BETWEEN COUNTABLY COMPACT AND ω-BOUNDED

ISTVÁN JUHÁSZ, LAJOS SOUKUP, AND ZOLTÁN SZENTMIKLÓSSY

ABSTRACT.

1. Introduction

Definition 1.1. Given a topological property P we say that a space X is **P**-*bounded* iff each subspace Y with property P has compact closure.

2. Preliminary results

Fact 2.1. *If X is a regular space, and $\chi(p,X) > \omega$ for all $p \in X$, then every subspace Y with $\mathrm{w}(Y) = \omega$ is nowhere dense.*

Definition 2.2. Given a topological space X, for $Y \subset X$ let

$$\mathrm{cl}_X^{\omega D}(Y) = \bigcup \{\overline{A}^X : A \in \bigl[Y\bigr]^\omega \text{ is discrete,}\}$$

and

$$\mathrm{cl}_X^{\omega nwd}(Y) = \bigcup \{\overline{A}^X : A \in \bigl[Y\bigr]^\omega \text{ is nowhere dense in } X\}$$

We say that $Y \subset X$ is *ω-D-closed (ω-nwd-closed)* iff $Y = \mathrm{cl}_X^{\omega D}(Y)$ $\bigl(Y = \mathrm{cl}_X^{\omega nwd}(Y)\bigr)$ respectively.

Lemma 2.3. *If Z is regular space, and $D \subset Z$ dense, then for each countable, discrete $S \subset \mathrm{cl}_Z^{\omega D}(D)$ there is a countable, discrete $A \subset D$ with $S \subset \overline{A}$. In particular, $\mathrm{cl}_Z^{\omega D}(D)$ is ω-D-closed.*

Proof of the lemma. Let $S = \{s_n : n \in w\} \subset \mathrm{cl}_Z^{\omega D}(D)$ be discrete in Z, and for all $s_n \in S$ fix a countable, discrete $A_n \subset D$ with $s_n \in \overline{A_n}$.

For each $n \in \omega$ fix an open set $U_n \ni s_n$ such that $s_i \notin \overline{U_n}$ for $i \ne n$.

Let

$$A = \bigcup_{n \in \omega} ((A_n \cap U_n) \setminus \bigcup_{m \le n} \overline{U}_m).$$

Clearly $A \subset D$ is countable.

If $x \in A_n \cap A$ then $x \in U_n \setminus \bigcup_{m \le n} \overline{U}_m$ and $(U_n \setminus \bigcup_{m \le n} \overline{U}_m) \cap A = (U_n \setminus \bigcup_{m \le n} \overline{U}_m) \cap A_n$, so x has a neighborhood U with $A \cap A = \{x\}$.

Finally $S \subset \overline{A}$, which was to be proved. □

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Lemma 2.4. *If Z is regular c.c.c space, and $D \subset Z$ dense, then for each countable, nowhere dense $S \subset \mathrm{cl}_Z^{\omega nwd}(D)$ there is a countable, nowhere dense $A \subset D$ with $S \subset \overline{A}$. In particular, $\mathrm{cl}_Z^{\omega nwd}(D)$ is ω-nwd-closed.*

Proof of the lemma. Let $S = \{s_n : n \in w\} \subset \mathrm{cl}_Z^{\omega nwd}(D)$ be nowhere dense in Z, and for all $s_n \in S$ fix a countable, nowhere dense $A_n \subset D$ with $s_n \in \overline{A_n}$.

Next fix a maximal family $\{U_m : m \in \omega\}$ of pairwise disjoint open sets in Z such that $\overline{U_m} \cap S = 0$ for all $m \in \omega$.

Let

$$A = \bigcup_{n \in \omega} (A_n \setminus \bigcup_{m \le n} U_m).$$

Clearly $A \subset D$ is countable. Since $A \cap U_m \subset \bigcup_{n<m} A_n$, we have that $A \cap U_m$ is nowhere dense. Since $U = \bigcup_{n\in\omega} U_n$ is dense open in Z, it follows that A is nowhere dense in Z.

Finally $s_n \notin \bigcup_{m\le n} \overline{U_m}$, so $s_n \in \overline{(A_n \setminus \bigcup_{m\le n} U_m)}$. Thus $S \subset \overline{A}$, which was to be proved. □

Theorem 2.5. *Every completely regular space X has an embeddings into a space F_X such that*
(1) $|F_X| = |X| + 2^\omega$.
(2) X is a closed and nowhere in F_X
(3) $F_X \setminus X$ is ω-bounded.
(4) if X is locally comppact, then so is F_X.

Moreover,

(i) If X was not ω-bounded, then F_X is not ω-nwd-bounded.
(ii) If X was not M_2-bounded, then F_X is not M_2-nwd-bounded.
(iii) If X was D-bounded, then F_X is also D-bounded.
(iv) If X was M_2-bounded, then F_X is also M_2-bounded.

Proof. Let $\mathrm{c}X$ be a compactification of X. If X is locally compact, jet $\mathrm{c}X$ be the one-point compactification αX of X Let Y be a compact space which contains a weak P-point y with $|Y| = 2^\omega$. Then the space

$$F_X = \big(\mathrm{c}X \times (Y \setminus \{y\})\big) \cup X \times \{y\}. \tag{2.1}$$

works.

If X is locally compact, then $F_x = (\alpha X \times Y) \setminus \{(\alpha, y)\}$. □

3. Positive theorems

Lemma 3.1. *Let X be a T_1 space, $\mathcal{U}$ be a family of pairwise disjoint open sets, and S be a dense subset of the open set $G = \bigcup \mathcal{U}$. We say that $D \subset S$ is* diagonal *iff $D \cap U$ is finite for all $U \in \mathcal{U}$. Let*

$$I(\mathcal{U}, S) = \{D \in [S]^\omega : D \textit{ is diagonal}\} \tag{3.1}$$

and

$$\mathcal{H}(\mathcal{U}, S) = \{D' : D \in I(U, S)]\} \textit{ and } H(\mathcal{U}, S) = \bigcup \mathcal{H}(\mathcal{U}, S) \tag{3.2}$$

(1) The family $I(\mathcal{U}, S)$ is a P-ideal. $\mathcal{H}(\mathcal{U}, S)$ is σ-directed. So for each $P \in \left[H(\mathcal{U}, S)\right]^\omega$ there is a diagonal set D with $P \subset D'$.
(2) If X is locally compact and regular, then

$$\overline{G} \setminus \bigcup \{\overline{U} : U \in \mathcal{U}\} \subset \overline{H(\mathcal{U}, S)}.$$

Moreover

$$\left\{p \in \overline{G} \setminus \bigcup \{\overline{U} : U \in \mathcal{U}\} : t(p, X) = \omega\right\} \subset H(\mathcal{U}, S).$$

Proof. (1) Assume that $\{D_n : n \in \omega\} \subset I(\mathcal{U}, S)$. Let

$$\mathcal{U}' = \{U \in \mathcal{U} : U \cap D_n \neq \emptyset \text{ some } n \in \omega \ \}.$$

Enumerate $\mathcal{U}'$ as $\{U_n : n < \omega\}$, and put

$$D = \bigcup_{n \in \omega} \Big(D_n \setminus \bigcup_{m \le n} U_m\Big).$$

Then $U_m \cap D \subset \bigcup_{k<m} D_k$, so $D \in \mathcal{D}$.

Moreover $D_n \setminus D \subset \bigcup_{k<n} U_k$, so $D_n \setminus D$ is finite.

(2) Write $H = \overline{G} \setminus \bigcup \{\overline{U} : U \in \mathcal{U}\}$. Let $p \in H$. Consider an arbitrary open set $V \ni p$. Since X is regular , there is an open set W such that $p \in W \subset \overline{W} \subset V$. Since $p \notin \overline{U}$ for $U \in \mathcal{U}$, there are infinitely many $U \in \mathcal{U}$ with $W \cap U \neq \emptyset$. Thus there is $D \in \mathcal{D}$ with $D \subset W$. Since X is countably compact, we have $\emptyset \neq D' \subset H \cap \overline{W} \subset V$, which proves $H \subset \overline{H(\mathcal{U}, S)}$.

Assume that $p \in H$ with $t(p, X) = \omega$. Then $p \in \overline{H(\mathcal{U}, S)}$ implies that there is $P \in \left[H(\mathcal{U}, S)\right]^\omega$ with $p \in \overline{P}$. Then $P \subset D'$ for some diagonal set D by (1). Thus $p \in D'$, and so $p \in H(\mathcal{U}, S)$. □

Lemma 3.2. *If X is a countably compact regular space, $S \subset X$ is dense, $P \in \left[S'\right]^\omega$ is nowhere dense, and $t(p, X) = \omega$ for all $p \in P$, then there is a countable discrete set $D \subset S$ with $P \subset D'$.*

Proof. Choose a maximal family $\mathcal{U}$ pairwise disjoint open subsets of X such that $p \notin \overline{U}$ for all $U \in \mathcal{U}$. Then $P \subset \overline{G} \setminus \bigcup \{\overline{U} : U \in \mathcal{U}\}$, so we can apply the previous lemma .(2) □

Corollary 3.3. *An ω-D-bounded regular, countable tight space is ω-nwd-bounded.*

Corollary 3.4. *An countably compact regular, countable tight space is D-generated.*

T2. Legyen X megsz. kpt T_3, U és S mint fent és U erősen diszj. is. Ekkor H(U,S) sűrű X G-ben.

Köv. Ha még t(X) = w is, akkor minden S sűrűhöz és A megsz. sss-höz van $D \in [S]^w$ diszkrét, melyre $A \subset cl(D)$.

Ebből trivi, h w-D-bdd = w-nwd-bdd megsz. szűk T_3 X-re, de az is hogy megsz. kpt. és megsz. szűk T_3 X D-generált.

Problem 1. *(a) Is there a Frechet (or first countable or sequential or ω-D-sequential) ω-D-bounded, but not M_2-bounded (not ω-bounded) space?*
(b) Is there a sequential ω-D-bounded, but not ω-nwd-bounded space?

Theorem 3.5. *(1) A countably compact, separable, regular space X with $\mathrm{w}(X) < \mathfrak{p}$ is compact.*
(2) An ω-D-bounded, separable, regular space X with $\mathrm{w}(X) < \mathrm{cov}(\mathcal{M})$ is compact.

Problem 2. *Is there a non-compact (not ω-nwd-compact) ω-D-bounded, separable, regular space X with $\mathrm{w}(X) = \mathrm{cov}(\mathcal{M})$?*

(1) nyilvan ismert reg. De ki csinalta?

Proof. Assume that X is a non-compact, separable, regular space. Then there is an open cover $\mathcal{U}$ of X such that $|\mathcal{U}| \le \mathrm{w}(X)$ and even the cover $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$ does not have a finite subcover.

Let D be a countable dense subset of X.

(1) Assume on the contrary that X is countably compact with $\mathrm{w}(X) < \mathfrak{p}$, and we derive a contradiction. Let

$$\mathcal{F} = \{D \cap (X \setminus \bigcup \mathcal{U}') : \mathcal{U}' \in \big[\overline{\mathcal{U}}\big]^{<\omega}\} \tag{3.3}$$

Since $X \setminus \bigcup \mathcal{U}' \ne \emptyset$ is open and D is dense, the family $\mathcal{F} \subset \big[D\big]^{\omega}$ is a filter, so $|\mathcal{F}| < \mathfrak{p}$ implies that $\mathcal{F}$ has a pseudointersection $F \in \big[D\big]^{\omega}$. Since X is countably compact, F has an accumulation point x. If $U \in \mathcal{U}$, then $F \subset^* X \setminus U$, so $x \in \overline{X \setminus U} = X \setminus U$. So $x \notin \bigcup U$. Contradiction.

(2) Assume on the contrary that X is ω-D-bounded with $\mathrm{w}(X) < \mathrm{cov}(\mathcal{M})$, and we derive a contradiction.

For each $d \in D$ pick $U_d \in \mathcal{U}$ with $d \in U_d$.

Let

$$P = \{\langle d_0, \dots, d_n\rangle \in \big[D\big]^{<\omega} \ : \ \forall i < n \ d_{i+1} \notin U_{d_i}\} \tag{3.4}$$

Let $\mathcal{P} = \langle P, \supseteq\rangle$. For $D' \subset D$ let

$$E_{D'} = \{\langle d_0, \dots, d_n\rangle \in P : d_i \in D' \text{ for some } i \le n \ \}. \tag{3.5}$$

Let

$$\mathcal{E} = \{E_{D \cap (X \setminus \bigcup \mathcal{U}')} : \mathcal{U}' \in \big[\overline{\mathcal{U}}\big]^{<\omega}\}. \tag{3.6}$$

Claim 3.5.1. *Every $E \in \mathcal{E}$ is dense in $\mathcal{P}$.*

Proof of the claim. Assume $E = E_{D \cap (X \setminus \bigcup \mathcal{U}')}$.

Let $\langle s_0, \dots, s_k\rangle \in P$. Since

$$\bigcup_{0 \le i < k} \overline{U}_{s_i} \cup \bigcup \mathcal{U}' \ne X, \tag{3.7}$$

so we can pick

$$d' \in D \cap (X \setminus \bigcup_{0 \le i < k} \overline{U}_{s_i} \cup \bigcup \mathcal{U}') \tag{3.8}$$

Then $\langle s_0, \dots, s_k, d\rangle \in E_{D\cap(X\setminus\bigcup\mathcal{U}')}$. □

Since $|\mathcal{E}| \le |\mathcal{U}| + \omega < \operatorname{cov}(\mathcal{M})$, there is an $\mathcal{E}$-generic filter $\mathcal{G}$. Let

$$\langle d_i : i < \omega\rangle = \bigcup D. \tag{3.9}$$

Then $U_{d_i} \cap D \subset \{d_0, \dots, d_i\}$, so $D' = \{d_i : i \in \omega\}$ is right separated, so it is discrete.

On the other hand, $\overline{D'}$ is not compact. Indeed, assume on the contrary that $\overline{D'}$ is compact. Then $\overline{D'} \subset \bigcup \mathcal{U}'$ for some $\mathcal{U}' \in \left[\mathcal{U}\right]^{<\omega}$. Then $E_{D\setminus\bigcup\{\overline{U}:U\in\mathcal{U}'\}}$ and $\mathcal{G}$ are disjoint, so $\mathcal{G}$ was not $\mathcal{E}$-generic. Contradiction. □

4. $\mathcal{D}$-forced like construction

Theorem 4.1. *There is a space X such that X is ω-nwd-bounded, M_2-bounded, but not ω-bounded.*

Proof. By [10], the space $\mathbf{2}^{2^\omega}$ has a a countable, dense, submaximal subset S. Pick $a \in S$, let $Y = S \setminus \{a\}$, and put $X = \operatorname{cl}^{\omega nwd}_{\mathbf{2}^{2^\omega}}(Y)$. Since S is nodec, $a \notin X$, so X is not ω-bounded.

By Lemma 2.4, X is ω-nwd-bounded.

By Fact 2.1, every subset $A \in \left[\mathbf{2}^{2^\omega}\right]^\omega$ with countable weight is nowhere dense. So X is M_2-bounded, as well. □

Theorem 4.2. *There is a space Y such that Y is M_2-bounded, but not ω-nwd-bounded.*

Proof. Let X be the a not ω-bounded, but ω-nwd-bounded and M_2-bounded space from theorem 4.1. Apply Theorem 2.5 to get the space F_X. By 2.5(i), F_X is not ω-nwd-bounded. By 2.5(i), F_X is M_2-nwd-bounded. □

Theorem 4.3. *Assume that κ is an infinite cardinal and $X = \{x_\alpha : \alpha < 2^\kappa\} \subset \mathbf{2}^{2^\kappa}$ is dense. Then there is a space $Y = \{y_\alpha : \alpha < 2^\kappa\} \subset \mathbf{2}^{2^\kappa}$ such that for each $I \in \left[2^\kappa\right]^\kappa$*

(1) if $\{x_i : i \in I\}$ is nowhere dense , then $\{y_i : i \in I\}$ is nowhere dense.
(2) if $\{x_i : i \in I\}$ is crowded, then

$$\overline{\{x_i : i \in I\}}^{\mathbf{2}^{2^\kappa}} \subseteq \overline{\{y_i : i \in I\}}^{\mathbf{2}^{2^\kappa}}. \tag{4.1}$$

(3) Disjoint discrete subsets of Y with cardinalities $\le \kappa$ has disjoint closure in $\mathbf{2}^{2^\kappa}$. In particular, every discrete subspace of Y with cardinality $\le \kappa$ is closed in Y.

Proof. Let $\{\langle I_\zeta, J_\zeta\rangle : \zeta < 2^\nu\}$ be a 2^κ-abundant enumeration of the set $\left[2^\kappa\right]^{\le\kappa} \times \left[2^\kappa\right]^{\le\kappa}$.

By transfinite induction on $\xi \le 2^\kappa$ we define sets

$$X^\xi = \{x^\xi_\alpha : \alpha < 2^\kappa\} \subset \mathbf{2}^{2^\kappa} \tag{4.2}$$

and an increasing sequence $\{\delta_\xi : \xi \le 2^\kappa\}$ ordinals as follows: Let $x^0_\alpha = x_\alpha$ for $\alpha < 2^\kappa$, and let $\delta_0 = 0$.

If ξ is limit, let $\delta_\xi = \sup_{\zeta<\xi} \delta_\zeta$ and let

$$x^\xi_\alpha(\nu) = \lim_{\zeta\to\xi} x^\zeta_\alpha(\nu). \tag{4.3}$$

Assume that $\xi = \zeta + 1$. If $I_\zeta \cap J_\zeta \ne \emptyset$, or $\{x^\zeta_i : i \in I_\zeta\}$ or $\{x^\zeta_i : i \in J_\zeta\}$ are not discrete, then let $\delta_\xi = \delta_\zeta$ and $X^\xi_\alpha = x^\zeta_\alpha$ for all $\alpha < 2^\kappa$.

Assume that $I_\zeta \cap J_\zeta = \emptyset$, $\{x^\zeta_i : i \in I_\zeta\}$ and $\{x^\zeta_i : i \in J_\zeta\}$ are discrete. Let $\delta_\zeta < \delta'_\xi < 2^\kappa$ such that

$$\{x^\zeta_i \restriction \delta'_\xi : i \in I_\zeta\} \text{ and } \{x^\zeta_i \restriction \delta'_\xi : i \in J_\zeta\} \text{ are discrete in } \mathbf{2}^{\delta'_\xi}. \tag{4.4}$$

If

$$\overline{\{x^\zeta_i : i \in I_\zeta\}} \cap \overline{\{x^\zeta_i : i \in J_\zeta\}} = \emptyset,$$

then pick a finite family $S_\zeta \in \left[Fn(2^\kappa,2)\right]^{<\omega}$ of finite function $s_\zeta \in Fn(2^\kappa,2)$ such that the basic open sets $\{[s] : s \in S_\zeta\}$ separate $\overline{\{x^\zeta_i : i \in I_\zeta\}}$ and $\overline{\{x^\zeta_i : i \in J_\zeta\}}$.

Let $\delta'_\xi \le \delta_\xi < 2^\kappa$ such that $\operatorname{dom}(s) \subset \delta_\xi$ for $s \in S_\zeta$.

Assume that

$$\overline{\{x^\zeta_i : i \in I_\zeta\}} \cap \overline{\{x^\zeta_i : i \in J_\zeta\}} \ne \emptyset.$$

Let $\delta_\xi = \delta'_\xi + 1$ and for $\alpha < 2^\kappa$ let

$$x^\xi_\alpha(\nu) = \begin{cases} x^\zeta_\alpha(\nu) & \text{if } \nu \ne \delta'_\xi \text{ or } \alpha \notin I_\zeta, \\ 0 & \text{if } \nu = \delta'_\xi \text{ and } \alpha \in I_\zeta. \\ 1 & \text{if } \nu = \delta'_\xi \text{ and } \alpha \in J_\zeta. \end{cases} \tag{4.5}$$

Let $y_\alpha = x^{2^\kappa}_\alpha$ for $\alpha < 2^\kappa$. We show that $Y = \{y_\alpha : \alpha < 2^\kappa\}$ satisfies the requirements.

Claim 4.3.1. *If $I \in \left[2^\kappa\right]^{\le\kappa}$, $s \in Fin(2^\kappa,2)$, $\nu \in \operatorname{dom}(s)$, $[s] \cap \{x^\nu_i : i \in I\} = \emptyset$, then there is $t \in Fin(2^\kappa,2)$ such that*

(a) $t \supset s$ and $\operatorname{dom}(t) \setminus \operatorname{dom}(s) \subset \nu$,
(b) $[t] \cap \{x^{\nu+1}_i : i \in I\} = \emptyset$.

Proof of the Claim. We can assume that $X^\nu \ne X^{\nu+1}$.

So there is $\zeta < 2^\kappa$ such that

(i) $\nu = \delta'_{\zeta+1}$, and so $\nu + 1 = \delta_{\zeta+1}$.
(ii) $\{x^\zeta_i \restriction \nu : i \in I_\zeta\}$ $\{x^\zeta_i \restriction \nu : i \in J_\zeta\}$ are discrete in $\mathbf{2}^\nu$.

By (ii), there is $r \in Fin(\nu,2)$ such that $r \supset s \restriction \nu$ and $[r] \cap \{x^\zeta_i \restriction \nu : i \in I_\zeta \cup J_\zeta\} = \emptyset$. Then $[r] \cap \{x^{\zeta+1}_i \restriction \nu : i \in I_\zeta \cup J_\zeta\} = \emptyset$ because $x^{\zeta+1}_i \restriction \nu = x^\zeta_i \restriction \nu$.

Thus $t = r \cup s$ satisfies the requirements. □

Claim 4.3.2. *If $I \in \left[2^\kappa\right]^{\le\kappa}$, $s \in Fin(2^\kappa,2)$, $[s] \cap \{x_i : i \in I\} = \emptyset$, then there is $t \in Fin(2^\kappa,2)$ such that $t \supset s$ and $[t] \cap \{x^{2^\kappa}_i : i \in I\} = \emptyset$.*

Proof of the Claim. Write $\mathrm{dom}(s) = \{\nu_0 < \cdots < \nu_n\}$. Let

$$J = \{\zeta < 2^\kappa : \delta_{\zeta+1} = \nu + 1 \text{ for some } \nu \in \mathrm{dom}(s)\}.$$

Write $J = \{\zeta_1 < \cdots < \zeta_m\}$. Using Claim 4.3.1 we can define a finite sequence $t_0, t_1, \ldots, t_m \in Fin(2^\kappa, 2)$ such that

(a) $s = t_0 \subset \ldots \subset t_m$,
(b) $\mathrm{dom}(t_{k+1}) \setminus \mathrm{dom}(t_k) \subset \delta'_{\zeta_{k+1}}$.
(c) $[t_{k+1}] \cap \{x_i^{\delta_{\zeta_{k+1}}} : i \in I\} = \emptyset$.

Then $t = t_m$ satisfies the requirements. □

By Claim 4.3.2 we have

Claim 4.3.3. *If $\{x_i : i \in I\}$ is nowhere dense for some $I \in \left[2^\kappa\right]^{\leq\kappa}$, then $\{y_i : i \in I\}$ is nowhere dense.*

So we verified (1).

Claim 4.3.4. *If $I \in \left[2^\kappa\right]^{\leq\kappa}$, $s \in Fin(2^\kappa, 2)$ and $\nu \in \mathrm{dom}(s)$ such that $\{x_i^\nu : i \in I\}$ is crowded, then there is $t \in Fin(2^\kappa, 2)$ such that*

(a) $t \supset s$ and $\mathrm{dom}(t) \setminus \mathrm{dom}(s) \subset \nu$,
(b) $[t] \cap \{x_i^{\nu+1} : i \in I\} \neq \emptyset$ is crowded.

Proof of the Claim. We can assume that $X^\nu \neq X^{\nu+1}$.

So there is $\zeta < 2^\kappa$ such that

(i) $\nu = \delta'_{\zeta+1}$, and so $\nu + 1 = \delta_{\zeta+1}$.
(ii) $\{x_i^\zeta \restriction \nu : i \in I_\zeta\}$ and $\{x_i^\zeta \restriction \nu : i \in J_\zeta\}$ are discrete in $\mathbf{2}^\nu$.

Since $\{x_i^\nu : i \in I\}$ is crowded, by (ii) there is $r \in Fin(\nu, 2)$ such that

(a) $r \supset s \restriction \nu$
(b) $[r] \cap \{x_i^\zeta \restriction \nu : i \in I_\zeta \cup J_\zeta\} = \emptyset$.
(c) $[r] \cap \{x_i^\nu : i \in I\}$ is crowded.

Then $[r] \cap \{x_i^{\zeta+1} \restriction \nu : i \in I_\zeta \cup J_\zeta\} = \emptyset$ because $x_i^{\zeta+1} \restriction \nu = x_i^\zeta \restriction \nu$.

Thus $t = r \cup s$ satisfies the requirements. □

Claim 4.3.5. *If $I \in \left[2^\kappa\right]^{\leq\kappa}$, $s \in Fin(2^\kappa, 2)$ such that $\{x_i : i \in I\} \subset [s]$ is crowded, then $\{y_i : i \in I\} \cap [s] \neq \emptyset$.*

Proof of the Claim. Write $\mathrm{dom}(s) = \{\nu_0 < \cdots < \nu_n\}$. Let

$$J = \{\zeta < 2^\kappa : \delta_{\zeta+1} = \nu + 1 \text{ for some } \nu \in \mathrm{dom}(s)\}.$$

Write $J = \{\zeta_1 < \cdots < \zeta_m\}$. Using Claim 4.3.4 we can define a finite sequence $t_0, t_1, \ldots, t_m \in Fin(2^\kappa, 2)$ such that

(a) $s = t_0 \subset \ldots \subset t_m$,
(b) $\mathrm{dom}(t_{k+1}) \setminus \mathrm{dom}(t_k) \subset \delta'_{\zeta_{k+1}}$.
(c) $[t_{k+1}] \cap \{x_i^{\delta_{\zeta_{k+1}}} : i \in I\}$ is non-empty crowded.

Then $t = t_m$ satisfies the requirements. □

By Claim 4.3.5 we have

Claim 4.3.6. *If $\{x_i : i \in I\}$ is crowded, then* $\overline{\{x_i : i \in I\}}^{\mathbf{2}^{2^\kappa}} \subset \overline{\{y_i : i \in I\}}^{\mathbf{2}^{2^\kappa}}$.

So we verified (2).

If $I, J \in \left[2^\kappa\right]^{\le\kappa}$, $I \cap J = \emptyset$, $\{y_i : i \in I\}$ and $\{y_i : i \in J\}$ are discrete, then there is $\nu < 2^\kappa$ such that $\{y_i \restriction \nu : i \in I\}$ and $\{y_i \restriction \nu : i \in J\}$ are discrete in $\mathbf{2}^\nu$. So, by the construction, there is a $\nu < 2^\kappa$ such that $y_i(\nu) = 0$ for all $i \in I$, and $y_i(\nu) = 1$ for all $i \in J$. Thus $\overline{\{y_i : i \in I\}} \cap \overline{\{y_i : i \in J\}} = \emptyset$. So we have (3).

So we proved Theorem 4.3. □

Theorem 4.4. *There is a dense subspace Y of $\mathbf{2}^{2^\kappa}$ with size 2^κ such that*
(1) Every discrete subset of Z of size $\le \kappa$ is closed in Z.
(2) for each $f \in \mathbf{2}^{2^\kappa}$ there is a nowhere dense $X_f \in \left[Z\right]^\kappa$ such that $f \in D'_f$.

Proof. The Cantor cube $\mathbf{2}^{2^\kappa}$ contains a dense subspace T of size κ.

Put $X = \mathbf{2}^\kappa \times T \subset \mathbf{2}^\kappa \times \mathbf{2}^{2^\kappa} \approx \mathbf{2}^{2^\kappa}$.

Write $X = \{x_\alpha : \alpha < 2^\kappa\}$

If $f \in \mathbf{2}^{2^\kappa}$, then f is an accumulation point of the nowhere dense crowded set $D_f = \{f \restriction \kappa\} \times T$. Write $D_f = \{x_\alpha : \alpha \in I_f\}$

Apply Theorem 4.3 for X to obtain a space $Y \subset \mathbf{2}^{2^\kappa}$.

Then $\{y_\alpha : \alpha \in I_f\}$ is nowhere dense and f is an accumulation point of $\{y_\alpha : \alpha \in I_f\}$.

So Y satisfies the requirements. □

Theorem 4.5. *There is a dense, separable, non-compact topological space $X \subset 2^{2^\omega}$ such that*

$$X = \mathrm{cl}^{\omega D}_{\mathbf{2}^{2^\omega}}(X) = \bigcup\left\{\overline{E}^{\mathbf{2}^{2^\omega}} : E \in \left[X\right]^\omega \text{ is discrete}\right\}, \tag{4.6}$$

but

$$\mathbf{2}^{2^\omega} = \mathrm{cl}^{\omega nwd}_{\mathbf{2}^{2^\omega}}(X) = \bigcup\left\{\overline{F}^{\mathbf{2}^{2^\omega}} : F \in \left[X\right]^\omega \text{ is nowhere dense}\right\}. \tag{4.7}$$

So X is ω-D-bounded, but not ω-nwd-bounded. Moreover there is no convergent sequence in X, so X is M_2-bounded.

Proof. Using Theorem 4.4 fix a dense subspace Y of $\mathbf{2}^{2^\omega}$ with size 2^ω such that
(1) Every closed subset of Z of size $\le \omega$ is closed discrete in Y.
(2) for each $f \in \mathbf{2}^{2^\omega}$ there is a nowhere dense $X_f \in \left[Y\right]^\omega$ such that $f \in D'_f$.

Pick $y \in Y$ and let $Z = Y \setminus \{z\}$

Let

$$X = \bigcup\{\overline{D} : D \subset Z \text{ is discrete}\}. \tag{4.8}$$

Then $y \notin X$, but y is an accumulation point of a countable nowhere dense subset of $Y \subset X$, so X is not compact.

The following claim is straightforward:

Claim 4.5.1. *If $E \subset X$ is a countable discrete set, then there is a countable discrete $D \subset Y \cup Y'$ with $E \subset \overline{D}$.*

Thus if $E \subset X$ is a countable discrete set, then E is contained in some compact set $\overline{D}^{\mathbf{2}(2)^{2^\omega}} \subset X$, so $\overline{E}$ is compact. Thus X is ω-D-bounded. □

The cardinal invariant $\bar{\mathfrak{o}}$ was introduced by Leathrum [6] as follows: $\bar{\mathfrak{o}}$ is the minimal size of a maximal almost disjoint family of antichains in the Cantor tree.

Theorem 4.6. *If $\bar{\mathfrak{o}} = \mathfrak{s} = \mathrm{cof}(\mathcal{M}) = \omega_1$, then there is a countable dense set D in 2^{ω_1} such that D is nodec and there is no convergent sequence in $\mathrm{cl}^{\omega nwd}_{2^{\omega_1}}(D)$.*

So, if $d \in D$, then $X = \mathrm{cl}^{\omega D}_{2^{\omega_1}}(D \setminus \{d\})$ is not ω-bounded, but ω-nwd-bounded and M_2-bounded.

Proof. $2^\omega = 2^{\omega_1}$ is the ground model.

Lemma 4.7. *If $\bar{\mathfrak{o}} = \mathfrak{s} = \omega_1$, then there is a sequence $\langle(\mathcal{U}_\alpha, \mathcal{V}_\alpha) : \alpha < \omega_1\rangle$ such that $\mathcal{U}_\alpha \cap \mathcal{V}_\alpha = \emptyset$ and $\mathcal{U}_\alpha \cup \mathcal{V}_\alpha \subset 2^{<\omega}$ is a family of pairwise disjoint basic open subset of 2^ω such that if $\mathcal{W} \subset 2^{<\omega}$ is an arbitrary infinite family of pairwise disjoint basic open sets then there is $\alpha < \omega_1$ such that $|\mathcal{W} \cap \mathcal{U}_\alpha| = |\mathcal{W} \cap \mathcal{V}_\alpha| = \omega$.*

Proof. $\bar{\mathfrak{o}} = \omega_1$ implies that there is a maximal almost disojoint family $\langle(\mathcal{T}_\alpha : \alpha < \omega_1\rangle$ of antichains in $2^{<\omega}$. For each antichain $\mathcal{T}_\alpha$ let $S_{\alpha,i} : i < \omega_1$ be a splitting family in $\mathcal{T}_\alpha$. Let $\langle(\mathcal{U}_\alpha, \mathcal{V}_\alpha) : \alpha < \omega_1\rangle$ be an enumeration of $(S_{\alpha,i}, T_\alpha \setminus S_{\alpha,i})$.

For $\mathcal{W}$ there is α such that $\mathcal{W} \cap T_\alpha$ is infinite. Now some $S_{\alpha,i}$ splits $\mathcal{W} \cap T_\alpha$. □

Induction in ω_1 steps:

$D_0 \subset 2^{\omega_1}$ is countable dense.

In general: modify S up to $\omega(1+\alpha)$.

$D_\alpha \subset 2^{\omega(1+\alpha)}$ is countable dense

We will guarantee: Enumerate a codinal subset of the nowhere dense subsets of D_α in ω_1 type: $\{E^\alpha_i : i < \omega_1\}$.

We also have $\left\langle(\mathcal{U}^\alpha_j, \mathcal{V}^\alpha_j) : j < \omega_1\right\rangle$

We guarantee in some step later:

(1) E^α_i is closed discrete in D_β
(2) there is a coordinate ζ such that $\mathcal{U}^\alpha_j \cap E^\alpha_i(\zeta) \equiv 0$ and $\mathcal{V}^\alpha_j \cap E^\alpha_i(\zeta) \equiv 1$.

Claim 4.7.1. *$D = D_{\omega_1}$ is nodec.*

Indeed, every nowhere denseappear in some intermediate steps, so it will be closed disrete later.

Claim 4.7.2. *There is no convergent sequence in $\mathrm{cl}^{\omega nwd}_{2^{\omega_1}}(D)$.*

Proof. Assume that $\{x_n\} \to x$.

Let $x_n \in \overline{S_n}$, S_n is nowhere dense, so S_n is discrete. By thinning out we can assume that there are $x_n \in W_n \in 2^{<\omega}$ basic open. We can assume $S_n \subset W_n$.

Then $S = \bigcup S_n$.

Then $S \subset E^\alpha_i(\zeta)$.

There are j both U^α_j and V^α_j contains infinitely many W_n.

Then is some step ζ we have $\mathcal{U}^\alpha_j \cap E^\alpha_i(\zeta) \equiv 0$ and $\mathcal{V}^\alpha_j \cap E^\alpha_i(\zeta) \equiv 1$. So for infinitely many n we have $S_n(\zeta) \equiv 0$, so $s_n(\zeta) = 0$.

Similarly for infinitely many n we have $S_n(\zeta) \equiv 1$, so $s_n(\zeta) = 1$. So the sequence $\{s_n\}$ can not convege.

□

□

Theorem 4.8. *It is consistent that $2^\omega = 2^{\omega_1}$ is large and there is M_2-bounded, but not ω-nwd-bounded space*

Proof. Let X be the ω-bounded, but ω-nwd-bounded and M_2-bounded space of size 2^ω from theorem 4.6 Apply Theorem 2.5 to get the space F_X. By 2.5(i), F_X is not ω-nwd-bounded. By 2.5(iv), F_X is M_2-nwd-bounded. □

5. Examples from special points in Cech-Stone compactifications

A point is ω-far if it is not in the closure of a countable discrete subset of X.

A point is remote if it is not in the closure of any nwd subset of X.

A point $p \in \beta X \setminus X$ is a remote point of X if p is not the limit of any nowhere dense subset of X. Remote points were introduced by Fine and Gillman [7].

Theorem 5.1 (Dow,[3])**.** *Every nonpseudocompact ccc space of π-weight ω_1 has a remote point.*

Theorem 5.2. *Let X be a completely regular space.*

(1) The space $\mathrm{cl}^{\omega D}_{\beta X} X$ is ω-D-bounded.

(2) If X is ccc then the space $\mathrm{cl}^{\omega nwd}_{\beta X} X$ is ω-nwd-bounded.

(3) If $\chi(x, \beta X) > \omega$ for all $x \in \beta X$, then $\mathrm{cl}^{\omega nwd}_{\beta X} X$ is M_2-bounded.

(4) If X is separable and X has an ω-far point, then $\mathrm{cl}^{\omega D}_{\beta X} X$ is not ω-bounded.

(5) If X is separable and X has a remote point, then $\mathrm{cl}^{\omega nwd}_{\beta X} X$ is not ω-bounded.

(6) If X is M_2 and X has a remote point, then $\mathrm{cl}^{\omega nwd}_{\beta X} X$ is not ω-M_2-bounded.

(7) If X is not pseurocompact and $\pi w(X) \le \omega_1$, then X has a remote point.

Theorem 5.3. *There is a* *locally compact NEM* *space X such that X is ω-nwd-bounded, M_2-bounded, but not ω-bounded.*

Proof. Let $Y = \omega \times 2^{\omega_1}$

By Theorem 5.1, there is a remote point x in βY\Y.

[8, Corollary 1.5.] : there is a remote point x in βY\Y

Let $X = \beta Y \setminus \{x\}$.

Let $D \subset Y$ be a countable dense set. By the lemma, X is nwd-bounded, and not ω-bounded, because $x \notin X$.

$X = \mathrm{cl}^{\omega nwd}_{\beta Y} Y$

Since $\chi(y, \beta Y) \ge \omega_1$ for all $y \in \beta Y$, a countable set is nowhere dense in βY by fact 2.1.

So X is M2 bounded, nwd-bounded, but not omega bounded because $D \subset X$ is dense.

Need: there is no first countable point in βY.

□

Theorem 5.4. *There is a locally compact space $X = \omega^* \setminus \{x\}$ such that X is not ω-nwd-bounded, but M_2-bounded.*

Proof. By vanMill, Handbook, or in [8] There is a point $x \in \omega^*$ such that

- x is not a weak P-point,

- $x \notin D'$ for all countable discrete $D \subset \omega^*$

Then $\omega^* \setminus \{x\}$ is an example. □

Theorem 5.5. *There is a 0-dimensional separable,* NEM locally compact *ω-nwd-bounded, but not M_2-bounded space X.*

Proof. Consider the space

$$X = \mathrm{cl}_{\beta\mathbb{Q}}^{\omega nwd}\,\mathbb{Q}.$$

By [2, Thm 1.5] a topological space has remote points if it has countable π-weight is not pseudocompact. Hence there is a point $\beta\mathbb{Q} \neq \mathrm{cl}_{\beta\mathbb{Q}}^{\omega nwd}\,\mathbb{Q}$. Since $\mathbb{Q}$ is M_2, X is not M_2-bounded.

However, by Lemma 2.4, Y is ω-nwd-bounded. □

Theorem 5.6. *There is an 0-dimensional, ω-D-bounded, but not M_2-nwd-bounded space X.*

Proof. Let X be an ω-D-bounded space which is not M_2-bounded from Theorem 5.5

Apply Theorem 2.5 to get the space F_X. By 2.5(ii), F_X is not M_2-nwd-bounded. By 2.5(iii), F_X is D-nwd-bounded. □

Theorem 5.7. *If $\mathfrak{p} = \mathrm{cof}(\mathcal{M})$, then there is a 0-dimensional locally compact separable non-compact topological space $X = \langle (\mathbb{C} \setminus \{0\}) \cup \mathfrak{p}, \tau\rangle$ such that*

(1) the subspace topologies on $(\mathbb{C} \setminus \{0\})$ and on $\mathfrak{p}$ are the natural euclidean and ordinal topologies, respectively,
(2) $(\mathbb{C} \setminus \{0\})$ is dense open in (so X is not compact),
(3) X is ω-nwd-bounded.
(4) $\chi(\alpha, X) \leq |\alpha| + \omega$ for $\alpha < \mathfrak{p}$.

In particular, if $\mathfrak{p} = \mathrm{cof}(\mathcal{M}) = \omega_1$, then X is first countable.

Theorem 5.8. *It is consisent that there is an 0-dimensional, ω-D-bounded, but not M_2-nwd-bounded space X of size 2^ω.*

Proof. Plug the space from Theorem 5.7 into the previous proof. □

Proof. Let X be the D-bounded, but not M_2-bounded space of size 2^ω from theorem 5.7 Apply Theorem 2.5 to get the space F_X. By 2.5(ii), F_X is not M_2-nwd-bounded. By 2.5(iii), F_X is D-nwd-bounded. □

Proposition 5.9. *There is a countably compact, non-compact, locally compact, separable space X with* $\mathrm{w}(X) = \mathfrak{t}$.

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{t}\}$ be a tower, and let X be the $\gamma\mathbb{N}$-space created from $\mathcal{A}$.

Replace the isolated points with copies of $[0, 1]$. □

The following example is well-known:

Proposition 5.10. *There is a crowded, dense, countably compact subspace X of ω^* with $|X| = 2^\omega$.*

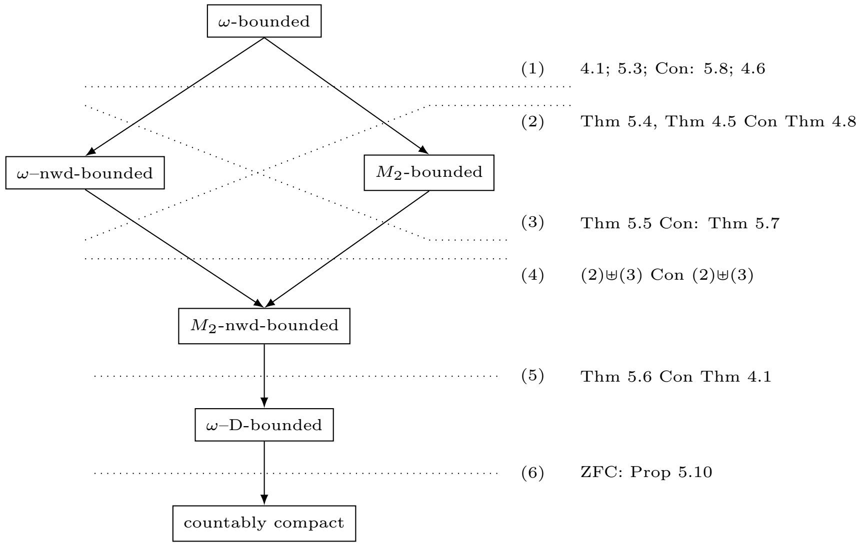

Figure 1. ZFC examples

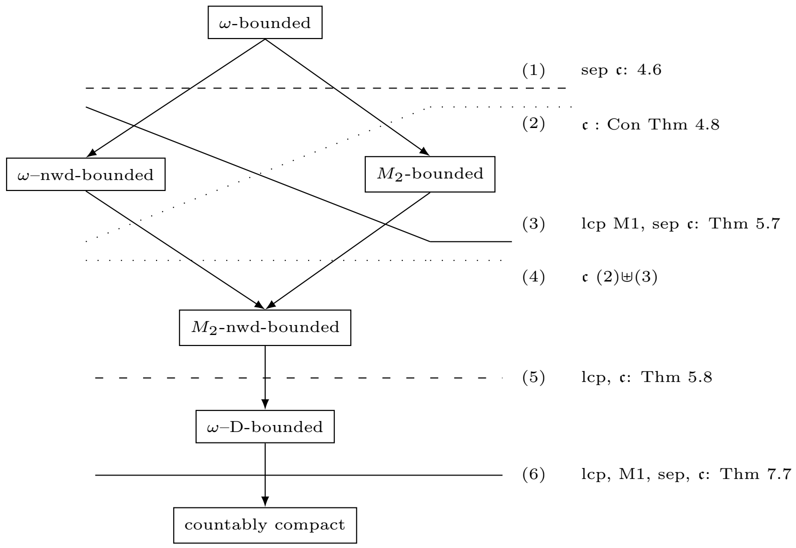

Figure 2. Consistent examples

6. Separation of boundedness properties

Problem 3. *Is there ZFC examples of cardinalities* $< 2^{2^\omega}$, *or* $\le 2^{\omega_1}$, *or* $\le 2^\omega$*?*

Problem 4. *We have locally compact example only for (2): theorem 5.4.*

Recall Problem 1: (a) Is there a Frechet (or first countable) ω-D-bounded, but not M_2-bounded (not ω-bounded) space?
(b) Is there a sequential ω-D-bounded, but not ω-nwd-bounded space?

7. Products

Theorem 7.1. *If X and Y are not ω-bounded, then $X \times Y$ is not ω-D-bounded.*

Proof. We can assume that $D = \{d_n : n \in \omega\} \in \left[X\right]^\omega$ and $E \in \left[Y\right]^\omega$ are countable , and $\mathcal{U}$ and $\mathcal{V}$ witness that $\overline{D}$ and $\overline{E}$ are not compact.

We construct $e_n \in E$ and $d_n \in U_n \in \mathcal{U}$ $e_n \in V_n \in \mathcal{V}$ such that $\langle d_n, e_n\rangle \in U_m \times V_m$ implies $n = m$.

Choose $e_n \in E \setminus \bigcup\{\overline{V_m} : m < n\}$ Then pick V_n and U_n such that $\langle d_i, e_i\rangle \notin U_n \times V_n$ for $n < i$.

Thus $\overline{\{\langle d_n, e_n\rangle\}}$ is compact, so its projection to X is also compact, but it contains D, so $\overline{D}$ is also compact. Contradiction.

□

Theorem 7.2. *If X is ω-D-bounded and Y is countably compact, then $X \times Y$ is countably compact.*

So if X is ω-D-bounded, then $X^n$is countably compact for all $n \in \omega$.

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Theorem 7.3 (vanMill, [7, Theorem 9.1])**.** *Let X be nonpseudocompact. Assuming $\mathfrak{b} = \mathbf{c}$, $\beta X \backslash X$ contains a point x such that if $F \subset \beta X \backslash x$ is countable and nowhere dense, then $x \notin cl\beta XF$.*

Theorem 7.4. *Under $\mathfrak{b} = \mathbf{c}$ there is a locally compact space X such that X is ω-nwd-bounded, M_2-bounded, but not ω-bounded.*

Proof. Let $Y = \omega \times 2^{\omega_1}$

By Theorem 4.1, there is a point x such that if F⊂βY\Y is countable and nowhere dense, then x∉clβYF.

Since Y is nowhere M2, a countable set is nowhere dense in βY.

So $\beta Y \setminus \{x\}$ is M2 bounded, nwd-bounded, but not omega bounded because Y is separable.

Need: there is no first countable point in βY.

□

Theorem 7.5. *If X is ω-D-bounded and $s(X) = \omega$, then X is a compact.*

Proof. A space X is compact iff the closure of any discrete space is compact. If $s(X) = \omega$, then there is no uncountable discrete subspace. □

Problem 5. *Is there a ω-D-bounded, but not ω-bounded space with $t(X) = \omega$?*

It is consistent that there is a countably compact HFD. This space can not be ω-D-bounded.

Is there in ZFC a countablytight, countably compact, but not ω-bounded (ω-D-bounded space)

Consistently there is a first countable example (mexico)

Lemma 7.6. *If X is countably compact $t(X) = \omega$, $\chi(p, X) \le \omega_1$, T_3, then there is a countable discrete D with $p \in D'$*

Proof. countably compact implies $\chi(p, X) = \psi\chi(p, X) = \omega_1$

cobnstruct a free sequence converge to p

Should stop in countably many steps. □

Theorem 7.7. *If CH + (t) then there is a locally compact, first countable, separable, not ω-D-bounded crowed space.*

Proof. locally compact, first countable, non-compact separable scattered space. osztasweski times [0,1] □

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
E-mail address: juhasz@renyi.hu

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
E-mail address: soukup@renyi.hu
URL: http://www.renyi.hu/~soukup

Eötvös University of Budapest
E-mail address: szentmiklossyz@gmail.com