Indestructible properties

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Combinatorial Set Theory

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2. A blackbox theorem

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3. *S* and *L*-spaces
4. Smooth graphs

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Theorem (Szentmiklossy)

*There is no **first countable** L-space under $MA_{\aleph_1}$.*

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Given a structure X and a property φ we say that **the property φ of X is c.c.c-indestructible**

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Given a structure X and a property φ we say that **the property φ of X is c.c.c-indestructible** if for each c.c.c poset Q we have $1_Q \Vdash_Q$ "X has property φ".

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1. Is Kunen's theorem sharp?
2. (Roitman) Can S and L-groups exit under $MA_{\aleph_1}$?

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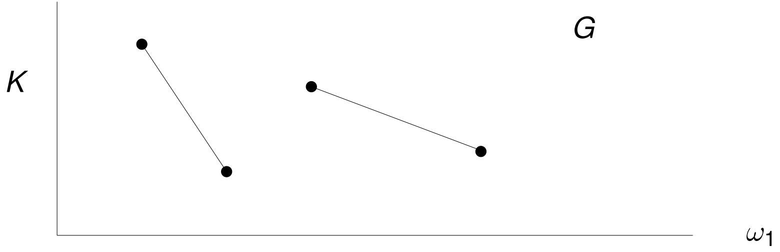

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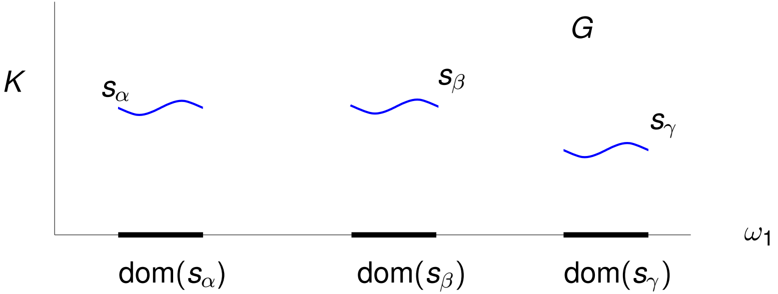

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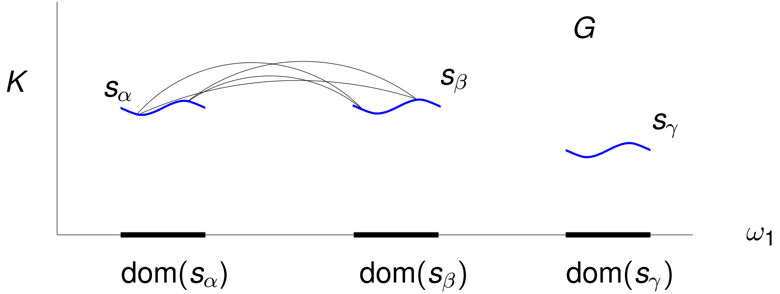

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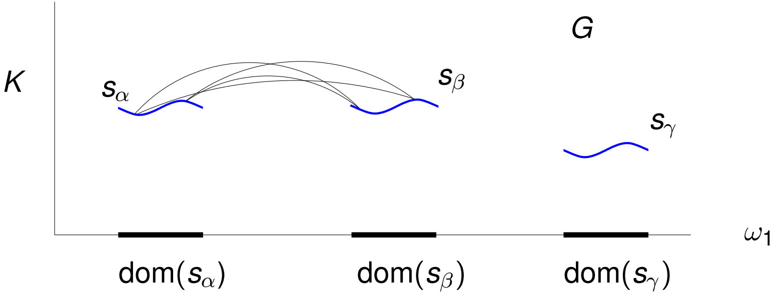

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Theorem

*Assume $2^{\omega_1} = \omega_2$. If G is a **strongly solid** graph on $\omega_1 \times K$, then for each $m \in \omega$ there is a c.c.c poset P of size ω_2 such that*

$$V^P \models \text{“}G \text{ is } \textbf{c.c.c-indestructibly } m\textbf{-solid.}\text{”}$$

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3. $(*)_m$ implies that G is m-solid

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2. $(*)_m$ is a **c.c.c-indestructible** property
3. $(*)_m$ implies that G is m-solid
4. if $V \models$ "G **is strongly solid**" then there is a **c.c.c. poset** P such that $V^P \models G$ **has property** $(*)_m$.

A blackbox theorem: a sample application

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Proposition

C is c.c.c indestructibly non-trivial in some generic extension.

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A graph $G = \langle \omega_1 \times K, E \rangle$ is ***m*-solid** if given any dom-disjoint sequence $\langle s_\alpha : \alpha < \omega_1 \rangle \subset \mathrm{Fn}_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that $[s_\alpha, s_\beta] \subset G$.

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S and L-spaces: Kunen's result is sharp

Theorem (Kunen)

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$V^{C_{\omega_1} * P} \models X^n$ is a c.c.c. indestructible S-space □

Theorem

If GCH holds then there is a c.c.c poset P such that

$$V^P \models \text{“}\mathbf{2}^{\omega_1} \text{ contains an S-group and an L-group, and } MA_{\aleph_1} \text{ holds”.}$$

Definition

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Theorem (–)

A first countable 0-dimensional O-space can exist under Martin's Axiom.

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous**

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I(G): isomorphism classes of induced uncountable subgraphs of G.

1. $|\,\mathrm{I}(G)|$ is small,
2. $G \cong G[A]$ for many $A \subset \omega_1$.

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G is **smooth** iff $|\,\mathrm{I}(G)| = 1$

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Theorem (Shelah, –)

Assume that GCH holds and every Aronszajn-tree is special. Then $|\mathrm{I}(G)| = 2^{\omega_1}$ for each non-trivial graph $G = \langle \omega_1, E \rangle$.

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Problem

Is there a non-trivial, almost smooth graph on ω_1?

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It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

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- C: Cohen graph on ω_1

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- G is strongly solid because C is the Cohen graph

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*It is consistent that there is a non-trivial, ω**-smooth** graph on ω_1.*

Smooth graphs

G is **κ-smooth** $\forall W \in [\omega_1]^{\omega_1}$ $\exists W' \in [W]^{<\kappa}$ s.t. $G \cong G[W \setminus W']$.

Fact

If a graph G is n-smooth for some $n \in \omega$, then G is complete or empty.

Theorem (Hajnal, Nagy, –)

*Under $\diamondsuit^+$ there is a non-trivial, ω_1 **-smooth** graph on ω_1.*

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Problem

*Is there a non-trivial, ω_1 **-smooth** graph on ω_1?*

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It is consistent with $MA_{\aleph_1}$ *that* $\omega_1 \not\to^* [\omega_1]^n_{k-bdd}$ *for each* $n, k \in \omega$, *i.e. there is an* k*-bounded colouring* $c : [\omega_1]^n \to \omega_1$ *such that* $\forall X \in [\omega_1]^{\omega_1}$ $\exists \alpha < \omega_1 \; |c^{-1}\{\alpha\} \cap [X]^n| = k$.

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Bibliography

- U. Abraham, S. Todorčevič, *Martin's Axiom and first countable S- and L-spaces*, HANDBOOK OF SET-THEORETICAL TOPOLOGY, e.d. K. Kunen and J. E. Vaughan, 327–345, North-Holland, New York, 1984.
- A. Hajnal, Zs. Nagy, L. Soukup, *On the number of non-isomorphic subgraphs of certain graphs without large cliques and independent subsets*, "A TRIBUTE TO PAUL ERDŐS ", e.d. A Baker, B. Bollobás, A. Hajnal, Cambridge University Press, 1990, p. 223-248
- S. Shelah, L. Soukup, *On the number of non-isomorphic subgraphs* , ISRAEL J. MATH, 86(1994) no 1-3, 349-371.
- L. SOUKUP, *Smooth Graphs*, Comment. Math. Univ. Carolin. 40 (1999), no. 1, 187–199.
- L. SOUKUP, *Indestructible properties of S- and L-spaces*. Topology and its Application. 112 (2001), no 3, pp. 245-257.