds for $k = l + 1$, we proceed by transfinite induction on the height α of X. The case $\alpha = 0$ is trivial. Suppose that $\alpha > 0$ and that the statement holds for spaces of height $< \alpha$ which are unions of at most k subparacompact scattered spaces. First, assume that $\alpha = \beta + 1$ is a successor ordinal. Let η be a basic ONA for X. Put $D = I_\beta(X)$. Let $Z = X \setminus \eta[D]$. Since Z is closed in X and $\mathrm{ht}(Z) < \alpha$, we infer that Z is D by the induction hypotheses. Let E be a closed discrete subset of Z such that $\bigcup\{\eta(x) \cap Z : x \in E\} = Z$. Then, it is easy to check that $D \cup E$ is a closed discrete subset of X and $\eta[D \cup E] = X$.

Now, assume that α is a limit ordinal and η is a basic ONA for X. Since each X_i is subparacompact, we have that for $1 \leq i \leq k$ there is a covering $\mathcal{P}_i = \bigcup\{\eta_{ij} : j \geq 0\}$ of X_i satisfying the following:

(1) Each element of $\mathcal{P}_i$ is a closed subset of X_i,
(2) $\mathcal{P}_i$ is a refinement of $\{\eta(x) \cap X_i : x \in X_i\}$,

(3) η_{ij} is discrete in X_i for every $j \geq 0$.

Now for every $n \geq 0$ let

$$\eta_n = \bigcup\{\eta_{in} : 1 \leq i \leq k\}.$$

Since $\eta(x)$ is a cone on x for every $x \in X$, by using (2), we deduce that $\mathrm{ht}(V) < \alpha$ for every $V \in \eta_n$. Hence, by using (1), we infer from the induction hypotheses that every element of η_n is D.

Proceeding by induction on $n \geq 0$ we define closed discrete subsets E_n, D_n of X with $E_n \subseteq D_n$ such that the following holds:

$(*)_n$ *(a)* $\bigcup\{\eta[D_m] : m \leq n\} \supseteq \bigcup(\eta_0 \cup \cdots \cup \eta_n)$,

(b) $D_n \setminus E_n \subseteq (\bigcup \eta_n) \setminus \eta[E_n]$,

(c) $\eta_0 \cup \cdots \cup \eta_n$ is locally finite at every point of $X \setminus (\eta[E_n] \cup \bigcup\{\eta[D_m] : m < n\})$,

(d) $D_n \cap \bigcup\{\eta[D_m] : m < n\} = \emptyset$.

First, assume $n = 0$. For $i \in \{1, \ldots, k\}$ let $F_{i0} = \{x \in X : \eta_{i0}$ is not locally finite at $x\}$. Clearly, for $1 \leq i \leq k$, F_{i0} is closed in X and $F_{i0} \subseteq X \setminus X_i$, and so F_{i0} is D by the induction hypothesis. Now let $F_0 = \bigcup\{F_{i0} : i \in \{1, \ldots, k\}\}$. Clearly, F_0 is the set of all $x \in X$ such that η_0 is not locally finite at x. Since F_0 is a finite union of closed D subspaces of X, we have that F_0 is also D. So, let E_0 be a closed discrete subset of F_0 such that $F_0 \subseteq \eta[E_0]$. Put $W_0 = \eta[E_0]$. Since every element of η_0 is D and W_0 is open, we may consider for every $V \in \eta_0$ a closed discrete subset D_V of $V \setminus W_0$ such that $\{\eta(x) \setminus W_0 : x \in D_V\}$ covers $V \setminus W_0$. Then, we put $D_0 = E_0 \cup \bigcup\{D_V : V \in \eta_0\}$. It is easy to check that D_0 is a closed discrete subset of X and that condition $(*)_0$ holds.

Now assume that $n \geq 1$. Let

$$U_n = \bigcup\{\eta[D_m] : m < n\}.$$

Proceeding as above, we have that $F_n = \{x \in X : \eta_n$ is not locally finite at $x\}$ is D, and so $F_n \setminus U_n$ is also D. Let E_n be a closed discrete subset of $F_n \setminus U_n$ such that $\eta[E_n] \supseteq F_n \setminus U_n$. Put $W_n = U_n \cup \eta[E_n]$. Then, for every $V \in \eta_n$ we consider a closed discrete subset D_V of $V \setminus W_n$ such that $\{\eta(x) \setminus W_n : x \in D_V\}$ covers $V \setminus W_n$. Now we set $D_n = E_n \cup \bigcup\{D_V : V \in \eta_n\}$. We have that D_n is a closed discrete subset of X and condition $(*)_n$ holds.

Finally, we define $D = \bigcup\{D_n : n \geq 0\}$. It is easy to see that D is as required. □

We do not know whether it is possible to prove in a direct way the result given in [9, Corollary 9]. For this, note that if we want to refine the argument given in the proof of Theorem 2.1, first we should show in a direct way that any locally compact space which is the union of finitely many subparacompact $\mathcal{C}$-scattered spaces has the D-property.

Let $\mathbf{K}_1$ be the class of all regular Lindelöf spaces, $\mathbf{K}_2$ be the class of all paracompact spaces and $\mathbf{K}_3$ be the class of all subparacompact spaces. Note that if every dense in itself space of $\mathbf{K}_i$ is D, then every space of $\mathbf{K}_i$ is D $(1 \leq i \leq 3)$. To check this point, assume that every dense in itself space of $\mathbf{K}_i$ is D and consider a space

X in $\mathbf{K}_i$. If X is scattered, we are done. Otherwise, consider the least ordinal β such that $X^\beta = X^{\beta+1}$. Let Z be the space X^β with the relative topology of X. Since Z is a dense in itself space of $\mathbf{K}_i$, we infer that Z is D. Assume that η is an ONA for X. Without loss of generality, we may assume that if $x \notin X^\beta$, then $\eta(x)$ is a cone on x in the space $X \setminus X^\beta$. Since Z is D, there is a closed discrete subset D of Z such that $Z \subseteq \eta[D]$. Let $Y = X \setminus \eta[D]$. Since Y is a scattered space of $\mathbf{K}_i$, we have that Y is D, and so there is a closed discrete subset E of Y such that $\eta[E] \supseteq Y$. Then, it is easy to see that $D \cup E$ is a closed discrete subset of X, and clearly $X = \eta[D \cup E]$.

Note that in contrast with Theorem 2.1 there is a large class of compact scattered spaces which are not hereditarily D. To check this point, for every ordinal α let $T(\alpha)$ be the ordinal α equipped with the order topology. Then, it is easy to check that if α is a limit ordinal of uncountable cofinality, then $T(\alpha+1)$ is compact scattered but $T(\alpha)$ is not D.

It was shown in [8] that every metacompact scattered space of countable height is D. However, it is not known whether every metacompact scattered space is D. Note that the notions of "paracompactness", "subparacompactness" and "metacompactness" are not equivalent on scattered spaces (see [3, Examples 4.2, 4.3 and 4.4]). Also, by slightly modifying the constructions explained in [3, Examples 4.2 and 4.3], we can show that for every ordinal $\alpha \geq 2$, there are spaces X_α, Y_α such that $\mathrm{ht}(X_\alpha) = \mathrm{ht}(Y_\alpha) = \alpha$, X_α is metacompact but not subparacompact, and Y_α is a metacompact subparacompact space which is not paracompact.

On the other hand, since the space constructed by van Douwen and Wicke in [6] is a regular scattered space of height ω which is not D, that space is an example of a disjoint union of countably many regular paracompact scattered spaces which does not have property D. However, we can prove the following result.

Theorem 2.2. *If a space X is the union of a countable collection of regular Lindelöf $\mathcal{C}$-scattered spaces, then X is D.*

In order to prove Theorem 2.2, we will use the *compact-open game* $G(X)$ for a space X, which is defined as follows. There are two players I and II, and there are ω moves in a play. In each move i of a play there are two steps. At the first step, player I chooses a compact subspace C_i of X; at the second step, player II chooses an open set V_i in X such that $C_i \subseteq V_i$. Player I wins the play, if $X = \bigcup_{i\geq 0} V_i$. Otherwise, player II wins.

Lemma 2.2. *If player I has a winning strategy in the compact-open game $G(X)$, then X is a D-space.*

Proof. Assume that player I has a winning strategy in the game $G(X)$. Towards showing that X is D, let η be an ONA for X. Then, we consider the following play $\langle C_0, V_0, C_1, V_1, \dots \rangle$ in the game $G(X)$: in the i-th move,

(1) player I uses his winning strategy to pick C_i,
(2) if $C_i \not\subseteq \bigcup\{V_j : j < i\}$, then we consider a finite subset F_i of $C_i \setminus \bigcup\{V_j : j < i\}$ such that $\eta[F_i] \supseteq C_i \setminus \bigcup\{V_j : j < i\}$ and we put $V_i = \bigcup\{V_j : j < i\} \cup \eta[F_i]$,
(3) if $C_i \subseteq \bigcup\{V_j : j < i\}$, then we let $V_i = \bigcup\{V_j : j < i\}$.

Put
$$a = \{i \in \omega : C_i \not\subseteq \bigcup\{V_j : j < i\}\}.$$

Since I uses his winning strategy in $G(X)$, we have $X = \bigcup_{i\in\omega} V_i$. But $\bigcup_{i\in\omega} V_i = \bigcup_{i\in a} V_i = \eta[\bigcup\{F_i : i \in a\}]$. Also, since $F_k \cap V_i = \emptyset$ for every $i, k \in a$ with $i < k$, we deduce that $\bigcup\{F_i : i \in a\}$ is a closed discrete subset of X. □

The converse of Lemma 2.2 is not true even for Lindelöf spaces, since it is well-known that the Sorgenfrey line S is a Lindelöf D-space and, by [10, Theorem 5.12], we have that player II has a winning strategy in the game $G(S)$. Note also that every uncountable discrete space X is an example of a scattered D-space such that player II has a winning strategy in the game $G(X)$.

Lemma 2.3. *Assume that $X = \bigcup\{X_n : n \geq 0\}$. Suppose that player I has a winning strategy in $G(X_n)$ for every $n \geq 0$. Then, player I has also a winning strategy in the game $G(X)$.*

Proof. Let $\{a_n : n \geq 0\}$ be a partition of ω in infinite subsets. We describe a winning strategy of player I in the game $G(X)$: play I just uses his winning strategy for $G(X_n)$ in the steps whose indexes are in a_n. □

Note that Lemma 2.3 can not be extended to most of the games studied in [10].

The proof for the following lemma is given in [10, Theorem 9.3].

Lemma 2.4. *If X is a regular Lindelöf $\mathcal{C}$-scattered space, then player I has a winning strategy in the game $G(X)$.*

Now, in order to prove Theorem 2.2, we obtain from Lemmas 2.3 and 2.4 that if X is a countable union of regular Lindelöf $\mathcal{C}$-scattered spaces, then player I has a winning strategy in the game $G(X)$, and hence we deduce from Lemma 2.2 that X has property D.

The following result is an immediate consequence of Theorem 2.2.

Corollary 2.1. *(a) If a space X is the union of a countable collection of regular Lindelöf scattered spaces, then X is D.*

(b) If a space X is the union of a countable collection of Lindelöf locally compact spaces, then X is D.

3. A result for locally finite unions

If a space X is a locally finite union of closed D subspaces, then X is D. It is not clear whether we can drop the assumption *closed* from the statement above. However, we can do this provided the subspaces are not just D, but player I has a winning strategy in the compact-open game !

Theorem 3.1. *Assume that X is a space and $\mathcal{Y}$ is a locally finite cover of X such that for each $Y \in \mathcal{Y}$, player I has a winning strategy in the game $G(Y)$. Then X is D.*

Proof. Let η be an ONA for X. We can assume that for each $x \in X$ the set

(∗) $\{Y \in \mathcal{Y} : Y \cap \eta(x) \neq \emptyset\}$ is finite.

Write $\mathcal{Y} = \{Y_\alpha : \alpha < \kappa\}$.

We will consider a play τ of $\kappa \cdot \omega$ moves such that for each $\alpha < \kappa$, in the moves $\kappa \cdot n + \alpha$ for $n \geq 0$ player I uses his winning strategy in the game $G(Y_\alpha)$. More precisely, for $\xi < \kappa \cdot \omega$ write $\xi = \kappa \cdot i_\xi + \alpha_\xi$ where $i_\xi < \omega$ and $\alpha_\xi < \kappa$. Then, in the ξ-move of the play τ we carry out the i_ξ-move in the game $G(Y_{\alpha_\xi})$ as follows.

Assume that we have C_ζ and U_ζ for $\zeta < \xi$. Then, player I uses his winning strategy in the game $G(Y_{\alpha_\xi})$ for the play

$$\langle\langle C_{\kappa\cdot i+\alpha_\xi}, U_{\kappa\cdot i+\alpha_\xi} \cap Y_{\alpha_\xi}\rangle : i < i_\xi\rangle$$

to get C_ξ. Now if $C_\xi \subseteq \bigcup\{U_\zeta : \zeta < \xi\}$, player II chooses $U_\xi = \bigcup\{U_\zeta : \zeta < \xi\}$. Otherwise, player II takes a finite subset F_ξ of $C_\xi \setminus \bigcup\{U_\zeta : \zeta < \xi\}$ such that $\eta[F_\xi] \supseteq C_\xi \setminus \bigcup\{U_\zeta : \zeta < \xi\}$ and then he chooses $U_\xi = \bigcup\{U_\zeta : \zeta < \xi\} \cup \eta[F_\xi]$.

Put

$$a = \{\xi \in \kappa \cdot \omega : C_\xi \not\subseteq \bigcup\{U_\zeta : \zeta < \xi\}\}$$

and

$$D = \bigcup\{F_\xi : \xi \in a\}.$$

Claim 1. *$\eta[D]$ covers X.*
Let $z \in X$. Pick $\alpha < \kappa$ with $z \in Y_\alpha$. Write $\alpha_i = \kappa \cdot i + \alpha$ for $i < \omega$ and consider the sequence

$$C_{\alpha_0}, U_{\alpha_0} \cap Y_\alpha, C_{\alpha_1}, U_{\alpha_1} \cap Y_\alpha, \ldots, C_{\alpha_i}, U_{\alpha_i} \cap Y_\alpha, C_{\alpha_{i+1}}, U_{\alpha_{i+1}} \cap Y_\alpha, \ldots$$

This sequence is a play in $G(Y_\alpha)$, where I played according with his winning strategy. Thus $z \in \bigcup\{U_{\alpha_i} : i < \omega\}$. Let $\xi = \min\{\zeta \in \kappa \cdot \omega : z \in U_\zeta\}$.

If $\xi \notin a$, then $U_\xi = \bigcup\{U_\zeta : \zeta < \xi\}$, and so $z \in U_\zeta$ for some $\zeta < \xi$, which contradicts the minimality of ξ. Thus $\xi \in a$, and so $U_\xi = \bigcup\{U_\zeta : \zeta < \xi\} \cup \eta[F_\xi]$. Hence, again by the minimality of ξ, we have that $z \in \eta[F_\xi]$..

Claim 2. *D is closed discrete.*
Let $z \in X$. We show that z is not an accumulation point of D. By Claim 1, we can take $\xi = \min\{\zeta \in a : z \in \eta[F_\zeta]\}$.

Then, $\eta[F_\xi] \cap D$ is finite. For this, suppose that $\sigma \in a$. Indeed, if $\sigma > \xi$ then $F_\sigma \cap \eta[F_\xi] = \emptyset$ by the way in which F_σ is chosen. Now, assume that $\sigma < \xi$ and $F_\sigma \cap \eta[F_\xi] \neq \emptyset$. Put $\sigma = \kappa \cdot i_\sigma + \alpha_\sigma$. It follows that $\eta[F_\xi] \cap Y_{\alpha_\sigma} \neq \emptyset$. Moreover, $i_\sigma \leq i_\xi$. By $(*)$, the set $A = \{\alpha \in \kappa : \eta[F_\xi] \cap Y_\alpha \neq \emptyset\}$ is finite. Hence, σ should be an element of the finite set $\{\kappa \cdot i + \alpha : \alpha \in A \wedge i \leq i_\xi\}$. So z has a neighbourhood which intersects D in a finite set, which proves the claim. □

As a consequence of Theorem 3.1 and Lemma 2.4 , we obtain the following result.

Theorem 3.2. *If a space X is the union of a locally finite collection of regular Lindelöf $\mathcal{C}$-scattered spaces, then X is D.*

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