PARTITIONS INTO POWER-HOMOGENEOUS PATHS

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1. Introduction

Our aim is to prove theorems stating that certain edge-colored graphs can be partitioned into monochromatic paths or powers of paths.

Investigations began in the 80s with a results of Rado [5] implying that the every r-edge colored ($r \in \omega$) complete graph on ω can be partitioned into r monochromatic paths with different colors. Rado's result extends to finite complete graphs with 2-edge colorings, however by increasing the number of colors one runs into difficulties. Indeed, Kathy Heinrich constructed colorings of K_n with $r \geqslant 3$ colors so that there is no r-partition of K_n to paths with different colors. However, A. Pokrovskiy [3] quite recently proved that one can partition a 3-colored K_n into 3 monochromatic paths. Again in 1986, Gyárfás [2] showed that for every $r \in \omega$ there is $f(r) \in \omega$ so that for any r-edge coloring of K_n there is a cover of K_n by $\leqslant f(r)$ monochromatic paths.

We extend these results by proving ...

Naturally, one would like to extend the above results to graphs with fewer edges and hence, we turn to complete bipartite graphs. Pokrovskiy [3] proves (also follows from Gyárfás, Lehel ???, see [3]) that for every 2-edge coloring of $K_{n,n}$ there is a 3-partition to monochromatic paths and this result is sharp. Furthermore, Haxell proved that for all $r \in \omega$ there is $C_r \in \omega$ so that every r-edge colored $K_{n,n}$ partitions into at most C_r-many monochromatic cycles (in particular, paths).

Again, we extend this line of research by proving that ...

2. Preliminaries

$$\mathrm{N}_G(v) = \{u \in \mathrm{V}(G) : uv \in \mathrm{E}(G)\}.$$

$$\mathrm{N}_G(A) = \bigcap\nolimits_{v \in A} \mathrm{N}_G(v).$$

Definition 2.1. Let G be a graph and $c : E(G) \to \nu$ a coloring of the edges of G. We say that the sequence of vertices $P = \langle p_\alpha : \alpha < \kappa \rangle$ of G is a *path* iff

(1) $(p_\alpha, p_{\alpha+1}) \in E(G)$ for all $\alpha < \kappa$,
(2) $\sup\{\alpha < \beta : (p_\alpha, p_\beta) \in E(G)\} = \beta$ for all limit $\beta < \kappa$.

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A path P is *monochromatic* in some color $i < \nu$ (or a *path in color* i) iff

(1′) $c(p_\alpha, p_{\alpha+1}) = i$ for all $\alpha < \nu$,
(2′) $\sup\{\alpha < \beta : (p_\alpha, p_\beta) \in E(G) \text{ and } c(p_\alpha, p_\beta) = i\} = \beta$ for all limit $\beta < \nu$.

Definition 2.2. Let G be a graph and $c : E(G) \to \nu$ a coloring of the edges of G and κ a cardinal. A subset $A \subseteq V(G)$ is *κ-connected in color i* for some $i < \nu$ iff for every $a, b \in A$ and $B \in [A]^{<\kappa}$ there is a finite, path $P = \langle p_j : j < k\rangle$ in $(A \setminus B) \cup \{a, b\}$ in color i with $p_0 = a$ and $p_{k-1} = b$.

The following lemma is well-known and easy.

tra **Lemma 2.3.** *If G is a copy of K_ω, moreover $c : E(G) \to r$ is a coloring of the edges of G with finitely many colors, then there is a function $d_c : \mathrm{V}(G) \to r$ and there is a color $j_c \in r$ such that*

($*$) *for each finite subset U of $\mathrm{V}(G)$ there is $v \in \mathrm{V}(G)$ such that $d(v) = j_c$ and $c(u, v) = d_c(u)$ for all $u \in U$.*

3. Partitions of hypergraphs

A *loose path* in a k-uniform hypergraph is a sequence of edges, $e_1, e_2, \ldots$ such that for $|e_i \cap e_{i+1}| = 1$ and $e_i \cap e_j = \emptyset$ for $i + 1 < j$

A *tight path* is a sequence of distinct vertices where every consecutive set of k vertices forms an edge.

A. Gyárfás and G.N. Sárközy, [1, Theorem 3.], proved the following result: *Suppose that the edges of a countably infinite complete k-uniform hypergraph are colored with r colors. Then the vertex set can be partitioned into monochromatic finite or one-way infinite loose paths of distinct colors.*

Theorem 3.1. *Suppose that the edges of the countably infinite complete k-uniform hypergraph on ω are colored with r colors. Then the vertex set can be partitioned into monochromatic finite or one-way infinite tight paths of distinct colors.*

Proof. The case $k = 2$ was proved by Rado in [5]. We imitate his proof.

Let $c : \left[\omega\right]^k \to r$. A set $T \subset r$ of colors is called *prefect* iff there are vertex disjoint finite paths $\{P_t : t \in T\}$ and there is an infinite set A such that for all $t \in T$

(a) P_t is a tight monochromatic path in color t
(b) if $1 \leqslant i < k$ and x is the last i vertices from P_t and $y \in \left[A\right]^{k-i}$, then $c(x \cup y) = t$.

Let T be a perfect set of colors with maximal number of elements.

Claim 3.1.1. *If the vertex disjoint finite paths $\{P_t : t \in T\}$ and the infinite set A satisfy (a) and (b), then for all $v \in \omega \setminus \bigcup_{i \in T} P_t$ there is a color $t \in T$ and a finite sequence $v_1 v_2 \ldots v_{k-1}$ from A, and an infinite set $A' \subset A$ such that the paths*

$$\{P_s : s \in T \setminus \{t\}\} \cup \{P_t^\frown v_1 v_2 \ldots v_{k-1} v\} \tag{3.1}$$

and A' satisfy (a) and (b).

Proof of the Claim. Define a new coloring $d : [A]^{k-1} \to r$ by the formula $d(x) = c(x \cup \{v\})$. By Ramsey Theorem, there is an infinite d-homogeneous set $B \subset A$ in some color t. Then $t \in T$, otherwise $T \cup \{t\}$ would be a bigger perfect set witnessed by $P_t = \{v\}$ and B.

Now pick $v_1 v_2 \dots v_{k-1}$ from B and let $A' = B \backslash \{v_0, v_1, v_{k-1}, v\}$. □

The Claim clearly implies the Theorem. □

4. An infinite version of a conjecture of Seymour

Seymour's Conjecture . *Let G be a finite graph of order $n \geqslant 3$, and let $k \in \omega$. If G has minimum degree*

$$\delta(G) \geqslant \frac{k}{k+1} n, \tag{4.1}$$

contains the kth power of a Hamiltonian cycle.

If $k + 1 \nmid n$, then the assumption (4.1) implies that $\mathrm{N}_G[A] \neq \varnothing$ for all $A \in [\mathrm{V}(G)]^{k+1}$.

Theorem 4.1. *Let G be a countably infinite graph, and let $k \in \omega$. If $\mathrm{N}_G[A]$ is infinite for all $A \in [\mathrm{V}(G)]^{k+1}$, then G contains the kth power of a Hamiltonian path.*

Proof.

Claim 4.1.1. *If the kth power of a finite path $P = x_0 \dots x_n$ is in G, then for all $v \in \mathrm{V}(G) \backslash P$ there is a a finite sequence $v_1 v_2 \dots v_{k-1}$ of vertices, such that the kth power of the finite path $P^\frown v_1 v_2 \dots v_{k-1} v$ is also in G.*

Proof of the Claim. By finite induction pick distinct vertices $v_1, \dots, v_{k-1}$ such that

$$v_i \in \mathrm{N}_G[\{x_{n-k+i}, \dots, x_n\} \cup \{v\} \cup \{v_j : j < i\}]. \tag{4.2}$$

□

Using the Claim, we can construct the rewuired Hamiltonian path inductively. □

5. Covers by ℓth powers of paths

Definition 5.1. Assume that H is a graph, $W \subset V(H)$, and $k \in \omega$. The game $\mathfrak{G}_k(H, W)$ is played by two players, Adam and Bob, as follows. The players choose pairwise disjoint finite subsets of $V(H)$ alternately:

$$A_0, B_0, A_1, B_1, \dots$$

Bob wins the game $\mathfrak{G}_k(H, W)$ if

(A) $W \subset \bigcup_{i \in \omega} A_i \cup B_i$, and
(B) $H[\bigcup_{i \in \omega} B_i]$ contains the kth power of a (finite or one way infinite) Hamiltonian path.

Claim 5.1.1. *If $H = (V, E)$ and $W \subset V$ then the following are equivalent:*

(1) for every $a, b \in W$ and $F \in [V \backslash \{a, b\}]^{<\omega}$ there is a path in from a to b in $V \backslash F$.
(2) Bob wins $\mathfrak{G}_1(H, W)$.

Proof. (1) $\Rightarrow$ (2): By our assumption, Bob can always connect an uncovered point of W to a previously constructed path.

(2) $\Rightarrow$ (1): Let Adam start with $A_0 = F$ and continue with $A_i = \emptyset$; the Hamiltonian path constructed by Bob's strategy will go through a and b. □

We will convert a winning strategy of Bob into a partition by the following

Lemma 5.2. *Suppose that $H = (V, E)$, $V = \cup\{W_i : i < N\}$ with $N \in \omega$ and let $H_i = (V, E_i)$ for some $E_i \subset E$. If Bob wins $\mathfrak{G}_k(H_i, W_i)$ for all $i < N$ then V can be partitioned into kth powers of paths $\{P_i : i < N\}$ so that edges between consecutive vertices of P_i are in E_i.*

Corollary 5.3. *Let $c : E(K_{\omega,\omega}) \to r$ for some $r \in \omega$. Then $K_{\omega,\omega}$ can be partitioned into at most $2r - 1$ monochromatic paths. Furthermore, for every $r \in \omega$ there is $c_r : E(K_{\omega,\omega}) \to r$ so that $K_{\omega,\omega}$ cannot be covered by less than $2r - 1$ monochromatic paths.*

Proof. Let us denote the two classes of $K_{\omega,\omega}$ by A and B. Fix a coloring c and ultrafilters U_A, U_B on A, B respectively; now, let $A_i = \{u \in A : \{v \in B : c(u, v) \in i\} \in U_B\}$ and similarly $B_i = \{v \in B : \{u \in A : c(u, v) \in i\} \in U_A\}$. Without loss of generality, we can suppose that $A_0 \in U_A$. Let H_i denote the graph on $A \cup B$ with edges $c^{-1}(i)$.

Claim 5.3.1. *Bob wins the games $\mathfrak{G}_1(H_0, A_0 \cup B_0), \mathfrak{G}_1(H_i, A_i)$ and $\mathfrak{G}_1(H_i, B_i)$ for $1 \leqslant i < r$.*

Proof. It is easy to see that Claim 5.1.1 can be applied in each case. □

This finishes the proof of the first part of the theorem by Lemma 5.2.

Next, we will construct our colorings c showing that the above results is sharp. Let $r \geqslant 2$, let $A = \cup\{A_i : i < r\}$ with A_0 infinite and $A_i = \{a_i\}$ for $1 \leqslant i < r$ and let $B = \cup\{B_i : i < r\}$ with each B_i infinite. Define the r-coloring c_r as follows: let

$$c_r \restriction A_i \times B_j = i + j \mod r \quad \text{for } i, j \in r.$$

Note that if P is a monochromatic path which covers some A_i then $|\{j < r : P \cap B_j \neq \emptyset\}| \leqslant 1$; furthermore P is finite and thus $B_j \backslash P \neq \emptyset$ if $1 \leqslant i < r$ and $j < r$. Similarly, if P is a monochromatic path which covers some B_i then $|\{j < r : P \cap A_j \neq \emptyset\}| \leqslant 1$ as well. Now it is easy to see that there is no c_r-monochromatic cover by less than $2r - 1$ paths. □

Theorem 5.4. *Assume that H is a countably infinite graph, $W \subset V(H)$, and $k \in \omega$. If there are subsets $W_0, \ldots, W_k$ of W such that $W_0 = W$ and*

$$W_{j+1} \cap N_H[F] \text{ is infinite}$$

for each $j < k$ and for all $F \in \left[\bigcup_{i \leqslant j} W_i\right]^{2k}$, then Adam wins that game $\mathfrak{G}_k(H, W)$.

Proof. Assume that $\mathrm{V}(H) = \omega$.

Define the relation $\sqsubset$ on $(k+1) \times \omega$ as follows:

$$\langle i, x\rangle \sqsubset \langle i, y\rangle \text{ and } \langle i, x\rangle \sqsubset \langle i+1, y+1\rangle \text{ for } x < y. \tag{5.1}$$

The $\sqsubset$-predecessors of $\langle i, y\rangle$ are

$$\{\langle i, x\rangle : x < y\} \cup \{\langle i+1, z\rangle : y < z\}. \tag{5.2}$$

Let $\{\langle i_n, x_n\rangle : n < \omega\}$ be an enumeration of $k \times \omega$ such that

$$\langle i_m, x_m\rangle \sqsubset \langle i_n, x_n\rangle \text{ implies } m < n. \tag{5.3}$$

In the stage n, Adam picks $a_{\langle i_n, x_n\rangle} \in W_{i_n} \setminus \bigcup_{i<n} A_i \cup B_i$ such that

$$a_{\langle i_n, x_n\rangle} a_{\langle i_m, x_m\rangle} \in \mathrm{E}(H) \text{ for } \langle i_m, x_m\rangle \sqsubset \langle i_n, x_n\rangle, \tag{5.4}$$

and

$$a_{\langle i_n, x_n\rangle} = \min(W_0 \setminus \bigcup_{i<n} A_i \cup B_i) \text{ provided } i_n = 0. \tag{5.5}$$

Let $A_n = \{a_{\langle i_n, x_n\rangle}\}$.

Then (A) holds by 5.5.

Let $\{\langle j_n, y_n\rangle : n < \omega\}$ be the lexicographical enumeration of $k \times \omega$. Then $\{a_{\langle j_n, y_n\rangle} : j < \omega\}$ is the kth power of a path. Indeed, assume that $0 \leqslant m < n \leqslant m + k$. Then either $\langle j_n, y_n\rangle \sqsubset \langle j_m, y_m\rangle$ or $\langle j_m, y_m\rangle \sqsubset \langle j_n, y_n\rangle$. Then 5.4 implies that $a_{\langle i_n, x_n\rangle} a_{\langle i_m, x_m\rangle} \in \mathrm{E}(H)$. □

Theorem 5.5. *(1) Given any coloring of the edges of K_ω with 2 colors, the vertices can be partitioned into 5 homogeneous path-square.*

(2) For each natural numbers k and r there is a natural number M such that given any coloring of the edges of K_ω with r colors, the vertices can be partitioned into M homogeneous k-power of a path apart from a finite set.

Proof of theorem 5.5(2). (2) We will use the notation of lemma 2.3. By induction on the length of sequences, for each finite sequence $s \in r^{\leqslant kr+1}$ define a set $A_s \subset \mathrm{V}(G)$ as follows:

- $A_\varnothing = \mathrm{V}(G)$.
- if A_s is defined, let

$$A_{s^\frown i} = \{u \in A_s : d_{c \restriction A_s}(u) = i\} \tag{5.6}$$

provided A_s is infinite. If A_s is finite, then let

$$A_{s^\frown 0} = A_s \text{ and } A_{s^\frown i} = \varnothing \text{ for } 1 \leqslant i < r\ . \tag{5.7}$$

Consider an arbitrary $s \in r^{kr+1}$ such that A_s is infinite. Then there is a color $i_s < r$ and there is a k-element subset $H_s = \{h_0 > h_1 > \cdots > h_k\}$ of $kr + 1$ such that $s(h_0) = s(h_1) = \cdots = i_s$. So, by theorem 5.4, the finite sequence

$$A_s, A_{s \restriction h_0}, \ldots, A_{s \restriction h_k} \tag{5.8}$$

witnesses that Adam has a winning strategy in the game $\langle G_{i_s}, A_s\rangle$, where $G_{i_s} = \langle \mathrm{V}(G), c^{-1}\{i_s\}\rangle$.

Playing the games

$$\{\langle G_{i_s}, A_s\rangle : A_s \text{ is infinite}\} \tag{5.9}$$

parallel, we can find at most r^{kr+1}-many kth power of vertex disjoint monochromatic paths which cover $V(G)$ apart from the finite set $\bigcup\{A_s : A_s \text{ is finite}\}$. □

To prove theorem 5.5(1) we need some preparation.

In [4, Corollary 1.10] Pokrovskiy proved the following: *Let $k \geqslant 1$. Suppose that K_n is colored with two colours. Then K_n can be covered with k disjoint red paths and a disjoint blue kth power of a path.*

sqr **Lemma 5.6.** *Assume that $P = v_0v_1\dots$ is a path such that*

$$N_G[\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}]\setminus P \tag{5.10}$$

is infinite for all $v_i \in P$. Then G contains a 2nd power of a path with covers P.

Proof. Pick pairwise disjoint vertices $w_0, w_1, \dots,$ from $V(G)\setminus P$ such that $w_i \in N_G[\{v_{2i}, v_{2i+1}, v_{2i+2}, v_{2i+3}\}]$.

Then

$$v_0\ v_1\ w_0\ v_2\ v_3\ w_1\ v_4 \dots v_{2i}\ v_{2i+1}\ w_i\ v_{2i+1}\ v_{2i+2}\ w_{i+1}\dots \tag{5.11}$$

is a 2nd power of a path with covers P. □

Proof of theorem 5.5(1). (1) Fix a coloring $c : \left[\omega\right]^2 \to 2$. Let $G_i = \left\langle \omega, c^{-1}\{i\}\right\rangle$ for $i < 2$.

We will use the notation of lemma 2.3.

Let $c_0 = c$ and

$$A_0 = \{v \in \omega : d_{c_0}(v) = j_{c_0}\}, \text{ and } B_0 = \omega\setminus A_0. \tag{5.12}$$

Let $c_1 = c_0\restriction B_0$ and

$$A_1 = \{v \in B_0 : d_{c_1}(v) = j_{c_1}\}, \text{ and } B_1 = B_0\setminus A_1. \tag{5.13}$$

Let $c_2 = c_1\restriction B_0$ and

$$A_2 = \{v \in B_1 : d_{c_2}(v) = j_{c_2}\}, \text{ and } B_2 = B_1\setminus A_2. \tag{5.14}$$

We can assuem that $j_{c_0} = 0$.

Case 1. *B_0 is finite*

$G[B_0]$ can be covered by two 0-paths P_0 and P_1 and a 1 square path Q_1. by [4, Corollary 1.10].

So by lemma 5.6 P_0 and P_1 can be covered by two 0-homogeneous squares of some paths R_0 and R_1. We can guarantee that R_0, R_1, Q_1 are vertex disjoint.

Since Adam wins $\mathfrak{G}_2(G_0, A_0)$, so $G[\omega\setminus R_0 \cup R_1 \cup Q_1]$ can be covered by one 0-homogeneous square of paths.

So G can be covered by 4 squares of paths.

Case 2. *B_0 is infinite, $j_{c_1} = 0$*

Case 3. *B_0 is infinite, $j_{c_1} = 1$, B_1 is finite*

Case 4. *$j_{c_1} = 1$ and B_1 is infinite*

Assume that $j_{c_2} = 1$

Adam wins $\mathfrak{G}_2(G_0, A_0)$, $\mathfrak{G}_2(G_1, A_1)$ and $\mathfrak{G}_2(G_{j_{c_2}}, A_2)$.

Moreover Adam also wins $\mathfrak{G}_2(G_{1-j_{c_2}}, B_2)$ witnesses by

- (B_2, A_2, A_1) if $j_{c_2} = 1$, and by

- (B_2, A_2, A_0) if $j_{c_2} = 0$.

So G can be covered by 4 squares of paths.

□

6. Partitioning the 2-edge colored K_{ω_1,ω_1}

Theorem 6.1. *Given any coloring of the edges of K_{ω_1,ω_1} with 2 colors, the vertices can be partitioned into finitely many ($\leqslant 11$??) monochromatic paths.*

Let us fix a coloring c for the rest of this section, and let $K_{\omega_1,\omega_1} = A \cup B$ the two classes.

Claim 6.1.1. *If there is a i-monochromatic copy of K_{ω_1,ω_1} in $A \cup B$ for some $i < 2$ then $A \cup B$ can be partitioned into 3 monochromatic paths.*

Proof. Let $A_0 \cup B_0$ denote the monochromatic K_{ω_1,ω_1} and extend $A_0 \cup B_0$ to Z which is a maximal ω_1-connected set in color i. Clearly, Z is a path in color i.

Now, it is easy to see that there is $A_1 \subset A_0$ and $B_1 \subset B_0$ so that

(1) $Z \backslash (A_1 \cup B_1)$ is a path in i,
(2) $A \backslash Z \cup B_1$ and $B \backslash Z \cup A_1$ are paths in color 0.

□

Hence, we can suppose that there is no monochromatic copy of K_{ω_1,ω_1} in $A \cup B$. Let

$$\Gamma_i = \{\alpha \in A : |N(\alpha, i)| \leqslant \omega\}$$

and

$$\Delta_i = \{\beta \in B : |N(\beta, i)| \leqslant \omega\}.$$

$\boxed{\text{s}}$ **Observation 6.2.** *For all $A' \in [A]^{\omega_1}$ and $B' \in [B]^{\omega_1}$ there are ω_1 independent edges of color i between A' and B' (for $i < 2$). Hence*

(1) $\min\{|\Gamma_i|, |\Delta_i|\} \leqslant \omega$ *if* $i < 2$,
(2) if $\alpha \in A \backslash \Gamma_i$ and $\beta \in B \backslash \Delta_i$ then there are ω_1 many vertex disjoint paths in color i between α and β.

Without loss of generality, we can suppose that $|\Gamma_0| \leqslant \omega$ and hence it suffices to consider the case when $\Gamma_0 = \varnothing$ by Theorem 5.3.

Claim 6.2.1. *If Γ_1 is uncountable then there is an uncountable $B' \subset B$ so that $\Gamma_1 \cup B'$ is a path in color 0 and $|N(\alpha, i) \backslash B'| = \omega_1$ if $\alpha \in A$, $i < 2$ and $|N(\alpha, i)| = \omega_1$.*

Therefore, we can further suppose that Γ_1 is empty as well. We need to consider 3 cases as follows:

Case 1: $|\Delta_0|, |\Delta_1| \leqslant \omega$,
Case 2: $|\Delta_0| = \omega_1$ while $|\Delta_1| \leqslant \omega$,
Case 3: $|\Delta_0| = |\Delta_1| = \omega_1$.

In **Case 1**, we can suppose that $|\Delta_i| = \emptyset$ by Theorem 5.3. Now $A \cup B$ is ω_1 connected in both colors by Observation 6.2 (2). Also, if $A \cup B$ is not a 0-trail then it is clearly a 1-trail (as shown many times before).

Second, in **Case 2**, we can suppose that $|\Delta_1| = \emptyset$ by Theorem 5.3. Now $|\Delta_0| = \omega_1$ implies that $A \cup B$ is a 1-trail while it is clearly 1-connected by $|\Delta_1| = |\Gamma_1| = \emptyset$; hence $A \cup B$ is a path.

Finally, we consider **Case 3**. We inductively build P_0, P_1 partitioning $A \cup B$ to monochromatic paths (P_i to be $(1-i)$-homogeneous) so that $P_i = \cup\{P_{i,\alpha} : \alpha < \omega_1\}$ and Δ_i is cofinal in $P_{i,\alpha}$. Limits points are easy to choose by the definition of Δ_i and in successor steps we can cover the remaining points using connectedness ensured by Observation 6.2 (2).

Problem 6.3. *Given an edge coloring of K_{ω_1,ω_1} with finitely many colors is there a partition of the vertices into finitely many monochromatic paths?*

7. Finitely many colors and finite partitions on ω_1

Let H_{ω_1,ω_1} denote the balanced bipartite graph with classes $A = \{a_\xi : \xi < \omega_1\}, B = \{b_\xi : \xi < \omega_1\}$ of size ω_1 such that

$$(a_\xi, b_\zeta) \in E(H_{\omega_1,\omega_1}) \text{ iff } \xi \leqslant \zeta.$$

We will call A the *main class* of H_{ω_1,ω_1}.

The following lemma will be of surprising relevance:

ver **Lemma 7.1.** *Let $r \in \omega$ and $c : E(H_{\omega_1,\omega_1}) \to r$. Then there are finitely many monochromatic and disjoint paths $\{P_i : i < N\}$ covering the main class of H_{ω_1,ω_1}.*

Proof. We prove by induction on r. Note that a monochromatic H_{ω_1,ω_1} is a paths which concludes the $r = 1$ case. Now, in general, it suffices to see that there are disjoint $\{Q_j : j < M\}$ such that Q_j is either countable or a monochromatic path or a copy of H_{ω_1,ω_1} colored with $\leqslant r-1$ colors. Fix a uniform ultrafilter U on B and let $A_j = \{a \in A : N(a,j) \in U\}$ for $j < r$. We can disregard those A_js which are countable; see theorem on countable bipartite partitions.

Claim 7.1.1. *If $|A_j| = \omega_1$ then one of the following holds*

(1) there is a club C so that for all $\alpha \in C$ there is $B_\alpha \in [N(x_\alpha, j)]^{\omega_1}$ (where $x_\alpha = \min A \backslash \alpha$)such that $\sup\{\delta \in \alpha : a_\delta \in A_j, c(a_\delta, b) = j\} = \alpha$ for all $b \in B_\alpha$, or

(2) there is a countable $E \subset A$ and uncountable B_j so that c colors $H_{\omega_1,\omega_1}[A_j \backslash E \cup B_j]$ with $r \backslash \{j\}$.

Now an easy induction shows that there are *disjoint* sets Q_j so that $A_j = Q_j \cap A$ and either Q_j has properties

(1) there is a club C so that for all $\alpha \in C$ there is $b_\alpha \in N(x_\alpha, j) \cap Q_j$ (where $x_\alpha = \min A \backslash \alpha$) such that $\sup\{\delta \in \alpha : a_\delta \in A_j, c(a_\delta, b_\alpha) = j\} = \alpha$, and

(2) for all $a, a' \in A_j$ there are ω_1-many vertex disjoint paths from a to a' inside Q_j

or Q_j is a copy of H_{ω_1,ω_1} colored with $\leqslant r-1$ colors (Case (1) and Case (2) respectively from above). In the first case, Q_j is not necessarily a path but we can cover A_j *inside* Q_j with a j-monochromatic path. $\square$

Corollary 7.2. *Suppose that the edges of a complete balanced bipartite graph are colored with finitely many colors. Then*

(1) *the vertices can be* covered *by finitely many monochromatic paths,*
(2) *the vertices can be* partitioned *into countably many monochromatic paths.*

Theorem 7.3. *Suppose that $G = (V, E)$ satisfies $|V| = \omega_1$ and $|V \backslash N_G(v)| \leqslant \omega$ for all $v \in V$. Given an edge coloring of G with finitely many colors the vertices of G can be partitioned into finitely many monochromatic paths.*

Proof. We prove by induction on r. Fix a coloring c with $r \in \omega$ colors, wlog $r \geqslant 2$. Naturally $\omega_1 = V$.

Case 1: suppose that there is $A \subset \omega_1$which is a copy of K_{ω_1} colored only by $\leqslant r-1$ colors. Pick $A' \subset A$ so that $|A'| = |\omega_1 \backslash A|$ and $\omega_1 \backslash A \cup A'$ forms a copy of H_{ω_1,ω_1} with main class $\omega_1 \backslash A$. By Lemma 7.1 there are finitely many disjoint paths $\{P_i : i < n\}$ inside $\omega_1 \backslash A \cup A'$ covering $\omega_1 \backslash A$. Finally, note that $W = \omega_1 \backslash \cup P_i \subset A$ hence the edges of W are only $r-1$-colored; by our inductional hypothesis, we can cover W by finitely many disjoint paths.

Case 2: suppose that there is no $A \subset \omega_1$which is a copy of K_{ω_1} colored only by $\leqslant r-1$ colors.

Observation 7.4. *If $A \in [\omega_1]^{\omega_1}$ and $i < r$ then A must be an i-trail.*

Proof. Otherwise, there is a uncountable $S \subset \omega_1$, a 1-1 increasing sequence $\{a_\alpha : \alpha \in S\} \subset A$ and $\lambda \in \omega_1$ so that $c \restriction (A \backslash \lambda \cap \alpha \times \{a_\alpha\}) \neq i$ and $\alpha < a_\alpha < \beta$ and $(a_\alpha, a_\beta) \in E$ if $\alpha < \beta \in S$. Clearly $\{a_\alpha : \alpha \in S\}$ would be a copy of K_{ω_1} colored with $\leqslant r-1$ colors. $\square$

Now find an uncountable $A \subseteq \omega_1$ so that A is ω_1-connected in some color, wlog in 0. Pick $A' \subset A$ so that $|A'| = |\omega_1 \backslash A|$, $A \backslash A'$ is uncountable and for all $x, y \in A$ there are ω_1-many vertex disjoint $x \to y$ 0-monochromatic paths P with $P \backslash \{x, y\} \subset A \backslash A'$.

Observation 7.5. *Suppose that W satisfies $A \backslash A' \subset W \subset A$. Then W is ω_1-connected in color 0.*

There is $A'' \subset A'$ so that $\omega_1 \backslash A \cup A''$ forms a copy of H_{ω_1,ω_1} with main class $\omega_1 \backslash A$. By Lemma 7.1 there are finitely many disjoint paths $\{P_i : i < n\}$ inside $\omega_1 \backslash A \cup A'$ covering $\omega_1 \backslash A$. Finally note that $W = \omega_1 \backslash \cup P_i$ is both a 0-trail and ω_1-connected in color 0 by the last two Observations. Hence W is a path in color 0.

$\square$

Corollary 7.6. *Given an edge coloring of K_{ω_1} with finitely many colors the vertices of K_{ω_1} can be partitioned into finitely many monochromatic paths.*

We will see in the next section that actually every 2-edge coloring of K_{ω_1} actually admits a 2-partition into monochromatic paths. However, the following is not known:

Problem 7.7. *Given a r-edge coloring of K_{ω_1} is there a partition of the vertices into $\leqslant r$ many monochromatic paths (of different colors)?*

8. Neighbors from the club filter

Naturally, one would like to have an answer for the following:

Problem 8.1. *Suppose that the graph $G = (\omega_1, E)$ satisfies that $N_G(v)$ is a club for all $v \in \omega_1$ and E is colored by finitely many colors. Is there a partition of ω_1 into finitely many monochromatic paths?*

We have two partial answers:

Claim 8.1.1. *Suppose that the graph $G = (\omega_1, E)$ satisfies that $N_G(v)$ is a club for all $v \in \omega_1$ and E is colored by finitely many colors. Then we can*

(1) partition ω_1 into countably many monochromatic paths,
(2) cover ω_1 by finitely many monochromatic paths.

Proof. First, we prove (1): let $\omega_1 = \cup\{S_n : n \in \omega\}$ so that S_n are disjoint and stationary. We can inductively define a sequence $\mathcal{P}_n$ so that

(1) each $P \in \mathcal{P}_n$ is a monochromatic path,
(2) $P \cap P' = \emptyset$ if $P \in \mathcal{P}_n$, $P' \in \mathcal{P}_m$,
(3) $\cup\mathcal{P}_n$ covers S_n.

If $\mathcal{P}_k$ is constructed for $k < n$ and $S_n \setminus \cup\{P : P \in \mathcal{P}_k, k < n\}$ is uncountable then there is a copy of H_{ω_1,ω_1} inside $S_n \cup S_{n+1}$ with main class $S_n \setminus \cup\{P : P \in \mathcal{P}_k, k < n\}$. Now apply Lemma 7.1 to get $\mathcal{P}_n$.

Second, we prove (2): let $\omega_1 = S_0 \cup S_1$ with S_i disjoint stationary. There are two copies of H_{ω_1,ω_1}, say X_i inside ω_1 so that the main class of X_i is S_i (and second class contained in S_{1-i}). Apply Lemma 7.1 for X_i to finish the proof. □

Theorem 8.2. *Suppose that $G = (\omega_1, E)$ is a graph so that $N_G(v)$ is a club for all $v \in \omega_1$. If c is a 2-edge coloring of G then ω_1 can be partitioned into 2 monochromatic paths.*

Proof. We distinguish two cases as follows:

Case 1: there is a monochromatic (say in color 0) copy (A, B) of H_{ω_1,ω_1} in G so that $A \cup B$ is stationary. If so, then extend $A \cup B$ to a maximal ω_1-connected C. Note that there is $D \subset C$ so that $(\omega_1 \setminus (A \cup B), D)$ is a 1-monochromatic copy of H_{ω_1,ω_1} while and $A \cup B \setminus D$ is still a 0-connected and still contains a 0-monochromatic copy of H_{ω_1,ω_1}. Observe that $A \cup B \setminus D$ is a 0-path while $\omega_1 \setminus (A \cup B) \cup D$ is a 1-path.

Case 2: there is no monochromatic copy (A, B) of H_{ω_1,ω_1} in G so that $A \cup B$ is stationary. The standard argument shows that every stationary $S \subset \omega_1$ is a trail in both colors. Pick an ultrafilter containing all clubs and using that find a stationary $S \subset \omega_1$ so that S is ω_1-connected in some color, say 0. Without loss of generality, it can be supposed that S is maximal with respect to being ω_1-connected in 0. Now, there is a non stationary $S_0 \subset S$ so that $\omega_1 \setminus S \cup S_0$ is a 1-monochromatic copy of H_{ω_1,ω_1} and so that $S \setminus S_0$ is ω_1-connected in 0. Note that $\omega_1 \setminus S \cup S_0$ is a 1-path and $S \setminus S_0$ is a 0-path (as ω_1-connected in 0 and stationary, so a 0-trail). □

Corollary 8.3. *Given any coloring of the edges of K_{ω_1} with 2 colors, the vertices can be partitioned into 2 monochromatic paths.*

Clearly, if $\{N_G(v) : v \in V\}$ forms a uniform filter on ω_1 then G is ω_1-connected. The following examples shows that being ω_1-connected is nowhere near enough to admit monochromatic partitions for arbitrary 2-edge colorings.

Example 8.4. *There is an ω_1-connected graph (V, E) with an 2-edge coloring so that there are no monochromatic cycles (of length $\geqslant 3$); in particular, there is no monochromatic path of size $\omega + 1$ and so there is no cover by countably many monochromatic paths.*

Proof. Let $\omega_1 = \cup\{S_n : n \in \omega\}$ so that S_n are pairwise disjoint and uncountable. Furthermore, let $S_n = \bigcup\{T^n_{x,y} : x \neq y \in \cup\{S_k : k < n\}\}$ with $T^n_{x,y}$ pairwise disjoint uncountable. Let $E_0 = \varnothing$ and

$$E_n = E_{n-1} \cup \bigcup\{\{x, y\} \times T^n_{x,y} : x \neq y \in \cup\{S_k : k < n\}\}$$

and let $E = \cup E_n$.

Clearly, (ω_1, E) is ω_1-connected. Now, define $c \restriction E_n \backslash E_{n-1}$ so that if $x < y \in \cup\{S_k : k < n\}$ and $z \in T_{x,y}$ then $c(x, z) = 0$ and $c(y, z) = 1$. It is not hard to see that there are no monochromatic cycles; if C is a cycle consider $N = \max\{n : C \cap E_n \backslash E_{n-1} \neq \varnothing\}$ and let $z \in C \cap E_N \backslash E_{N-1}$. By the definition of the coloring, the two edges in C on z are colored with different colours. □

9. Path decompositions on higher cardinals

Definition 9.1. A graph $G = (V, E)$ is called κ-complete iff $|V| \geqslant \kappa$ and

$$|V \backslash N_G(x)| < \kappa$$

for all $x \in V$.

Our final goal is to prove Corollary 9.10, that every finite-edge coloured κ-complete graph can be partitioned into finitely many monochromatic paths.

9.1. **Preliminaries.** We will make use of the following definition:

Definition 9.2. A graph $G = (V, E)$ is of *type* $H_{\kappa,\kappa}$ iff $V = A \cup B$ where $A = \{a_\xi : \xi < \kappa\}, B = \{b_\xi : \xi < \kappa\}$ and

$$(a_\xi, b_\zeta) \in E(G) \text{ iff } \xi \leqslant \zeta < \kappa.$$

We will call A the *main class* and (A, B) with the inherited ordering is called the $H_{\kappa,\kappa}$-decomposition of G.

Note that A, B in the $H_{\kappa,\kappa}$-decomposition are not necessarily disjoint.

We will prove in Theorem 9.9 that every finite-edge coloured graph of type $H_{\kappa,\kappa}$ contains a monochromatic path of size κ; furthermore, if the two classes are disjoint then the main class is covered by finitely many disjoint monochromatic paths. This result will be used in the proof of Corollary 9.10.

e **Observation 9.3.** *Suppose that $G = (V, E)$ is κ-complete (for an arbitrary infinite κ) and let $X, Y \in [V]^\kappa$. Then there is $F \subset E$ so that $(X \cup Y, F)$ is of type $H_{\kappa,\kappa}$ and there is a $H_{\kappa,\kappa}$-decomposition $A \cup B$ so that $X = A$ and $B \subset Y$.*

Proof. If κ is regular then we can take an arbitrary enumeration $X = \{a_\xi : \xi < \kappa\}$ and $\{b_\xi : \xi < \kappa\} \subset Y$ are easily constructed by induction so that $(a_\xi, b_\zeta) \in E$ for $\xi \leqslant \zeta < \kappa$.

If kappa is singular then let $\mu = cf(\kappa)$ and let $(\kappa_\alpha)_{\alpha<\mu}$ increase to κ. Let $X_\alpha = \{x \in X : |V \backslash N(x)| < \kappa_\alpha\}$ and list X as $\{a_\xi : \xi < \kappa\}$ so that for all $\lambda < \kappa$ there is $\alpha < \mu$ with $\{a_\xi : \xi < \lambda\} \subset X_\alpha$. That is,

$$|Y \cap \bigcap\{N(a_\xi) : \xi < \lambda\}| = \kappa$$

for all $\lambda < \kappa$. Now $\{b_\xi : \xi < \kappa\} \subset Y$ is easily constructed by induction so that $(a_\xi, b_\zeta) \in E$ for $\xi \leqslant \zeta < \kappa$. □

ype **Observation 9.4.** *Suppose that $G = (V, E)$ is of type $H_{\kappa,\kappa}$ with main class V. Then there is a κ-complete graph embedded in G.*

Proof. If A, B is the $H_{\kappa,\kappa}$ decomposition then we have $B \subset A = V$ and B is the κ-complete subgraph. □

For a path P and $x <_P y \in P$ let $P \restriction [x, y)$ denote the segment of P from x to y (excluding y itself). For a set A and i-monochromatic path P we say that *P is concentrated on A* iff

$$N(x, i) \cap A \cap P \restriction [y, x) \neq \varnothing$$

for all limit element $x \in P$ and y below x in P. Clearly, if P is concentrated on A then for every $x \in P$ there is $x' \in P \cap A$ above x so that $P \restriction [x, x')$ is finite.

Observation 9.5. *Suppose that G is of type $H_{\kappa,\kappa}$. Then there is a path of order type κ which is concentrated on the main class of G.*

An elementary submodel M is *covering* iff for all $A \subset M$ with $|A| < |M|$ there is $A' \in M$ so that $A \subset A'$. If $M = \cup\{M_\xi : \xi < \mu^+\}$ where $\{M_\xi : \xi < \mu^+\}$ is an ϵ-chain of elementary submodels, $|M_\xi| = \mu$ then M is covering.

Observation 9.6. *Suppose that the graph $A \subset V(G)$ is κ-saturated in some color i with respect to a coloring c. If $M = \cup\{M_\xi : \xi < \nu\}$ where $\{M_\xi : \xi < \nu\}$ is an ϵ-chain of covering elementary submodels and $|M| = \lambda \leqslant \kappa$ so that $G, c \in M_0$ then $M \cap A$ is λ-saturated in color i inside $M \cap V(G)$.*

Proof. If $|M| = \lambda^+$ with $|M_\xi| \leqslant \lambda$ then the claim is trivial. If $|M| = \lambda$ with $|M_\xi| = \lambda$ then it suffices to prove that $N \cap A$ is λ-saturated in color i inside N for all covering submodels N of size λ, which is easy to see.

Otherwise, we can suppose that $|M_\xi| > |M_{<\xi}|$ where $M_{<\xi} = \cup\{M_\zeta : \zeta < \xi\}$. Take $a, a' \in A \cap M$ and $\tilde{A} \in [A]^{<\lambda}$. There is $\nu_0 < \nu$ so that $|M_\xi| > |\tilde{A}|$ for all $\xi \in \nu \backslash \nu_0$. There is $X_\xi \in M_\xi$ of size $< |M_\xi|$ so that

$$M_{<\xi} \cup (\tilde{A} \cap M_\xi) \subset X$$

by covering; so we can find $|M_\xi|$-many disjoint i-monochromatic paths in $M_\xi \backslash X$ connecting a to a'. This holds for all for all $\xi \in \nu \backslash \nu_0$ so we found λ many □

9.2. **Outline of the proof.** Let $(IH)_{\kappa,r}$ denote the statement that for any r-edge colouring of a graph G of type $H_{\kappa,\kappa}$ with main class A, we can find a monochromatic paths concentrated on A of size κ. Let $(IH)_\kappa$ denote $(IH)_{\kappa,r}$ for all $r \in \omega$; we will prove that $(IH)_\kappa$ holds for all κ which will imply the decomposition theorem for κ-complete graphs.

1 **Lemma 9.7.** *Let κ be infinite with $\mu = cf(\kappa)$. Suppose that c is an r-edge coloring of a graph $G = (V, E)$ where G is of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition A, B. Suppose that A is κ-saturated in a color $i < r$ and for all $\tilde{A} \in [A]^{<\kappa}$ and $\lambda < \kappa$ there is an i-monochromatic path P of size at least λ disjoint from $\tilde{A}$ which is concentrated on A. Then either*

- *(a) A is covered by an i-monochromatic path concentrated on A, or*
- *(b) there is $\tilde{A} \in [A]^{<\kappa}$ so that $A \setminus \tilde{A}$ is covered by a graph H of type $H_{\kappa,\kappa}$ with main class $A \setminus \tilde{A}$ so that $c \restriction E(H) \neq i$.*

2 **Lemma 9.8.** *Let κ be infinite. Suppose that c is an r-edge coloring of a graph $G = (V, E)$ where G is of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition A, B and $I \in [r]^{<r}$. Suppose A is κ-saturated in all colors $i \in I$. If $(IH)_{\kappa,r-1}$ then either*

- *(a) there is an $i \in I$ such that for all $\tilde{A} \in [A]^{<\kappa}$ and $\lambda < \kappa$ there is an i-monochromatic path P of size at least λ disjoint from $\tilde{A}$ which is concentrated on A, or*
- *(b) there is $\tilde{A} \in [A]^{<\kappa}$ so that there is a partition $A \setminus \tilde{A} = \cup\{A_j : j \in r \setminus I\}$ where $|N(x, j) \cap N(x', j)| = \kappa$ for all $x, x' \in A_j$ and $j \in r \setminus I$.*

Before proving the lemmas, let us show that they imply our main results.

f **Theorem 9.9.** *$(IH)_\kappa$ holds for all infinite κ, i.e. if G is a graph of type $H_{\kappa,\kappa}$ with a finite-edge colouring then we can find a monochromatic path of size κ concentrated on the main class of G.*

Moreover, for any r-edge colouring of the graph $H_{\kappa,\kappa}$ we can cover the main class by finitely many disjoint monochromatic paths.

Proof. We prove $(IH)_\kappa$ by induction on κ. We can suppose that $\kappa > \omega$ by previous results. Now we prove $(IH)_{\kappa,r}$ by induction on r. Note that $r = 1$ is trivial so let $r > 1$.

Fix an r-edge colouring and start applying Lemma 9.8 to find a decreasing sequence $A_0 \supseteq A_1 \supseteq \dots$ and a 1-1 sequence $i_0, i_1, \dots$ from r so that A_j is κ-saturated in color i_j. If A_j satisfies (a) of Lemma 9.8 with $I = \{i_0, \dots, i_j\}$ then applying Lemma 9.7 finishes the proof (either by finding a j-monochromatic path or by invoking the inductional hypothesis). If A_j satisfies (b) of Lemma 9.7 then one piece of the given partition defines A_{j+1}. If this induction goes on to define A_r then we have a set of size κ which is κ-saturated in all colors. By the inductional hypothesis for smaller graphs, we certainly have $i \in r$ so that for all $\tilde{A} \in [A]^{<\kappa}$ and $\lambda < \kappa$ there is an i-monochromatic path P of size at least λ which is concentrated on A. Thus applying Lemma 9.7 finishes the proof as before.

We prove the second statement by induction on κ and r as well; note that $\kappa = \omega$ or $\kappa > \omega$ and $r = 1$ are trivial. We will construct a finite tree $T \subset \omega^{<\omega}$ and subsets $\{Q_t : t \in T\}$ of $V = V(H_{\kappa,\kappa})$ so that every branch of tree has length $r + 1$ and

(1) $A = \dot{\cup}\{A \cap Q_t : t \in T, |t| = k\}$ for all $k < ht(T)$,
(2) Q_t is a copy of $H_{\lambda,\lambda}$ for some $\lambda \leqslant \kappa$ for all $t \in T$

and for all $t \in T$ we have either

(3) $A \cap Q_t$ is covered by finitely many disjoint monochromatic paths inside Q_t, or
(4) t is injective with $\mathrm{ran}(t) \subset r$ and $Q_t \cap A$ is κ-saturated in color i inside Q_t for all $i \in \mathrm{ran}(t)$.

We will prove that such a construction can be carried out and so (3) must hold for every $t \in T$ with $|t| = r + 1$; this finishes the proof.

We start by setting $Q_\varnothing = V$. Suppose Q_t is constructed with $|t| \leqslant r$. If (3) holds then set $Q_{t^\frown 0} = Q_t$; note that (3) holds whenever $|Q_t| < \kappa$ by induction. Otherwise, suppose that (4) holds and we distinguish two cases: suppose that there is an $i \in \mathrm{ran}(t)$ so that for all $\tilde{A} \in [A]^{<\kappa}$ and $\lambda < \kappa$ there is an i-monochromatic path P of size at least λ in $Q_t \backslash \tilde{A}$ which is concentrated on A. Note that this is the case if $|t| = r$ by the inductional hypothesis. Then Lemma 9.7 gives that $A \cap Q_t$ is either a single path or covered by an $\leqslant r - 1$ coloured copy of $H_{\kappa,\kappa}$ modulo a set of size $< \kappa$. In the latter case, we can clearly partition $Q_t = Q_{t^\frown 0} \cup Q_{t^\frown 1}$ so that

(i) $Q_{t^\frown 0}$ is a copy of $H_{\lambda,\lambda}$ with $\lambda < \kappa$,
(ii) $Q_{t^\frown 1}$ is a copy of $H_{\kappa,\kappa}$ so that c admits only $\leqslant r - 1$ coulours.

Note that $Q_{t^\frown 0}$ and $Q_{t^\frown 1}$ both satisfy (3) by induction.

In the second case there is no $i \in \mathrm{ran}(t)$ so that for all $\tilde{A} \in [A]^{<\kappa}$ and $\lambda < \kappa$ there is an i-monochromatic path P of size at least λ in $Q_t \backslash \tilde{A}$ which is concentrated on A; in particular, $|t| < r$. Apply Lemma 9.8 with $I = \mathrm{ran}(t)$ and get a decomposition $Q_t \cap A = \cup\{A_j : j \in r \backslash I\} \cup \tilde{A}$ so that $\tilde{A}$ has size $< \kappa$ and A_j is κ-saturated in color j inside Q_t for all $j \in r \backslash I$. It is clear that we can find a partition

$$Q_t = \cup\{Q_{t^\frown j} : j \in r \backslash I\} \cup Q_{t^\frown r}$$

so that $A_j = Q_{t^\frown j} \cap A$ for $j \in r \backslash I$ and $\tilde{A} = Q_{t^\frown r} \cap A$ which satisfies (1),(2) and (3) or (4) respectively; note that we would run into trouble finding this partition if the two classes are not disjoint.

□

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Corollary 9.10. *Suppose that c is a finite-edge colouring of a κ-complete graph $G = (V, E)$ on κ vertices. Then there is a partition of the vertices into finitely many disjoint monochromatic paths.*

Proof. We prove by induction on κ and r as before; we can suppose that $\kappa > \omega$ and $r > 1$.

Case 1: there is a κ-complete subgraph $W \subset V$ which is only coloured by $\leqslant r - 1$ colours. Find a subset $U \subset W$ so that $V \backslash W \cup U$ is covered by a copy of $H_{\lambda,\lambda}$ with $\lambda \leqslant \kappa$ and main class $V \backslash W$ while $W \backslash U$ still has size κ; in particular, $W \backslash U'$ is still κ-complete for all $U' \subset U$. Apply Theorem 9.9 to cover $V \backslash W$ inside $V \backslash W \cup U$ with finitely many disjoint monochromatic paths $\mathcal{P}$ and apply the inductional hypothesis to partition $W \backslash \cup \mathcal{P}$.

Case 2: not Case 1. We construct $A_j \subset V$, $t_j \in r^j$ and finite sets of disjoint monochromatic paths $\mathcal{P}_j$ for $j \leqslant r+1$ so that

(1) $A_j \supseteq A_{j+1}$, $\mathcal{P}_j \subset \mathcal{P}_{j+1}$ and $t_j \subset t_{j+1}$,
(2) A_j is either κ-complete (equivalenty, has size κ) or empty,
(3) A_j is κ-saturated in color $t(j')$ inside $A_{j'}$ for all $j' \leqslant j$,
(4) $V \backslash A_j$ is covered by $\mathcal{P}_j$.

It suffices to have $A_j = \varnothing$ for some j. We will see that this happens for some $j \leqslant r+1$. Let $A_\varnothing = V$ and $\mathcal{P}_0 = I_0 = \varnothing$. Suppose we constructed A_j for some $j < r$ and $A_j \neq \varnothing$. Note that A_j is of type $H_{\kappa,\kappa}$ by Observation 9.3 and we can have a $H_{\kappa,\kappa}$-decomposition with main class A_j. Let $I_j = \operatorname{ran}(t_j) \in [r]^{<r}$.

Case 2/a: there is an $i \in I_j$ so that for all $\tilde{A} \in [A]^{<\kappa}$ and $\lambda < \kappa$ there is an i-monochromatic path P of size at least λ in $A_j \backslash \tilde{A}$ which is concentrated on A_j. We claim that A_j is covered by a single i-monochromatic path P, i.e. let $A_{j+1} = \varnothing$ and $\mathcal{P}_{j+1} = \mathcal{P}_j \cup \{P\}$. Otherwise, we apply Lemma 9.7 to find a graph H in A_j of type $H_{\kappa,\kappa}$ which is coloured only by $\leqslant r-1$ colours. This cannot happen by our assumption (that Case 1 fails) and Observation 9.4.

Case 2/b: not Case 2/a. Note that A_j is of type $H_{\kappa,\kappa}$ with main class A_j so Lemma 9.8 implies that we can find $A \in [A_j]^\kappa$ which is κ-connected in some color $i \in r \backslash I_j$. We can select $U \subset A$ so that $A_j \backslash A \cup U$ is of type $H_{\lambda,\lambda}$ with $\lambda \leqslant \kappa$ and main class $A_j \backslash U$ while $A_j \backslash U'$ is κ-connected in colour i for all $U' \subset U$. Now apply Lemma 9.7 to cover $A_j \backslash A$ with finitely many disjoint monochromatic paths $\mathcal{P}$ inside $A_j \backslash A \cup U$. Let $A_{j+1} = A_j \backslash \cup \mathcal{P}$, $\mathcal{P}_{j+1} = \mathcal{P}_j \cup \mathcal{P}$ and $t_{j+1} \supseteq t_j$ with $t_{j+1}(j) = i$.

If $A_r \neq \varnothing$ then note that Case 2/a must have failed at all previous steps so by induction A_r must satisfy Case 2/a with the last colour $i = t_r(r)$. Thus we have $A_{r+1} = \varnothing$ which finishes the proof.

□

9.3. **Proving the main lemmas.** Our first goal is to prove Lemma 9.8.

2 **Lemma 9.11.** *Suppose that $\omega \leqslant \lambda \leqslant cf\kappa$ and $(IH)_{\lambda,r-1}$ holds. Suppose that c is an r-edge coloring of a graph $G = (V,E)$ where G is of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition A, B. Let $I \in [r]^{<r}$ and suppose that A is λ-saturated in all colours $i \in I$. Then either*

(a) there is an $i \in I$ and an i-monochromatic path of size λ concentrated on A, or

(b) there is $\tilde{A} \in [A]^{<\lambda}$ so that
$$|B \backslash N(a, I)| = \kappa$$
for all $a \in [A \backslash \tilde{A}]^{<\omega}$.

Moreover, (b) implies that $A \backslash \tilde{A}$ can be partitioned into $A \backslash \tilde{A} = \cup\{A_j : j \in r \backslash I\}$ where $|N(x,j) \cap N(x',j)| = \kappa$ for all $x, x' \in A_j$ and $j \in r \backslash I$.

Proof. Suppose (b) fails. Then inductively build a sequence of pairwise disjoint finite sets $\{a_\xi : \xi < \lambda\} \subset [A]^{<\omega}$ and sequence of points $Y = \{y_\xi : \xi < \lambda\}$ so that
$$y_\zeta \in N(a_\xi, I)$$

for all $\xi \leqslant \zeta < \lambda$. Now let $a'_\xi = \cup\{a_{\xi+i} : i < |I|+1\}$ and $y'_\xi = y_{\xi+|I|+1}$ for $\xi < \lambda$ limit. Note that for all limit ordinals $\xi \leqslant \zeta < \lambda$ there is an $i \in I$ so that

$$|\{x \in a'_\xi : c(x, y'_\zeta) = i\}| \geqslant 2.$$

By thinning out, we can suppose that for all $i \in I$, $\xi < \lambda$ and $x, x' \in a'_\xi$ there are λ-many disjoint finite i-monochromatic path from x to x' which avoid $Y' = \{y'_\xi : \xi < \lambda\}$ and all other points of $X = \cup\{a'_\xi : \xi < \lambda\}$.

Define a coloring of $H_{\lambda,\lambda}$ by $d(\xi, \zeta) = i$ iff $|\{x \in a'_\xi : c(x, y'_\zeta) = i\}| \geqslant 2$ and i is minimal such. $(IH)_{\lambda,r-1}$ implies the existence of a monochromatic path Q of size λ concentrated on the main class of $H_{\lambda,\lambda}$. It is easy to see now that using Q we can define a monochromatic path of size λ in our original graph which is concentrated on A.

Finally, let $\mathcal{U}$ be a uniform ultrafilter on B containing the filter $\{B \setminus N(a, I) : a \in [A \setminus \tilde{A}]^{<\omega}\}$. Define $A_j = \{x \in A \setminus \tilde{A} : N(x, j) \in \mathcal{U}\}$ for $j \in r$. Clerly $A \setminus \tilde{A}$ is partitioned into $\cup\{A_j : j \in r\}$ but note that $A_i = \emptyset$ if $i \in I$.

□

Proof of Lemma 9.8. Note that if κ is regular then applying Lemma 9.11 to $\lambda = \kappa$ gives us Lemma 9.8.

Suppose that κ is singular with $cf(\kappa) = \mu$ and pick a sequence of regular cardinals κ_α increasing to κ. We can suppose that (a) fails so say that there are no i-monochromatic paths of size κ_0. Write $H_{\kappa,\kappa}$ as an increasing union $\cup\{V_\alpha : \alpha < \mu\}$ so that V_α is of type $H_{\kappa_\alpha,\kappa_\alpha}$ so that $A \cap V_\alpha$ is κ_α-saturated in all colours $i \in I$. Apply Lemma 9.11 to each V_α with $\lambda = \kappa_0$ and select $\tilde{A}_\alpha \in [V_\alpha \cap A]^{<\kappa_0}$ so that

$$|B \setminus N(a, I)| \geqslant \kappa_\alpha$$

for all $a \in [A \cap V_\alpha \setminus \tilde{A}_\alpha]^{<\omega}$. Note that $\tilde{A} = \cup\{\tilde{A}_\alpha : \alpha < \mu\}$ has size $\mu \cdot \kappa_0$ less than κ and that

$$|B \setminus N(a, I)| \geqslant sup\{\kappa_\alpha : \alpha < \mu\} = \kappa$$

for all $a \in [A \setminus \tilde{A}]^{<\omega}$. The ultrafilter trick described at the end of Lemma 9.11 finishes the proof.

□

We now turn our attention to proving Lemma 9.7. We need to prove some claims first:

aim **Claim 9.11.1.** *Suppose that $\kappa \geqslant cf(\kappa) > \omega$, c is an r-edge coloring of a graph $G = (\kappa, E)$ where G is of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition A, B. Suppose that $\{M_\alpha : \alpha < \mu\}$ is a continuous ϵ-chain of elementary submodels with $|M_\alpha| = \kappa_\alpha < \kappa$ so that κ_α is a subset and element of M_α. If $i \in r$ then either*

(a) there is club $C \subset \mu$ so that for every $\alpha \in C$ there is $x \in A \setminus M_\alpha, y \in B \setminus M_\alpha$ with $c(x, y) = i$ and

$$|N(y, i) \cap A \cap M_\alpha \setminus M_{\alpha'}| \geqslant \omega$$

for all $\alpha' < \alpha$, or

(b) there is $\tilde{A} \in [A]^{<\kappa}$ so that $A \setminus \tilde{A}$ is covered by the main class of an $\leqslant r-1$-coloured graph H of type $H_{\kappa,\kappa}$.

Proof. Suppose that (a) fails; i.e. there is a stationary set $S \subset \mu$ so that for all $x \in A \backslash M_\alpha, y \in B \backslash M_\alpha$ with $c(x,y) = i$ we have

$$|N(y,i) \cap A \cap M_\alpha \backslash M_{\alpha'}| < \omega$$

for some $\alpha' < \alpha$.

Note that

Observation 9.12. *If there is an $\alpha \in S$ and $\lambda < \kappa$ so that*

$$|N(x,i)| \leqslant \lambda$$

for every $x \in A \backslash M_\alpha$ then (b) holds with $\tilde{A} = A \cap M_\alpha$.

Otherwise, we distinguish two cases:

Case 1: κ is regular. Note that there is a club $D \subset \kappa$ so that $M_\alpha \cap \kappa \in \kappa$ if $\alpha \in D$ and so we can suppose that $S \subset D$. Also, note that $\{M_\alpha \cap \kappa : \alpha \in S\}$ is stationary in κ.

Select $y_\alpha \in B \backslash M_\alpha$ so that

$$|N(y_\alpha, i) \cap A \cap M_\alpha \backslash M_{\alpha'}| < \omega$$

for some $\alpha' < \alpha$. That is, there is $\delta_\alpha < M_\alpha \cap \kappa$ so that

$$N(y_\alpha, i) \cap A \cap M_\alpha \subset \delta_\alpha.$$

Apply Fodor's pressing down lemma to the regressive function $M_\alpha \cap \kappa \to \delta_\alpha$ and find stationary $T \subset S$ and $\delta \in \kappa$ so that $\delta_\alpha = \delta$ for all $\alpha \in T$. It is easy to see that (b) is satisfied with $\tilde{A} = \delta$.

Case 2: κ is singular. Without loss of generality $\kappa_\alpha > \mu$ if $\alpha \in S$. Select $Y_\alpha \in [B \backslash M_\alpha]^{\kappa_\alpha^+}$ so that there is finite F_α and $\delta_\alpha < \alpha$ with

$$F_\alpha = N(y,i) \cap A \cap M_\alpha \backslash M_{\delta_\alpha}$$

for all $y \in Y_\alpha$. The importance here is that δ_α and $N(y,i) \cap A \cap M_\alpha \backslash M_{\delta_\alpha}$ does not depend on $y \in Y_\alpha$ which can be done as there are only $|\alpha|$ choices for δ_α and κ_α choices for F_α while κ_α^+ choices for y.

Apply Fodor's pressing down lemma to the regressive function $\alpha \to \delta_\alpha$ and find stationary $T \subset S$ and $\delta \in \mu$ so that $\delta_\alpha = \delta$ for all $\alpha \in T$. It is easy to see that $\tilde{A} = M_\delta \cup \bigcup\{F_\alpha : \alpha \in T\} \in [A]^{<\kappa}$ satisfies (b).

□

Claim 9.12.1. *Suppose that $\kappa > \omega$ has countable cofinality, c is an r-edge coloring of a graph $G = (\kappa, E)$ where G is of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition A, B. If $i \in r$ and $(M_n)_{n \in \omega}$ is an ϵ-chain of covering elementary submodels of regular size $< \kappa$ with $\kappa \subset \cup M_n$ then either*

(a) there are infinitely many $n \in \omega$ such that there is $x \in A \backslash M_n, y \in B \backslash M_n$ with $c(x,y) = i$ and

$$|N(y,i) \cap A \cap M_n| \geqslant \omega,$$

or

(b) there is $\tilde{A} \in [\kappa]^{<\kappa}$ so that $A \backslash \tilde{A}$ is covered by an $\leqslant r-1$-coloured H of type $H_{\kappa,\kappa}$.

Proof. Pick any ϵ-chain of covering elementary submodels $(M_n)_{n\in\omega}$ as above and suppose that (a) fails. Without loss of generality, we can suppose that if $x \in A\backslash M_n, y \in B\backslash M_n$ with $c(x,y) = i$ then $N(y,i) \cap M_n$ is finite for all $n \in \omega$.

Observation 9.13. *If there is $n \in \omega$ and $\lambda < \kappa$ so that $N(x,i) \leqslant \lambda$ for all $x \in A\backslash M_n$ then (b) holds with $\tilde{A} = A \cap M_n$.*

Otherwise, we can select $Y_n \in [M_{n+1}\backslash M_n]^{|M_n|^+}$ and finite $a_n \subset A\cap M_n$ so that $N(y,i) \cap M_n = a_n$ for all $y \in Y_n$. Let $\tilde{A} = \cup\{a_n : n \in \omega\}$ and $Y = \cup\{Y_n : n \in \omega\}$. It easy to see now that $A\backslash\tilde{A}$ is covered by H of type $H_{\kappa,\kappa}$ so that $V(H) \cap B \subset Y$. □

Before proving Lemma 9.7 we need the following

ego **Lemma 9.14.** *Suppose that c is an r-edge colouring of a graph $G = (V,E)$, $i \in r$ and $A \subset V$ is κ-saturated in a color i with $a \in A$. Furthermore, suppose that there is an i-monochromatic path P of order type κ so that P is concentrated on A. Then A is actually covered by an i-monochromatic path Q so that Q is concentrated on A, has order type κ and first point a. Moreover, for any given $K \in [A]^{cf(\kappa)}$ we can construct Q with the above properties so that K is cofinal in Q.*

Proof. We prove by induction on κ; the result clearly holds for $\kappa = \omega$ so suppose that $\kappa > \omega$ and we proved for cardinals $< \kappa$. Also, fix $a \in A$, $K \in [A]^{cf(\kappa)}$ and i-monochromatic path P concentrated on A; note that we don't need to worry about K if κ is regular as every subset of A of size κ will be cofinal in Q.

Find a continuous ϵ-chain of elementary submodels $(M_\alpha)_{\alpha<\mu}$ with $\mu = cf(\kappa)$ so that

(i) $M_0 = \emptyset$, $\kappa_\alpha = |M_\alpha| < \kappa$ and $M_{\alpha+1}$ is covering for all $\alpha < \mu$,
(ii) if κ is a limit then make κ_α strictly increasing with $\kappa_{\alpha+1}$ regular,
(iii) if $\kappa = \lambda^+$ then let $\kappa_\alpha = \lambda$,
(iv) let $M_\alpha \cap \kappa \in \kappa$ if κ is regular,
(v) κ_α is an element and subset of M_α for $\alpha > 0$,
(vi) P, A, a and everything relevant is in M_1.

Let us enumerate K as $\{k_\alpha : \alpha < \mu\}$ if κ is singular.

Observation 9.15. *There is a sequence $\{R_\alpha : \alpha < \mu\}$ of disjoint, finite i-monochromatic paths with union R so that*

(1) $R_0 = \{a\}$,
(2) R_α is a path from a point x_α which terminates in an element of A and $R_\alpha \subset M_{\alpha+1}$,
(3) $|N(x_\alpha,i) \cap A \cap M_\alpha\backslash M_{\alpha'}| \geqslant cf(\kappa_\alpha)$ for all $\alpha' < \alpha < \mu$.
(4) $k_\alpha \in R_\alpha$ for $\alpha < \mu$ if κ is singular.
(5) $A \cap M_{\alpha+1}\backslash M_\alpha$ is κ_α-saturated even in $V \cap M_{\alpha+1}\backslash(M_\alpha \cup R)$,
(6) $M_{\alpha+1}\backslash M_\alpha \cap R$ is finite for all $\alpha < \kappa$ if κ is regular.

Proof. First, suppose that κ is regular. Let $x_\alpha = \min P\backslash M_\alpha$ and $x'_\alpha = \min P \cap A\backslash M_\alpha$ with $R_\alpha = P \restriction [x_\alpha, x'_\alpha]$ for limit $\alpha < \mu$. Let $x_{\alpha+1} \in M_{\alpha+1}\backslash(M_\alpha \cup R_\alpha)$ be a $\kappa_{\alpha+1}$-limit of P and $x'_{\alpha+1}$ be the first element of $P \cap A$ above $x_{\alpha+1}$. (1)

and (2) are clearly satisfied; it suffices to check (3) which will be satisfied by our choices of x_α and the fact that $M_{\alpha'} \cap P$ is an initial segment of P below x_α. (4) and (5) holds as the R_α's are separated by the models.

Now, suppose that $\kappa > \mu$. Select κ_α^+-limit x_α of P and let x'_α be the first element of $P \cap A$ above x_α; extend $P \restriction [x_\alpha, x'_\alpha]$ into a finite i-monochromatic path R_α which terminates in k_α. Note that we can actually suppose that $\{R_\alpha : \alpha < \mu\}$ is a subset and element of M_1. Now (3) is satisfied by the fact that $|M_{\alpha'}| < \kappa_\alpha$ and (5) holds as $\cup\{R_\alpha : \alpha < \mu\}$ has size less than κ. □

Let $R = \cup\{R_\alpha : \alpha < \mu\}$ as above. We construct a continuous increasing sequence $(Q_\alpha)_{\alpha<\mu}$ of i-monochromatic paths so that

(a) Q_α covers $A \cap M_\alpha \backslash R$ inside M_α,
(b) Q_α is of order type κ_α and is concentrated on A,
(c) Q_α end extends $Q_{\alpha'}$ for all $\alpha' < \alpha < \mu$,
(d) the first element of Q_0 is a,
(e) $N(x_\alpha, i) \cap A \cap M_{\alpha+1} \backslash M_\alpha$ is cofinal in $Q_{\alpha+1}$,
(f) R_α is an initial segment of $Q_{\alpha+1} \backslash Q_\alpha$ while $R_\alpha \cap Q_{\alpha'} = \varnothing$ for all $\alpha' < \alpha$.

Clearly, $Q = \cup\{Q_\alpha : \alpha < \mu\}$ will be an i-monochromatic path that covers A and is concentrated on A.

Let $Q_0 = \varnothing$. If $\alpha < \mu$ is limit and we constructed $Q_{\alpha'}$ for $\alpha' < \alpha$ then let $Q_\alpha = \cup\{Q_{\alpha'} : \alpha' < \alpha\}$. Suppose we constructed Q_α and we wish to extend it into $Q_{\alpha+1}$.

Observation 9.16. *$Q_\alpha{}^\frown R_\alpha$ is an i-monochromatic path concentrated on A.*

Proof. Recall that x_α was the first element of R_α so we need to check that

$$N(x_\alpha, i) \cap A \cap Q_\alpha$$

is cofinal in Q_α. First, if α is a limit this amounts to checking that

$$N(x_\alpha, i) \cap A \cap M_\alpha \backslash M_{\alpha'} \neq \varnothing \text{ for all } \alpha' < \alpha$$

as any selection from $\{A \cap M_\alpha \backslash M_{\alpha'}\}$ is cofinal in Q_α. This clearly holds by the choice of x_α.

Second, if $\alpha = \alpha' + 1$ then property (e) from above ensures that x_α is a valid continuation of $Q_{\alpha'}$.

□

Therefore, our goal now is to cover $A \cap M_{\alpha+1} \backslash (M_\alpha \cup R)$ by a path P_α inside $V \cap M_{\alpha+1} \backslash (M_\alpha \cup R)$ where P_α is concentrated on A, starts at x'_α (the endpoint of R_α), has order tpye $\kappa_{\alpha+1}$ and $N(x_{\alpha+1}, i) \cap A \cap M_{\alpha+1} \backslash M_\alpha$ is cofinal in P_α. We then define

$$Q_{\alpha+1} = Q_\alpha{}^\frown R_\alpha{}^\frown P_\alpha$$

which will finish the induction and our proof. Naturally, we will apply the inductional hypothesis on $\kappa_{\alpha+1} < \kappa$ to conctruct P_α with cofinal set $K_\alpha \in [N(x_{\alpha+1}, i) \cap A \cap M_{\alpha+1} \backslash M_\alpha]^{cf(\kappa_{\alpha+1})}$. Note that $A \cap M_{\alpha+1} \backslash (M_\alpha \cup R)$ is $\kappa_{\alpha+1}$-saturated in the graph $V \cap M_{\alpha+1} \backslash (M_\alpha \cup R)$ and $x'_\alpha \in A$ by the construction of the R.

Hence, it suffice to show that there is a path P' inside $V \cap M_{\alpha+1} \backslash (M_\alpha \cup R)$ which is concentrated on A and has order type $\kappa_{\alpha+1}$. If κ is regular then $P \cap M_\alpha$ is an initial segment of P defined in $M_{\alpha+1}$ and hence we can select a $\kappa_{\alpha+1}$-limit z of our original path P above M_α that is in $M_{\alpha+1}$ and note $P \restriction z \subset M_{\alpha+1}$. Also, $P \cap (M_\alpha \cup R)$ must be bounded below z in P by point (5) in our construction of R. Hence, there is $w \in P \restriction z$ so that

$$P' = P \restriction [w, z) \subset M_{\alpha+1} \backslash (M_\alpha \cup R)$$

which we wanted to show. If κ is singular then the first $\kappa_{\alpha+1}$-limit z of P is in $M_{\alpha+1}$ and the same argument works.

□

Finally we arrive at the

Proof of Lemma 9.7. We start by assuming that (b) fails.

We distinguish 2 cases depending on the value of $cf(\kappa)$.

Case 1: suppose that $\kappa > cf(\kappa) = \omega$. Recall Claim 9.12.1 and take an ϵ-chain of covering elementary submodels $(M_n)_{n\in\omega}$ of uncountable regular size $< \kappa$ with $\kappa \subset \cup M_n$ so that there is an infinite $N \subset \omega$ so that for all $n \in N$ there is $x_n \in A \backslash M_n, y_n \in B \backslash M_n$ with $c(x_n, y_n) = i$ and

$$|N(y_n, i) \cap A \cap M_n| \geqslant \omega.$$

Without loss of generality $N = \omega$ and $x_n, y_n \in M_{n+1}$.

Observation 9.17. *There is a sequence $\{R_n : n \in \omega\}$ of pairwise disjoint finite i-monochromatic path so that*

(1) R_n is a path from a point $u_n \in M_n$ with last two points y_n, x_n,
(2) $|N(u_n, i) \cap A \cap M_n| = \kappa_n$.

Proof. First, find distinct $z_n \in N(y_n, i) \cap A \cap M_n$ for $n \in N$. We build R_n inductively on $n \in N$. Now, in each model M_n we can find an i-monochromatic path Q_n concentrated on A, of order type $\kappa_n + \omega + 1$ and disjoint from $\cup\{R_k : k < n\}$; we can take u_n to be its κ_n-limit point. Find $v_n \in Q_n \cap A$ above u_n. We can extend $Q_n \restriction [u_n, v_n]$ into a path R_n terminating in y_n, x_n and avoiding $\cup\{R_k : k < n\}$. This is done using saturation. □

Let $R = \cup\{R_n : n \in N\}$. If $P_n \subseteq M_n$ is an i-monochromatic path of order type κ_n which covers $A \cap M_n \backslash R$ and is disjoint from R_n then $P_n{}^\frown R_n$ is still an i-monochromatic path; also, if P_n is concentrated on A then so does $P_n{}^\frown R_n$. Hence it suffice to find P_n as above so that P_{n+1} starts from $\{x_n\}$ (the terminal point of R_n). Then $P = \cup\{P_n{}^\frown R_n : n \in N\}$ is a path as desired.

It is easy to see that Lemma 9.14 can be applied to the construction of P_n.

Case 2: $\kappa \geqslant \mu = cf(\kappa)$. By Claim 9.11.1 and the failure of (b) we can find a continuous ϵ-chain of elementary submodels $(M_\alpha)_{\alpha<\mu}$ so that

(i) $M_0 = \varnothing$, and M_α is the limit of covering elementary submodels for all $\alpha \in \mu \backslash 1$,
(ii) if $\kappa = \lambda^+$ then let $\kappa_\alpha = |M_\alpha| = \lambda$,
(iii) if κ is a limit then let κ_α strictly increasing,
(iv) let $M_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$ if $\kappa = \mu$

(v) κ_α is an element and subset of M_α for $\alpha > 0$,
(vi) P, A, a, C and everything relevant is in M_1,
(vii) for every $\alpha \in \mu$ there is $x \in A \backslash M_\alpha, y \in B \backslash M_\alpha$ with $c(x,y) = i$ and

$$|N(y,i) \cap A \cap M_\alpha \backslash M_{\alpha'}| \geqslant \omega$$

for all $\alpha' < \alpha$

We construct a continuous increasing sequence $(Q_\alpha)_{\alpha<\mu}$ of i-monochromatic paths and finite i-monochromatic paths $(R_\alpha)_{\alpha<\mu}$ so that

(a) Q_α is inside M_α and $Q_\alpha \cup R_\alpha$ covers covers $A \cap M_\alpha$,
(b) Q_α is of order type κ_α and is concentrated on A,
(c) Q_α end extends $Q_{\alpha'}$ for all $\alpha' < \alpha < \mu$,
(d) R_α starts with a point y_α and terminates in an element of A,
(e) $|N(y_\alpha, i) \cap A \cap M_\alpha \backslash M_{\alpha'}| \geqslant cf(\kappa_\alpha)$ if $\alpha' < \alpha < \kappa$,
(f) R_α is an initial segment of $Q_{\alpha+1} \backslash Q_\alpha$.

Let $Q_0 = \varnothing$ and R_0 a single element of $A \cap M_1$. For limit α simply let $Q_\alpha = \cup\{Q_{\alpha'} : \alpha' < \alpha\}$. By (vii) above, we can find $x_\alpha \in A \cap M_{\alpha+1} \backslash M_\alpha, y_\alpha \in B \cap M_{\alpha+1} \backslash M_\alpha$ connected by an edge coloured i so that $R_\alpha = (y_\alpha, x_\alpha)$ satisfies the above conditions.

Suppose we constructed Q_α and R_α and we wish to extend it into $Q_{\alpha+1}$. Let us define $R_{\alpha+1}$ first, i.e. the end of the new extension.

Observation 9.18. *There is a finite i-monochromatic path $R_{\alpha+1} \subset V \backslash (M_\alpha \cup R_\alpha)$ from an element $u \in M_{\alpha+1}$ into a $v \in A \cap M_{\alpha+2} \backslash M_{\alpha+1}$ so that $N(u.i) \cap A \cap M_{\alpha+1} \backslash M_\alpha$ has size at least $cf(\kappa_{\alpha+1})$.*

Proof. If κ is limit then pick an i-monochromatic path P of order type $\kappa_{\alpha+1} + \omega + 1$ in $M_{\alpha+1}$ which is inside $M_{\alpha+1} \backslash M_\alpha$; this can be done as $\kappa_{\alpha+1}^+$ is still less than κ. Let u be the $\kappa_{\alpha+1}$-limit of P and v' the first element of $P \cap A$ above u. Pick arbitrary $v \in A \cap M_{\alpha+2} \backslash M_{\alpha+1}$ and extend $P \restriction [u, v']$ into a finite i-monochromatic path $R_{\alpha+1} \subset V \backslash (M_\alpha \cup R_\alpha)$ terminating in v.

Now, suppose that $\kappa = \lambda^+$. There is a chain of elementary submodels $\{N_\xi : \xi < \lambda\} \in M_{\alpha+1}$ with $N = \cup\{N_\xi : \xi < \lambda\}$ so that $M_\alpha \subset N_0$ and there is $x \in A \backslash N, y \in B \backslash N$ with $c(x,y) = i$ and

$$|N(y,i) \cap A \cap N \backslash N_\xi| \geqslant \omega.$$

Cleary, $N \subset M_{\alpha+1}$ as N has size λ and also we can take x, y as above in $M_{\alpha+1}$. Note that we have

$$|N(y,i) \cap A \cap N \backslash M_\alpha| \geqslant cf(\lambda)$$

so we can extend the path (y,x) into R_α terminating in an arbitrary element of $A \cap M_{\alpha+2} \backslash M_{\alpha+1}$.

□

Now use Lemma 9.14 to cover the rest of $A \cap M_{\alpha+1}$ by an i-monochromatic path P' concentrated on A, of order type $\kappa_{\alpha+1}$ so that there is $K_\alpha \in [N(y_\alpha, i) \cap A \cap M_{\alpha+1} \backslash M_\alpha]^{cf(\kappa_{\alpha+1})}$ cofinal in P';note that $P^\frown R'_{\alpha+1}$ is also an i-monochromatic path concentrated on A.

□

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