Infinite Combinatorics

From Finite to Infinite

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Outline

Survey some finite problems and their infinite counterparts.
Methods of proofs

1. From Finite to Infinite
   - Connectedness
   - Spanning trees
   - Normal spanning trees
   - Pseudo-winners in tournaments
   - Unfriendly partitions
   - Splitting antichains in posets
   - Multi-way cuts
   - Chromatic number of product of graphs
2. Case study: quasi kernels and quasi sinks
3. Classical theorems
   - Euler lines and Euler circles
   - Covering and matching

Connectedness

Finite case

Theorem

A finite graph $G = (V, E)$ is connected iff for each $x, y \in V$ there is an x-y-path.

General (infinite) case

Theorem

A graph $G = (V, E)$ is connected iff for each $x, y \in V$ there is an x-y-path.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Same theorems, same proofs Finite=Infinite

Spanning trees

Finite case

Theorem

Every finite connected graph $G = (V, E)$ has a spanning tree.

General (infinite) case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

Zorn lemma : $\langle \mathcal{T}, \subset \rangle$ has a maximal element T

Let T be a maximal connected subtree of G.

There is no edge between T and $V \setminus T$.

$T = V$.

Same theorems, almost the same proofs, Finite≡Infinite

Normal spanning trees

Defi nition

A **normal spanning tree** of a connected graph $G = (V, E)$ is a **rooted subtree** T of G such that for each edge $(x, y) \in E$ the endpoints x and y are comparable in the rooted tree order.

depth-first search tree

Normal spanning trees

Finite case

Theorem

Every finite connected graph has a normal spanning tree.

General (infinite) case

Observation

$K_{\aleph_1}$ does not have a normal spanning tree.

Theorem

Every countable connected graph has a normal spanning tree.

Proof

Depth-first algorithm does not work for certain graphs: the algorithm may not visit all the vertices. Some inductive proof works.

Finite$\equiv$countable$\neq$ uncountable

Pseudo-winners in tournaments

Defi nition

Let $T = (V, E)$ be a tournament and let $t \in V$.
t is a ***pseudo-winner*** iff for each $y \in V$ there is a **path of length at most** 2 which leads from t to y

Pseudo-winners in tournaments

Finite case

Theorem

Every finite tournament has a pseudo-winner.

Proof

If t has maximal out-degree then t is a pseudo-winner.
Let $v \in V$. If $(t, v) \in E$ then OK.
If $(v, t) \in E$ then
$t \in Out(v) \setminus Out(t)$, so
$Out(t) \not\subset Out(v)$. If
$s \in Out(t) \setminus Out(v)$ the tsv is a directed path of length 2.

Infinite case

Observation

No pseudo-winner in $\langle \mathbb{Z}, < \rangle$.

Theorem

A tournament T contains a pseudo-winner or $\exists x \neq y \in V$ s.t. $T = Out(x) \cup In(y)$.

Proof

If y is not a pseudo-winner witnessed by x, then
$T = Out(x) \cup In(y)$.

Pseudo-winners in tournaments

Finite $\neq$ infinite, but infinite case is easy provided you know what you have to prove

Unfriendly partitions

Defi nition

Let $G = (V, E)$ be a graph. A partition (A, B) of V is called **unfriendly** iff every vertex has at least as many neighbor in the other class as in its own.

Observation

Every finite graph has an unfriendly partition.

Proof:

Take a partition having maximal number of edges between the classes of the partition.

Theorem

Every locally finite graph has an unfriendly partition.

Proof: locally finite graphs have unfriendly partitions

Gödel's Compactness Theorem

Theorem (Gödel)

A theory T has a model provided every finite subset T' of T has a model.

Proof: locally finite graphs have unfriendly partitions

Gödel's Compactness Theorem

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.
Formulas: ψ: $\forall x \ \big(R_A(x) \leftrightarrow \neg R_B(x)\big)$
for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put
$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$
$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$
Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim

Every $T' \in [T]^{<\omega}$ has a model.

Let $W = \{v : c_v \text{ occurs in } T'\}$. Then $G[W]$ has an unfriendly partition (A, B). Let M be the following model: the underlying set M is W, c_v is interpreted as v for $v \in W$, and R_A is interpreted as A and B is interpreted as B.

Let M be a model of T, and let $A = \{v \in V : M \models R_A(c_v)\}$ and

Unfriendly partitions

Unfriendly Partition Conjecture

Every graph has an unfriendly partition.

Theorem (Shelah)

There is an uncountable graph without an unfriendly partition.

Unfriendly Partition Conjecture, Revised

Every countable graph has an unfriendly partition.

Theorem (Shelah)

Every graph has a partition into three pieces such that every vertex has at least as many neighbor in the two other classes as in its own.

Unfriendly partitions

If $G = (V, E)$ is countable and every $v \in V$ has infinite degree then G has an unfriendly partition.

Problem

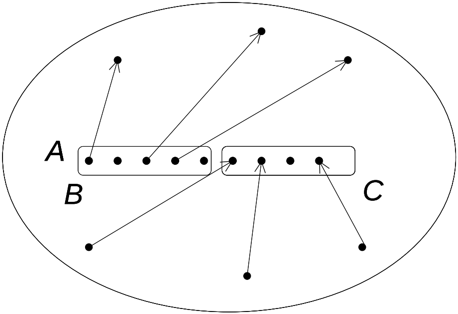
Let G be a finite graph, and a and b are vertices such that $d_G(a, b) \geq 10^{10^{10}}$. Is there an unfriendly partition of (A, B) of G such that $a \in A$ and $b \in B$?

Unfriendly partitions

finite case is trivial,
locally finite
the hardest case is the “mixed” countable case

Definition

Let $P = \langle P, \leq \rangle$ be a poset, $A \subset P$ be a **maximal antichain**.
$A \subset P$ **splits** iff A has a **partition** $A = B \cup^* C$ such that $P = B^{\uparrow} \cup C^{\downarrow}$.



Theorem (P. L Erdős - Niall Graham (1993))

In a finite Boolean lattice every max. antichain splits.

Theorem

*If P is a finite poset and A is a maximal antichain then the question **"Does A split?"** is NP-complete.*

Defi nition

$y \in P$ is a **cutting point** *iff* $\exists x, z \in P$ s.t. $x <_P y <_P z$ and $[x, z] = [x, y] \cup [y, z]$.
P is **cut-free** if there is no cutting point in P cutting point.

Theorem (Ahlswede, P. L. Erdős, N. Graham(1995))

In a finite cut-free poset every max. antichain splits.

Theorem (Ahlswede, P. L. Erdős, N. Graham(1995))

In a finite cut-free poset every finite maximal antichain splits.

Theorem (P. L. Erdős, –)

In a cut-free poset every finite maximal antichains split.

keep finite certain key structures

No proofs with Gödel Compactness Theorem!

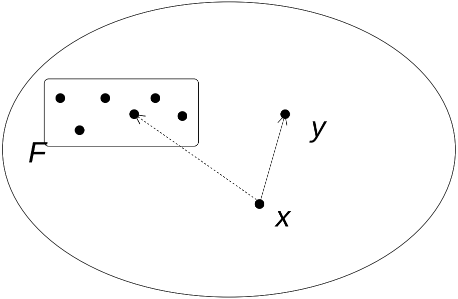
If P is cut-free, $Q \subset P$ then Q is not necessarily cut-free

Theorem (Ahlswede, Khachatrian)

There is a maximal, (infinite) non-splitting antichain A in $\left\langle [\omega]^{<\omega}, \subset \right\rangle$.

Definition

A poset $\mathcal{P}$ is **loose** iff for each $x \in P$ and $F \in [P]^{<\omega}$ if $x \notin F^{\uparrow}$ then there is $y \in x^{\uparrow} \setminus \{x\}$ such that $y \notin F^{\updownarrow}$.



Defi nition

A poset $\mathcal{P}$ is **loose** iff for each $x \in P$ and $F \in [P]^{<\omega}$ if $x \notin F^{\uparrow}$ then there is $y \in x^{\uparrow} \setminus \{x\}$ such that $y \notin F^{\updownarrow}$.

Theorem (P. L. Erdős, –)

A countable, cut-free, loose poset $\mathcal{P} = \langle P, \leq \rangle$ contains a maximal infinite non-splitting antichain A, moreover if $B, C \subset A$ with $B^{\uparrow} \cup C^{\downarrow} = P$ then $B = A$ and C is infinite.

Multi-way cuts

Multi-way Cut Problem

Given a graph $G = (V, E)$ and disjoint subsets $(A_0, A_1, \dots A_n)$ of V determine the minimal size of a family F of edges separating A_i and A_j whenever $i \neq j$.

G is a finite tree

Multi-way cuts

Theorem

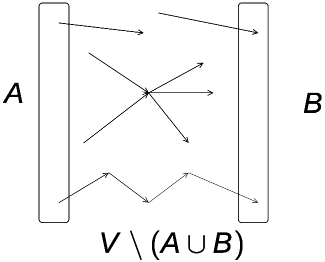
Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup b)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.



Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed path, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

Proof.

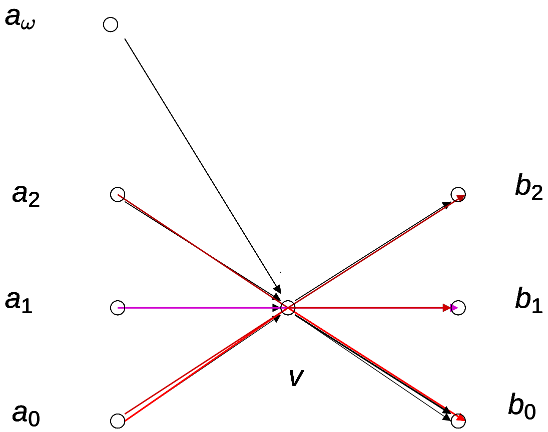
G is countable: easy induction: if P is an A-B-path then $G - P$ satisfies (1)–(3)

G is uncountable may got stuck at some point

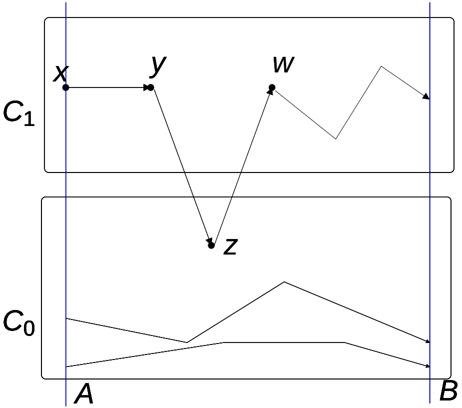
□

Multi-way cuts

Infi nite case



$G = (V, E)$, $A, B \subset V$, $|V| = |A| = \omega_1$
Inductive construction, but using the right enumeration
Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$
Enumerate $A = \{a_\xi : \xi < \omega_1\}$ such that $C_0 \cap A = \{a_0, a_1, \ldots\}$,
$C_1 \cap A = \{a_\omega, a_{\omega+1}, \ldots\}$,
By transfinite induction find disjoint families $\mathcal{P}_\alpha$ of A-B paths in
$G[\cup\{C_\xi : \xi \leq \alpha\}]$ such that $\mathcal{P}_\alpha$ covers $C_\alpha \cap A$.



(1) If $v \in C_\alpha$ with $|Out(v)| \leq \omega$ then $Out(v) \subset \cup\{C_\xi : \xi \leq \alpha\}$

(2) If $v \in C_\alpha$ with $|Out(v)| > \omega$ then $Out(v) \cap C_\alpha$ is infinite

(3) If $v \in C_\alpha$ with $|In(v)| \leq \omega$ then $In(v) \subset \cup\{C_\xi : \xi \leq \alpha\}$

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

- if $v \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.
- if $v \in C_\alpha$ with $|Out(x)| \leq \omega$ then $Out(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.
- if $v \in C_\alpha$ with $|Out(x)| > \omega$ then $In(x) \cap C_\alpha$ is infinite.

How to get such a partition? How to get the right properties of such a partition?

Elementary submodels

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \dots$
Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.
$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering
Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G \in M_0$. i.e.

(1) M_α is a countable elementary submodel of $\mathcal{H}(\theta)$ for $\alpha < \omega_1$

(2) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$,

(3) $M_\alpha = \bigcup \{M_\beta : \beta < \alpha\}$ provided α is limit

(4) $G \in M_0$.

Let $C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

Multi-way cuts

the uncountable case is much harder than the countable case

Chromatic number of product of graphs

Hedetniemi's Conjecture

If G and H are finite graphs then $\chi(G \times H) = \min\{\chi(G), \chi(M)\}$.

Theorem (El-Sahar, Sauer)

If $\min\{\chi(G), \chi(H)\} \geq 4$ *then* $\chi(G \times H) \geq 4$.

Hedetniemi's conjecture for countable chromatic graphs

Fact

If $\chi(G) \geq \aleph_0$ then for each $n \in \mathbb{N}$ there is a finite subgraph G' of G with $\chi(G) \geq n$.

Fact

If $\chi(G), \chi(H) \geq \aleph_0$, but $\chi(G \times H) = k \in \mathbb{N}$ then for each $n \in \mathbb{N}$ there are a finite graphs G' and $H'$with $\chi(G'), \chi(H') \geq n$ such that $\chi(G' \times H') \leq k$.

Fact

Assume that for each $k \in \mathbb{N}$ there is $f(k) \in \mathbb{N}$ such that $\chi(G \times H) \geq k$ whenever $\chi(G), \chi(H) \geq f(k)$. Then $\chi(G \times H) \geq \aleph_0$ whenever $\chi(G), \chi(H) \geq \aleph_0$.

Hedetniemi's conjecture for uncountable chromatic graphs

Theorem (Hajnal)

There are two ω_1-chromatic graphs G and H on ω_1 s. t. $\chi(G \times H) = \omega$.

Theorem (–)

It is consistent with GCH that there are two ω_2-chromatic graphs G and H on ω_2 s. t. $\chi(G \times H) = \omega$.

Problem

Is it consistent with GCH that there are two ω_3-chromatic graphs G and H on ω_3 s. t. $\chi(G \times H) = \omega$?

Problem

Find any bound lower bound for $\chi(G \times H)$ when G and H are infinite!

Hedetniemi’s conjecture for infinite graphs

Uncountable case is simpler than countable or finite.

Case Study

P. L. Erdös, A. Hajnal, –

Splitting antichains in posets

P poset: G_P comparability digraph

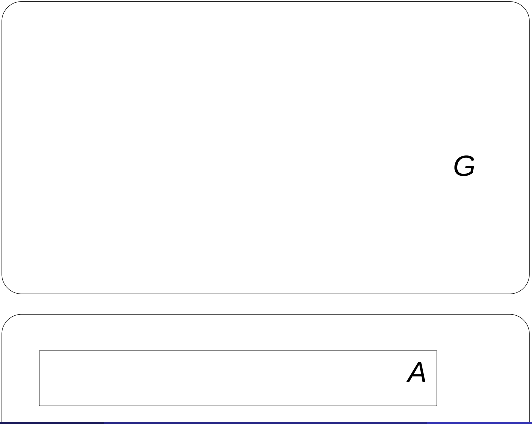
$A \subset P$ is an **antichain** in P iff it is an **independent set** in G_P

$A \subset P$ splits iff $A = B \cup^* C$ s.t in G_P we have $P = Out(B) \cup In(C)$

Introduction

Theorem (Chvatal, Lovász)

Every finite ***digraph*** *(i.e. directed graph) contains an* ***independent set*** *A such that for each point v there is a* ***path of length at most*** *2 from some point of A to v.*



In and Out

Defi nition

Assume that $G = (V, E)$ is a **digraph** and $A \subset V$. For $n \in \mathbb{N}$ let

$$\mathrm{In}_{\mathrm{n}}^{\mathrm{G}}(A) = \{v \in V : \text{there is a path of length at most } n \text{ which leads from } v \text{ to some points of } A\},$$

$$\mathrm{Out}_n^G(A) = \{v \in V : \text{there is a path of length at most } n \text{ which leads from some points of } A \text{ to } v\}.$$

Quasi-kernels and quasi-sinks

Defi nition

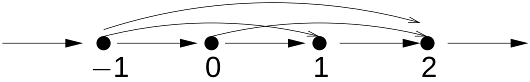
Let $G = (V, E)$ be a digraph.
An **independent set** A is a **quasi-kernel** if and only if $V = \mathrm{Out}_2^G(A)$.
An **independent set** B is a **quasi-sink** if and only if $V = \mathrm{In}_2^G(A)$.

Theorem (Chvatal, Lovász)

Every finite digraph $G = (V, E)$ contains a ***quasi-kernel (quasi-sink)****.*

Infinite tournaments

Plain generalization fails even for **infinite tournaments:** the tournament $(\mathbb{Z}, E_<)$ is a counterexample.



vertex set: integers

(x, y) is an **edge** if and only if $x < y$.

Splitting maximal antichains:

Given a digraph $G = (V, E)$ **find a maximal independent set** $A \subset X$ **with partition** (B, C) **s.t.** $G = \mathrm{Out}_2^G(B) \cup \mathrm{In}_2^G(C)$**!**

Not possible even in $(\mathbb{Z}, E_<)$

Given a digraph $G = (V, E)$ **find two disjoint independent sets** B **and** C **s.t.** $G = \mathrm{Out}_2^G(B) \cup \mathrm{In}_2^G(C)$**!**

The original conjecture

Problem

For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$.

Find counterexample!
Use copies of $(\mathbb{Z}, <)$ and some rule between copies
Did not work
Use copies of $(\mathbb{Z}, <)$, $(\mathbb{N}, <)$ and $\overline{K_{\aleph_0}}$
Did not work

A stepping-up theorem

Definition

$G = (V, E)$ is a **digraph**, $n, k \in \mathbb{N}$:
$G \in \mathfrak{In}_k \iff \exists$ an **independent set** $A \subset V$ s. t. $V = \mathrm{In}^G_k(A)$,
$G \in \mathfrak{Out}_n \iff \exists$ an **independent set** $B \subset V$ s.t. $V = \mathrm{Out}^G_n(B)$.
$G \in \mathfrak{In}_n\text{-}\mathfrak{Out}_k \iff \exists$ **partition** (V_1, V_2) of V s.t $G[V_1] \in \mathfrak{In}_n$ and $G[V_2] \in \mathfrak{Out}_k$.
A directed graph G is ***hereditary in*** $\mathfrak{In}_n\text{-}\mathfrak{Out}_k$ if and only if the **induced subgraphs** of G are all in $\mathfrak{In}_n\text{-}\mathfrak{Out}_k$.

Theorem

If G has a partition $(A_1, \ldots A_k)$ such that each $G[A_i]$ is ***hereditary in*** $\mathfrak{In}_1\text{-}\mathfrak{Out}_1$ *(for example $G[A_i]$ is isomorphic to $(\mathbb{Z}, <)$, to $(\mathbb{N}, <)$, or to a graph without edges) then $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$.*

Hunting for counterexamples

Try other constructions!

Theorem

*If $G = (V, E)$ is a digraph and $\mathrm{In}_1^G(x)$ **is finite** for each $x \in V$ then $G \in \mathfrak{Out}_2$.*

Theorem

*If the **chromatic number** of G is finite then $G \in \mathfrak{Out}_2$.*

Hunting for counterexamples

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. If $K_n \not\subset \widetilde{G}$ for some $n \geq 2$ then $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$. Especially, if the **chromatic number** of $\widetilde{G}$ is finite then $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$*

Theorem

*If $G = (V, E)$ is a digraph such that $\widetilde{G}$ is **locally finite** then $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$.*

A theorem and a conjecture

Theorem

*For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$.*

In the positive theorems we obtained $G \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$!

Conjecture

Every directed graph is in $\mathfrak{In}_2\text{-}\mathfrak{Out}_2$.

Hunting for counterexamples,2

Structure theorems for infinite tournaments:

Theorem

For an infinite tournament T the followings are equivalent:

(1) $T \in \mathfrak{Out}_3$

(2) $T \in \mathfrak{Out}_n$ *for some* $n \geq 3$

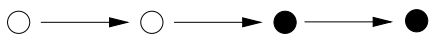
Problem: $T \in \mathfrak{Out}_2$ **iff** $T \in \mathfrak{Out}_3$?

Digraphs generated by finite structures

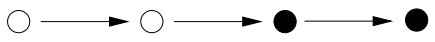
Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

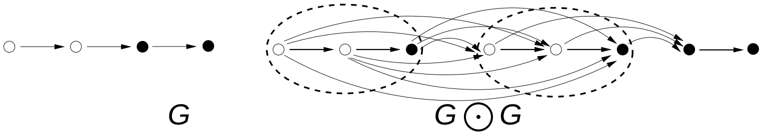
Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are “inherited” from G in the natural way.



G



Digraphs generated by a finite structure



$G[T_G]$ is an **induced subgraph** of $(G \odot G)[T_{G \odot G}]$.
Now we can repeat the procedure above using $G \odot G$ instead of G to get $(G \odot G) \odot (G \odot G)$.
Hence we obtain a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of digraphs with terminal vertices, $G_n = \langle V_n, E_n, T_n \rangle$ s. t. $G_0[T_0] \subset G_1[T_1] \subset G_2[T_2] \subset \dots$
Take

$$G^\infty = \bigcup \{G_n[T_n] : n \in \mathbb{N}\}.$$

Digraphs generated by a finite structure

Theorem

Let $G = (V, E, T)$ be a ***finite tournament with terminal vertices****.*
T. F. A. E:

(i) $G^\infty \in \mathfrak{Out}_3$,

(ii) $\mathrm{In}_1(v) \neq \{v\}$ *for each* $v \in V \setminus T$.

T. F. A. E.:

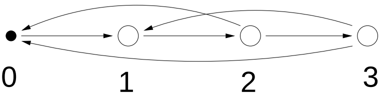
(a) $G^\infty \in \mathfrak{Out}_2$

(b) *there is* $v \in T$ *with* $V = \mathrm{Out}_2^G(v)$.

Theorem

There is an finite tournament G such that $G^\infty \in \mathfrak{Out}_3 \setminus \mathfrak{Out}_2$.

G :



Digraphs generated by a finite structure

Theorem

If $G = \langle V, E, T \rangle$ is a finite digraph with terminal vertices then $G^\infty \in \mathfrak{In}_2\text{-}\mathfrak{Out}_2$.

Euler-circle

Theorem

A connected graph has an Euler circle iff every vertex has even degree. A connected graph has an Euler line iff exactly two vertices have odd degree.

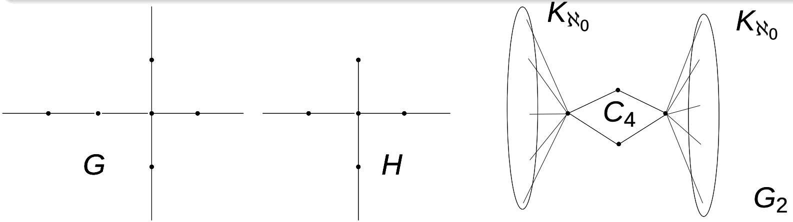
Defi nition

Euler circle in an infinite graph: a chain Z extending infinitely in both directions and containing each edge of G exactly once.
Euler line in an infinite graph: a chain Z extending infinitely in one directions and containing each edge of G exactly once.

Problem (König)

When does a countable infinite graph G contain an Euler circle/line?

Plain generalization fails for infinite graphs:



Euler-circle

Theorem (Erdös, Pál; Grünwald, T.; Vazsonyi, E., 1938)

A countable graph G has en Euler-line iff

- (i) *G is connected,*
- (ii) *G contains a vertex of either infinite or odd order, and at most one vertex of odd order,*
- (iii) *each G-H with H finite has at most one infinite component.*

A countable graph G has en Euler-circle iff

1. *G is connected;*
2. *G contains no vertex of odd order;*
3. *If H is any finite subgraph of G, G-H has at most two infinite components;*
4. *If all vertices of a finite H have in H the same, even, order, then G-H has only one infinite component.*

Euler-circles/lines in directed graphs

Theorem

A connected digraph has an Euler circle iff the in-degree and the out-degree are the same for each vertex.

Theorem (Nash-Williams)

Characterization of infinite digraphs having Euler-circles/lines

Covering and Matching: Basic Definitions

- If G is bipartite with bipartition $V = W \cup^* M$ write $G = (M, W, E)$
- A matching is a set of disjoint edges
- A cover is a set of vertices meeting all edges
- If $G = (V, E)$ is a graph, $A \subset V$ and $F \subset E$ let $F[A] = \{v \in V : \exists a \in A \ \{v, a\} \in F\}$.
- A matching F covers $A \subset V$ iff $A \subset F[V]$.
- $A \subset V$ is matchable into $B \subset V$ iff there is a matching F such that F covers A and $F[A] \subset B$.
- A matching F in a bipartite graph $G = (M, W, E)$ is an espousal iff F covers M.
- $G = (M, W, E)$ is espousable iff it has an espousal.

Menger's Theorem

Menger's Theorem

If $G = (V, E)$ is a finite graph, $A, B \subset V$ then

$$\min\{|X| : X \subset V \text{ separates } A \text{ and } B\} = \max\{|\mathcal{P}| : \mathcal{P} \text{ is a family of disjoint } A\text{-}B\text{-paths}\}.$$

A weak version of Menger's Theorem for infinite graphs

Theorem (Erdős)

If $G = (V, E)$ is a graph, $A, B \subset V$, then
$\min\{|X| : X \subset V$ separates A and $B\} =$
$\max\{|\mathcal{P}| : \mathcal{P}$ is a family of disjoint A-B-paths$\}$.

Proof.

$\kappa = \min$, $\lambda = \max$. $\min \geq \max$. λ is finite: Menger.

λ is infinite

$\mathcal{P}$ = a maximal family of pairwise disjoint A-B-paths. Zorn lemma!

$X = \cup\mathcal{P}$. X separates, so $|X| \geq \kappa$. $\mathcal{P}$ is infinite, so $|X| = |\mathcal{P}| \leq \lambda$. So $\lambda \geq \kappa$.

□

Erdős: too many points in X!

Erdős-Menger Conjecture

Menger's Theorem

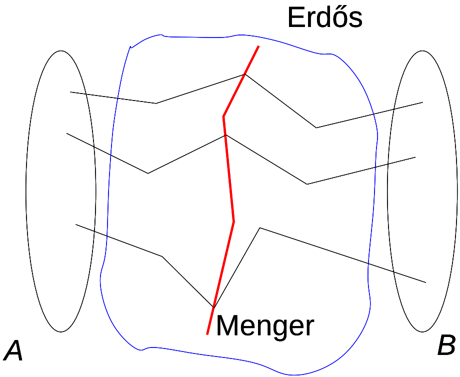
If $G = (V, E)$ is a finite graph, $A, B \subset V$ then

$$\min\{|X| : X \subset V \text{ separates } A \text{ and } B\} = \max\{|\mathcal{P}| : \mathcal{P} \text{ is a family of disjoint } A\text{-}B\text{-paths}\}.$$

Menger's Theorem, reformulated

If $G = (V, E)$ is a finite graph, $A, B \subset V$ then there is an A-B-separating set X and there is a family $\mathcal{P}$ of disjoint A-B-paths such that $|P \cap X| = 1$ for each $P \in \mathcal{P}$.

Erdős-Menger Conjecture



Erdős-Menger Conjecture

If $G = (V, E)$ is a graph, $A, B \subset V$ then there is an A-B-separating set X and there is a family $\mathcal{P}$ of disjoint A-B-paths such that $|P \cap X| = 1$ for each $P \in \mathcal{P}$.

Hall's Theorem

In a finite bipartite graph $G = (M, W, E)$ the set M has a matching iff $|E(A)| \geq |A|$ for each $A \subset M$.

Plain generalization fails for infinite graphs.

Playboy example.

$M = \{m_i : i \geq 0\}$

$W = \{w_i : i \geq 1\}$

$E = \{(m_i, w_i) : i \geq 1\} \cup \{(m_0, w_i) : i \geq 1)\}$

Problem: there is $A \subsetneq M$ such that every matching of A covers W.

1-obstruction

Hall's Theorem

Defi nition

Let $G = (M, W, E)$ be a bipartite graph. G has **property** Q provided

(Q) If G is inespousable then there is $X \subset M$ such that X is unmatchable but $E(X)$ is matchable into X.

Observation

"Every finite bipartite graph has property Q" $\Longrightarrow$ Hall's Theorem

If G is an unmatchable finite bipartite graph then $\exists X \subset M$ s.t such that X is unmatchable but $E(X)$ is matchable into X, then $|E(X)| < |X|$.
Infinite Hall: Every bipartite graph has property Q.

König's Theorem

Definition

Let $G = (M, W, E)$ be a bipartite graph. G has **property P** provided

(P) There is a matching F and a cover C such that $|e \cap C| = 1$ for each $e \in F$.

Observation

"Every finite bipartite graph has property P" $\Longleftrightarrow$ König's Theorem

Infinite König: Every bipartite graph has property P.

Aharoni's results

Aharoni's Theorem

T.F.A.E.

(1) Every bipartite graph has property P

(2) Every bipartite graph has property Q

Theorem (Aharoni, 1984)

If a bipartite graph $G = (M, W, E)$ is inespousable then there is $X \subset M$ such that X is unmatchable but $E(X)$ is matchable into X

Theorem (Aharoni, 1984)

Every bipartite graph $G = (M, W, E)$ has a matching F and a cover C such that $|e \cap C| = 1$ for each $e \in F$.

Bipartite graphs without espousal

Type 1 problem:
Playboy example:
$G = (M, W, E)$, $A \subsetneq M$, every matching of A covers W.
Type 2 problem: Regressive graph $G_{reg} = (M, W, E)$
$M = \{m_\beta : \omega \leq \beta < \omega_1\}$
$W = \{w_\alpha : \alpha < \omega_1\}$
$E = \{(m_\beta, w_\alpha) : \alpha < \beta < \omega_1\}$

Claim

No espousal in G.

Assume on the contrary that
$F = \{(m_\beta, w_{f(\beta)}) : \beta < \omega_1$ is a limit ordinal$\}$ is an espousal.
Then f is regressive function on a closed unbounded set:
Lázár's Theorem: f is constant on an unbounded set.
(Fodor's Lemma, pressing down lemma)

Structure theorem for inespousable graphs

Theorem (Podowski, Steffens)

A countable bipartite graph is espousable iff it is not 1*-obstructed* ***(there is no Type 1 problem)****.*

Theorem (Aharoni, Nash-Williams, Shelah)

A bipartite graph is espousable iff it is not obstructed ***(there is no Type 1 or Type 2 problem)****.*

Erdős-Menger Conjecture

Erdős-Menger Conjecture: many partial results

Theorem (Aharoni, Berger, 2005)

If $G = (V, E)$ is a graph, $A, B \subset V$ then there is an A-B-separating set X and there is a family $\mathcal{P}$ of disjoint A-B-paths such that $|P \cap X| = 1$ for each $P \in \mathcal{P}$.

Summary

Many finite problems have infinite counterparts.

Similar, but not the same.

Deep set-theory is not a must.

Infinite is not less interesting than finite.