

# The joy of elementary submodels

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# Introduction

How to use **elementary submodels** to prove theorems in infinite combinatorics?

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- Basic concepts
- Easy applications
- Simplified proofs
- New results and problems

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- $\mathcal{B} \cup \{A\}$  is a larger  $\Delta$ -system with kernel  $D$  than  $\mathcal{B}$ . **Contradiction.**

# Easy applications: Fodor lemma

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- If  $|G| = \omega_1$ : partition  $G$  into pieces  $\{G_\alpha : \alpha < \omega_1\}$  s.t.  $G_\alpha$  is countable and has no odd cuts.

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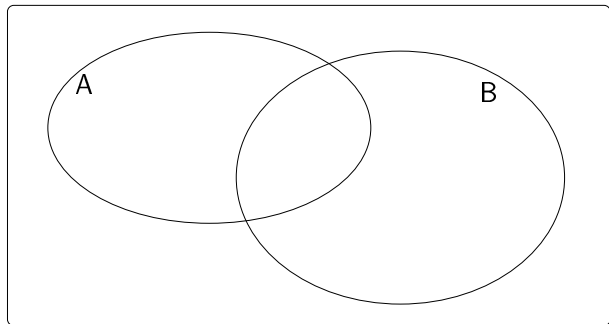
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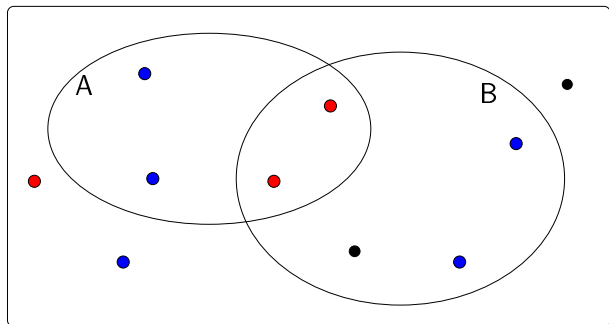
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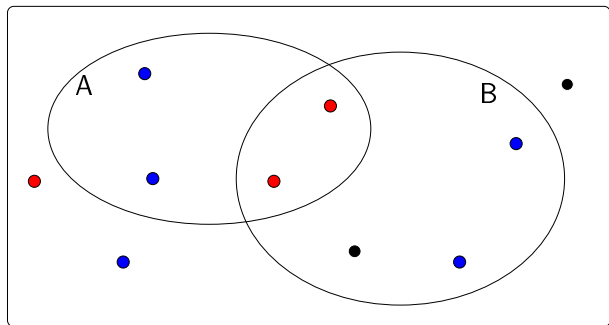
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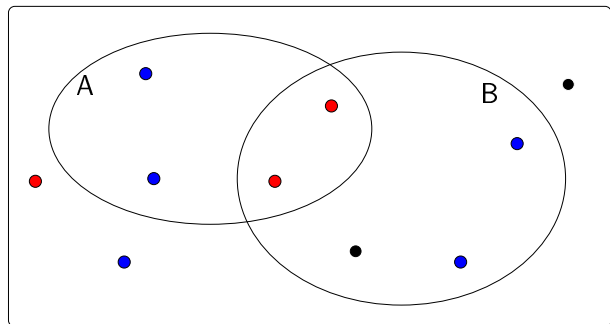
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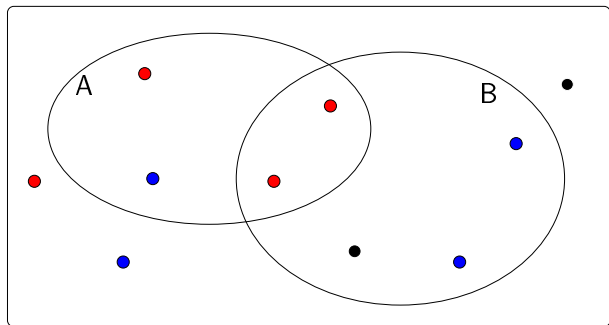
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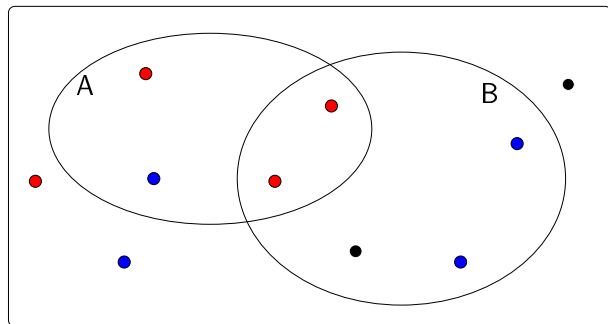
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- $X = \omega_1 \times k$

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## Theorem

If  $\kappa \not\rightarrow [\kappa]_{\kappa}^2$  then  $\chi_{\text{CF}}[\kappa, \ell, k + 1] = \kappa$  for  $k + 1 \leq \ell \leq 2k$ .

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**Question:** Assume that  $\mathcal{A} \subset [\lambda]^\kappa$  is a  $\mu$ -almost disjoint family. Is  $\mathcal{A}$  essentially disjoint?

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Elementary submodels

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Elementary submodels+ Shelah's revised GCH

- $\rho^{[\nu]} = \rho$  iff there is a family  $\mathcal{B} \subset [\rho]^{<\nu}$  of size  $\rho$  such that for all  $u \in [\rho]^\nu$  there is  $\mathcal{P} \in [\mathcal{B}]^{<\nu}$  such that  $u \subset \cup \mathcal{P}$ .

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- **Shelah's Revised GCH theorem:** If  $\rho \geq \beth_\omega$ , then  $\rho^{[\nu]} = \rho$  for each large enough regular  $\nu < \beth_\omega$ .