Forcing and Combinatorics, 3.

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Erdős-Rado Theorem

$\kappa \to (\lambda)^n_\sigma$ iff for all $f : [\kappa]^n \to \sigma$ there is $A \in [\kappa]^\lambda$ s.t. $|f''[A]^n| = 1$

Theorem (Erdős-Rado)

$(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. $\quad (2^{2^\kappa})^+ \to (\kappa^+)^3_\kappa \quad (exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$

Almost disjoint families

- **there is an almost disjoint family $\mathcal{A} \subset [\omega]^{\omega}$ of cardinality 2^{ω}**
- **Is there an almost disjoint family $\mathcal{B} \subset [\omega_1]^{\omega_1}$ of size 2^{ω_1}?**
- almost disjoint: $B \cap B' \leq \omega$ for $B \neq B' \in \mathcal{B}$.
- **there is an almost disjoint family $\mathcal{B} \subset [\omega_1]^{\omega_1}$ of cardinality ω_2**
- $2^{\omega_1} \geq \omega_3$.
- $2^{\omega} = \omega_1 \implies \exists$ almost disjoint family $\mathcal{B} \subset [\omega_1]^{\omega_1}$ of size 2^{ω_1}
- $2^{\omega} > \omega_1$.
- **Con(ZFC $+\ 2^{\omega} = 2^{\omega_1} = \omega_3 +$**
  there is no almost disjoint family $\mathcal{B} \subset [\omega_1]^{\omega_1}$ of size ω_3)

$2^{\omega_1} = \omega_3 + \nexists$ a.d. family $\mathcal{B} \subset [\omega_1]^{\omega_1}$ of size ω_3

- $\mathcal{M}$ c.t.m. $\mathcal{M} \models 2^{\omega_1} = \omega_2$.
- $P = \mathbb{FIN}(\omega_3 \times \omega)$ $|\mathbb{P}| = \omega_3$, c.c.c.
- $\mathcal{M}[\mathcal{G}] \models 2^{\omega_1} \geq 2^{\omega} \geq \omega_3$.
- $\mathcal{G} \ni p \Vdash \langle A_\alpha : \alpha < \check{\omega}_3 \rangle \subset [\check{\omega}_1]^{\check{\omega}_1}$ **is almost disjoint.**
- Let $D_{\alpha,\beta} = \{\xi < \omega_1 : \exists p_\xi \in \mathbb{P}\ p_\xi \Vdash \check{\xi} = \sup(A_{\check{\alpha}} \cap A_{\check{\beta}})\}$.
- $\{p_\xi : \xi \in D_{\alpha,\beta}\}$ is an antichain: $|D_{\alpha,\beta}| \leq \omega$. $g(\alpha, \beta) = \sup D_{\alpha,\beta}$
- $g : [\omega_3]^2 \to \omega_1$. Erdős-Rado: $(2^{\omega_1})^+ \to (\omega_2)^2_{\omega_1}$
- $\exists X \in [\omega_3]^{\omega_2}\ \exists \eta < \omega_1\ \forall \alpha, \beta \in X\ g(\alpha, \beta) = \eta$.
- $\mathcal{M}[\mathcal{G}] \models \{A_\alpha \setminus (\eta + 2) : \alpha \in X\}$ **is a disjoint family**

- $\mathcal{G} \ni p \Vdash \langle A_\alpha : \alpha < \check{\omega}_3 \rangle \subset \left[\check{\omega}_1\right]^{\check{\omega}_1}$ is almost disjoint.
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- $g : \left[\omega_3\right]^2 \to \omega_1$. Erdős-Rado: $(2^{\omega_1})^+ \to [\omega_2]^2_{\omega_1}$
- $\exists X \in \left[\omega_3\right]^{\omega_2}\ \exists \eta < \omega_1\ \forall \alpha, \beta \in X\ g(\alpha,\beta) = \eta$.
- $\mathcal{M}[\mathcal{G}] \models \{A_\alpha \setminus (\eta+2) : \alpha \in X\}$ **is a disjoint family**
- Assume $\zeta \ge \eta + 1$, $\alpha, \beta \in X$ and $\mathcal{M}[\mathcal{G}] \models \zeta \in A_\alpha \cap A_\beta$.
- $\mathcal{M}[\mathcal{G}] \models \eta + 1 \le \zeta \le \xi = \sup(A_\alpha \cap A_\beta)$
- $\eta + 1 \le \zeta \le \xi$ and $\exists q \in \mathcal{G}\ (q \le p)\ q \Vdash \check{\xi} = \sup(A_{\check{\alpha}} \cap A_{\check{\beta}})$.
- $\xi \in D_{\alpha,\beta}$ so $\xi \le g(\alpha,\beta) = \eta < \eta + 1 \le \zeta \le \xi$

How to construct a topological space?

- $X = \langle X, \tau \rangle$ is a topological space
- $\mathcal{N} \subset \mathcal{P}(X)$ is a **network** iff for each $x \in U \in \tau$ there is $N \in \mathcal{N}$ with $x \in N \subset U$.
- $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a network$\}$.
- $nw(X) \le |X|, w(X)$
- $Y \subset X$ is **weakly separated** iff there is a neighbourhood assignment $f : Y \to \tau$ s.t. for each $x \ne y \in Y$ then either $x \notin f(y)$ or $y \notin f(x)$.
- $R(X) = \sup\{|Y| : Y \subset X$ is weakly separated$\}$

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- $R(X) = \sup\{|Y| : Y \subset X$ is weakly separated$\}$
- $R(X) \leq nw(X)$
- There are spaces with $nw(X) > R(X)$ in many models.
- **OPEN: Is it consistent that $nw(X) \leq R(X)$ for each regular space?**

How to approximate a topological space?

$\mathcal{N}$ is a **network** iff for each $x \in U \in \tau$ there is $N \in \mathcal{N}$ with $x \in N \subset U$.
Y is **weakly separated** iff $\exists f : Y \to \tau$ s.t. $x \notin f(y)$ or $y \notin f(x)$ for $x \neq y \in Y$.

- **Con(there is a first countable, 0-dimensional topological space $X = \langle \omega_1, \tau \rangle$ such that $nw(X) = \omega_1$ and $R(X) = \omega$).**
- If $\mathcal{M}$ **is a ctm**, then there is a **c.c.c poset** P in $\mathcal{M}$ such that if $\mathcal{G}$ is an $\mathcal{M}$-generic filter in P then in $\mathcal{M}[\mathcal{G}]$ there is a first countable, 0-dimensional topological space $X = \langle \omega_1, \tau \rangle$ such that $nw(X) = \omega_1$ and $R(X) = \omega$.

$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle,\ \chi(X) = \omega,\ nw(X) = \omega_1,\ R(X) = \omega$

$\mathcal{N}$ is a **network** iff for each $x \in U \in \tau$ there is $N \in \mathcal{N}$ with $x \in N \subset U$.
Y is **weakly separated** iff $\exists f : Y \to \tau$ s.t. $x \notin f(y)$ or $y \notin f(x)$ for $x \neq y \in Y$.

- $p = \langle A_p, n_p, \{U_p(\alpha, i) : \alpha \in A, i \in n\} \rangle \in P$ iff
  - $A_p \in [\omega_1]^{<\omega}$, $n_p \in \omega$, $\alpha \in U_p(\alpha, i) \subset A$
- Idea: $U_{\mathcal{G}}(\alpha, i) = \cup\{U_p(\alpha, i) : p \in \mathcal{G}\}$.
- $p \leq q$ iff
  - $A_p \supset A_q$, $n_p \geq n_q$ $U_p(\alpha, i) \cap A_q = U_q(\alpha, i)$ for $\langle \alpha, i \rangle \in A_q \times n_q$
  - if $U_q(\alpha, i) \subset U_q(\beta, j)$ then $U_p(\alpha, i) \subset U_p(\beta, j)$
  - if $U_q(\alpha, i) \cap U_q(\beta, j) = \emptyset$ then $U_p(\alpha, i) \cap U_p(\beta, j) = \emptyset$
- $\mathbb{P} = \langle P, \leq \rangle$ satisfies c.c.c.
- $X = \langle \omega_1, \tau \rangle$ **is first countable, 0-dimensional.**
- $nw(X) = \omega_1$? $R(X) = \omega$?

$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle$, $\chi(X) = \omega$, $nw(X) = \omega_1$, $R(X) = \omega$

$\mathcal{N}$ is a **network** iff for each $x \in U \in \tau$ there is $N \in \mathcal{N}$ with $x \in N \subset U$.
Y is **weakly separated** iff $\exists f : Y \to \tau$ s.t. $x \notin f(y)$ or $y \notin f(x)$ for $x \neq y \in Y$.

- $X = \langle \omega_1, \tau \rangle$, $\{U(\alpha, i) : \alpha < \omega_1, i < \omega\}$ base
- Need to prove: $nw(X) = \omega_1$ **and** $R(X) = \omega$
- **If X is σ-discrete then** $nw(X) = \omega_1$
- **If $X$is σ-second-countable then** $R(X) = \omega$
- It is impossible!
- Wait a minute:
  - If $\mathcal{M}[\mathcal{G}][\mathcal{H}_0] \models X$ **is σ-discrete** then $\mathcal{M}[\mathcal{G}] \models nw(X) = \omega_1$.
  - If $\mathcal{M}[\mathcal{G}][\mathcal{H}_1] \models X$ **is σ-second-countable** then $\mathcal{M}[\mathcal{G}] \models R(X) = \omega$.

$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle$, $\chi(X) = \omega$, $nw(X) = \omega_1$, $R(X) = \omega$

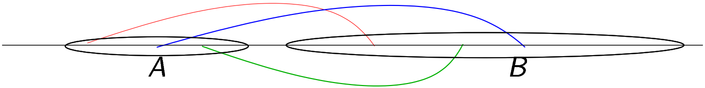
- $X = \langle \omega_1, \tau \rangle$, $\{U(\alpha, i) : \alpha < \omega_1, i < \omega\}$ base
- Need to prove: $nw(X) = \omega_1$ **and** $R(X) = \omega$**in** $\mathcal{M}[\mathcal{G}]$
- Find c.c.c posets Q_0 and Q_1 in $\mathcal{M}[\mathcal{G}]$ such that
  - if $\mathcal{H}_0$ is $\mathcal{M}[\mathcal{G}]$-generic in Q_0 then $\mathcal{M}[\mathcal{G}][\mathcal{H}_0] \models X$ **is** σ**-discrete**
  - if $\mathcal{H}_1$ is $\mathcal{M}[\mathcal{G}]$-generic in Q_1 then $\mathcal{M}[\mathcal{G}][\mathcal{H}_1] \models X$ **is** σ**-second-countable**
- Let $\langle B, f, g \rangle \in Q_0$ iff
  - $B \in [\omega_1]^{<\omega}$, $f, g : B \to \omega$
  - if $f(\beta) = n$ then $f(\beta') \neq n$ for each $\beta' \in U(\beta, g(\beta)) \setminus \{\beta\}$
- Ordering: extension
- It works.

$\mathcal{M}[\mathcal{G}] \models X = \langle \omega_1, \tau \rangle$, $\chi(X) = \omega$, $nw(X) = \omega_1$, $R(X) = \omega$

- $X = \langle \omega_1, \tau \rangle$, $\{U(\alpha, i) : \alpha < \omega_1, i < \omega\}$ base
- Need to prove: $nw(X) = \omega_1$ **and** $R(X) = \omega$
- Find c.c.c posets Q_0 and Q_1 in $\mathcal{M}[\mathcal{G}]$ such that
  - if $\mathcal{H}_0$ is $\mathcal{M}[\mathcal{G}]$-generic in Q_0 then $\mathcal{M}[\mathcal{G}][\mathcal{H}_0] \models X$ **is σ-discrete**
  - if $\mathcal{H}_1$ is $\mathcal{M}[\mathcal{G}]$-generic in Q_1 then $\mathcal{M}[\mathcal{G}][\mathcal{H}_1] \models X$ **is σ-second-countable**
- Let $\langle B, f, n, g \rangle \in Q_1$ iff
  - $B \in [\omega_1]^{<\omega}$, $f : B \to \omega$, $g : B \times n \to \omega$
  - if $f(\beta) = f(\beta') = n$ and $g(\beta, i) = g(\beta', j)$ then $U(\beta, i) \cap f^{-1}\{n\} = U(\beta', j) \cap f^{-1}\{n\}$
- Ordering: extension
- It works.

Negative partition relations

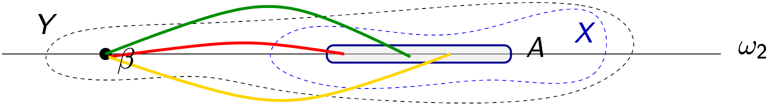
f **establishes** $\omega_2 \not\to \left[(\omega_1 : \omega)\right]^2_{\omega_1}$ iff $\forall A \in \left[\omega_2\right]^{\omega_1}$ $\forall B \in \left[\omega_2\right]^{\omega}$ if $\sup(A) < \min(B)$ then $f''[A, B] = \omega_1$.



Con (GCH + $\exists f \vdash \omega_2 \not\to [(\omega_1;\omega)]^2_{\omega_1}$).

- Assume CH. Define $\mathcal{P} = \langle P, \preceq \rangle$ σ-complete, ω_2-c.c.
- Underlying set : $\langle X, c, \mathcal{A}, \xi \rangle$
  - $X \in [\omega_2]^\omega$, $c : [X]^2 \to \omega_1$
  - $\mathcal{A} \subset P(X)$, $|\mathcal{A}| \le \omega$,
  - $\xi \in \omega_1$
- $\langle Y, d, \mathcal{B}, \zeta \rangle \preceq \langle X, c, \mathcal{A}, \xi \rangle$ iff
  - $Y \supset X$, $d \supset c$, $\mathcal{B} \supset \mathcal{A}$, $\zeta \ge \xi$

(1) $\forall A \in \mathcal{A} \quad \forall \beta \in (Y \setminus X) \cap \min A \quad \xi \subset d''[\{\beta\}, A]$.



A ZFC theorem

- $\omega \to (\omega)^2_2$
- $\omega_1 \not\to (\omega_1)^2_2$
- (Baumgartner, Hajnal) $\omega_1 \to (\alpha)^2_2$ **for** $\alpha < \omega_1$
- Step 1: If $MA_{\aleph_1}$ holds then $\omega_1 \to (\alpha)^2_2$ for $\alpha < \omega_1$.
- Fact: In any ctm $\mathcal{M}$ there is a c.c.c poset P s.t. if $\mathcal{G}$ is $\mathcal{M}$-generic filter then $\mathcal{M}[\mathcal{G}] \models MA_{\aleph_1}$
- Step 2: $\mathcal{M} \models f : [\omega_1]^2 \to 2$;

- (Baumgartner, Hajnal) $\omega_1 \to (\alpha)^2_2$ **for** $\alpha < \omega_1$
- Step 1: If $MA_{\aleph_1}$ holds then $\omega_1 \to (\alpha)^2_2$ for $\alpha < \omega_1$.
- Fact: In any ctm $\mathcal{M}$ there is a c.c.c poset P s.t. if $\mathcal{G}$ is $\mathcal{M}$-generic filter then $\mathcal{M}[\mathcal{G}] \models MA_{\aleph_1}$
- Step 2: $\mathcal{M} \models f : [\omega_1]^2 \to 2$;
- $\alpha = \{\alpha_n : n < \omega\}$.
- Define $Q = \langle Q, \leq \rangle$ as follows. $\sigma \in Q$ iff
  - σ is an injective function, $\mathrm{dom}(\sigma) = \{\alpha_0, \ldots, \alpha_n\}$ **for some** $n < \omega$
  - $\mathrm{ran}(\sigma) \subset \omega_1$, σ is order preserving
  - $\mathrm{ran}(\sigma)$ is f-homogeneous
- **Q is ill founded iff $\exists A \subset \omega_1$ f-homogeneoues,** $\mathrm{tp}(A) = \alpha$
- **Q is well-founded in $\mathcal{M}$ iff it is well-founded in $\mathcal{M}[\mathcal{G}]$**
- $\exists A$ in $\mathcal{M}[\mathcal{G}]$, so Q is ill-founded in $\mathcal{M}[\mathcal{G}]$ so Q is ill-founded in $\mathcal{M}$, so $\exists A$ in $\mathcal{M}$