Forcing without tears

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Forcing as a black box method: Basic concept

Prove that $ZFC \nvdash CH$

first order language: $\in$ binary predicate

Prove that if ZFC is consistent then $ZFC + \neg CH$ **is consistent**

$Con(T)$ iff T is **consistent**, i.e. $T \nvdash \psi \wedge \neg\psi$.

Con(ZFC) $\Longrightarrow$ **Con(ZFC** $+ \varphi$)

$\mathcal{M} \models ZFC \quad \xrightarrow{\text{FORCING}} \quad \mathcal{M}[\mathcal{G}] \models ZFC + \varphi$

ground model — generic extension

Gödel Incompleteness Theorem:
within ZFC one cannot produce a model of ZFC

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- $\mathbb{P} = \langle P, \leq \rangle \in M$ poset, with greatest element $1_{\mathbb{P}}$.
- $\mathbb{P}$ is a **forcing notion**, $p \in \mathbb{P}$ is a **condition**
- $\mathcal{G} \subset P$ is a **filter**

  (1.) $1_{\mathbb{P}} \in \mathcal{G}$

  (2.) $\forall p, q \in \mathcal{G}\ \exists r \in \mathcal{G}\ r \leq p, q$

  (3.) $\forall p \in \mathcal{G}$ if $p \leq q$ then $q \in \mathcal{G}$
- $D \subset P$ is **dense** iff $\forall p \in P\ \exists d \in D\ d \leq P$.
- Assume that $\mathcal{D}$ is a **family of sets**.

  A filter $\mathcal{G} \subset P$ is a **$\mathcal{D}$-generic filter in $\mathbb{P}$** iff for each $D \in \mathcal{D}$ if D is dense in P then $\mathcal{G} \cap D \neq \emptyset$.

Theorem (Rasiowa-Sikorski lemma)

*If $\mathbb{P}$ is partially ordered set and $\mathcal{D}$ is a **countable** family then there exists a $\mathcal{D}$-generic filter $\mathcal{G}$ in $\mathbb{P}$.*

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$\mathcal{M}[\mathcal{G}] \neq \mathcal{M}$.

Observation

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that

$$\forall p \in P\ \exists q, q' \leq p\ q \text{ and } q' \text{ are } \textbf{incompatible} \text{ in } \mathbb{P}.$$

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The main theorem

$val_{\mathcal{G}}(y) = \{val_{\mathcal{G}}(x) : \exists p \in \mathcal{G}\ \langle x, p\rangle \in y\}$
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Theorem

*Let $\mathcal{M} = \langle M, \in\rangle$ be a countable transitive model of ZFC, $\mathbb{P} \in M$ be a poset, and $\mathcal{G}$ be an **$\mathcal{M}$-generic filter in** $\mathbb{P}$.*

(1) $M[\mathcal{G}]$ *is a countable, transitive set,*

(2) $M \subset M[\mathcal{G}]$ *and* $\mathcal{G} \in \mathcal{M}[\mathcal{G}]$

(3) $On^{\mathcal{M}[\mathcal{G}]} = On^{\mathcal{M}}$

(4) $\mathcal{M}[\mathcal{G}] = \langle M[\mathcal{G}], \in\rangle$ *is a model of ZFC*

Absoluteness

$On^{\mathcal{M}} = On^{\mathcal{M}[\mathcal{G}]}$

- Consider the polinomial $f(x) = x^2 + 1$
- $\mathbb{R}[x] \models$ "$f(x)$ is irreducible"
- $\mathbb{C}[x] \models$ "$f(x)$ is not irreducible"
- **The statement $f(x)$ is irreducible is not "absolute".**

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- Consider the polinomial $f(x) = x^2 + 1$
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- α is an ordinal
- α is limit ordinal
- α is a successor ordinal
- $\alpha = \omega$
- $\mathcal{P} = \langle P, \leq \rangle$ is a poset

Fact

If $\mathcal{N} = \langle N, \in \rangle$ is a c.t.m. then $On^{\mathcal{N}} = \mathcal{N} \cap On$ and so $On^{\mathcal{N}}$ is a countable ordinal.

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If $\mathcal{N} = \langle N, \in \rangle$ is a c.t.m. then $On^{\mathcal{N}} = \mathcal{N} \cap On$ and so $On^{\mathcal{N}}$ is a countable ordinal.

If M is a c.t.m then the formula α is a countable ordinal is not absolute.

Absoluteness

φ is **absolute for M** iff $\forall x_1, \dots x_n \in M$ ($\varphi(x_1, \dots, x_m)$ iff $\langle \mathbf{M}, \in \rangle \models \varphi(x_1, \dots, x_n)$.

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The main theorem

Theorem

*Let $\mathcal{M} = \langle M, \in \rangle$ be a countable transitive model of ZFC, $\mathbb{P} \in M$ be a poset, and $\mathcal{G}$ be an $\mathcal{M}$-**generic filter in** $\mathbb{P}$.*

(1) $M[\mathcal{G}]$ *is a countable, transitive set,*

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(3) $On^{\mathcal{M}[\mathcal{G}]} = On^{\mathcal{M}}$

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- $val_{\mathcal{G}}(y) = \{val_{\mathcal{G}}(x) : \exists p \in \mathcal{G}\ \langle x, p\rangle \in y\}$
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Definition

Assume that

- $\varphi(x_1, \ldots, x_n)$ is a formula with at most $x_1, \ldots, x_n$ free
- $\mathcal{M} = \langle M, \in\rangle$ is a countable transitive model of ZFC,
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- Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset.
- $A \subset P$ is an **antichain** iff $p \perp q$ for each $p \neq q \in A$
- $\mathbb{P}$ satisfies the **countable chain condition (c.c.c.)** iff every antichain $A \subset \mathbb{P}$ is countable.
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*Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then*

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*Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then*

- *$\mathcal{M}$ and $\mathcal{M}[\mathcal{G}]$ have the same cardinals,*
- *$\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{M}[\mathcal{G}]}(\delta)$ for each ordinal $\delta \in On^{\mathcal{M}} = On^{\mathcal{M}[\mathcal{G}]}$*

Cardinals in $\mathcal{M}[\mathcal{G}]$.

- Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset.
- $A \subset P$ is an **antichain** iff $p \perp q$ for each $p \neq q \in A$
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A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

- Assume $\mathcal{M}[\mathcal{G}] \models \lambda < \kappa = \mathrm{cf}^{\mathcal{M}}(\delta) \wedge f : \lambda \to \delta$ **is cofinal**
- for some $\lambda < \kappa$ and $\dot{f} \in M$
  $\mathcal{M}[\mathcal{G}] \models val_{\mathcal{G}}(\dot{f}) : val_{\mathcal{G}}(\check{\lambda}) \to val_{\mathcal{G}}(\check{\delta})$ **is cofinal**
- there is $p \in P$ s.t. $p \Vdash \dot{f} : \check{\lambda} \to \check{\delta}$ **is cofinal**
- for each $\alpha < \lambda$ let $E_\alpha = \{\xi < \delta : \exists q^\alpha_\xi \le p \ q^\alpha_\xi \Vdash \dot{f}(\check{\alpha}) = \check{\xi}\}$
- $\{q^\alpha_\xi : \xi \in E_\alpha\}$ is an antichain in $\mathbb{P}$, so $|E_\alpha| \le \omega$.
- So $E = \cup\{E_\alpha : \alpha < \lambda\} \in [\delta]^{\le \lambda}$ is bounded in δ. $\zeta = \sup E < \delta$.
- Let $\mathcal{G} \ni p$ be generic in $\mathbb{P}$.
- $\mathcal{M}[\mathcal{G}] \models$ there are $\alpha < \lambda$ and $\zeta < \eta < \delta$ s.t. $val_{\mathcal{G}}(\dot{f})(\alpha) = \eta$.
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- $\exists q \in \mathcal{G}$ $q \le p \wedge$ $q \Vdash \dot{f}(\check{\alpha}) = \check{\eta}$.
- $\eta \in E_\alpha \subset E \subset \zeta$. Contradiction

A c.c.c forcing extension preserves cofinalities.

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A c.c.c forcing extension preserves cofinalities.

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A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

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A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

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A c.c.c forcing extension preserves cofinalities.

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A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

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- $\eta \in E_\alpha \subset E \subset \zeta$. Contradiction

A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

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- $\eta \in E_\alpha \subset E \subset \zeta$. Contradiction

A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

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- $\eta \in E_\alpha \subset E \subset \zeta$. Contradiction

A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

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A c.c.c forcing extension preserves cofinalities.

Assume that $\mathcal{M} = \langle M, \in \rangle$ is a transitive model of ZFC, and $\mathbb{P} \in M$ is a partially ordered set such that $\mathcal{M} \models \mathbb{P}$ **satisfies c.c.c.**, then $\mathrm{cf}^{\mathcal{M}}(\delta) = \mathrm{cf}^{\mathcal{N}}(\delta)$.

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Cardinal exponentiation in $\mathcal{M}[\mathcal{G}]$

Theorem

If $\mathcal{M}$ is a c.t.m, $\mathbb{P} \in M$ is a forcing notion, $\mathbb{P}$ satisfies κ-c.c., λ is a cardinal in $\mathcal{M}$ then $(2^\lambda)^{\mathcal{M}[\mathcal{G}]} \le \left(\left(|P|^{<\kappa}\right)^\lambda\right)^{\mathcal{M}}$

- for $x \in M$ and $\alpha < \lambda$
  choose a maximal antichain $A_{x,\alpha} \subset D_{x,\alpha} = \{p \in P : p \Vdash \check{\alpha} \in x\}$.
- $x \mapsto \langle A_{x,\alpha} : \alpha < \lambda \rangle \in \left([P]^{<\kappa}\right)^\lambda$
- **if** $\langle A_{x,\alpha} : \alpha < \lambda \rangle = \langle A_{y,\alpha} : \alpha < \lambda \rangle$ **then** $val_{\mathcal{G}}(x) \cap \lambda = val_{\mathcal{G}}(y) \cap \lambda$.
  - Assume $\alpha \in (val_{\mathcal{G}}(x) \setminus val_{\mathcal{G}}(y)) \cap \lambda$. $\exists p \in \mathcal{G}\ p \Vdash \check{\alpha} \in x \setminus y$
  - $\exists q \in A_{x,\alpha}$ q and p are compatible in $\mathbb{P}$. Let $r \le p, q$
  - $r \le q \in A_{x,\alpha} = A_{y,\alpha}$, so
  - $r \le p$ so $q \Vdash \check{\alpha} \notin y$.

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- $p \Vdash^* a \in b$ iff $\exists c\ \exists q \geq p\ (\langle c, q\rangle \in a \wedge p \Vdash^* a = c)$.
- $p \Vdash^* a \neq b$ iff $\exists c\ \exists q \geq p$
  $(\langle c, q\rangle \in a \wedge p \Vdash^* c \notin b) \vee (\langle c, q\rangle \in b \wedge p \Vdash^* c \notin a)$.
- $p \Vdash^* \neg\varphi$ iff $\neg\exists q \leq p\ q \Vdash^* \varphi$
- $p \Vdash^* \varphi \vee \psi$ iff $p \Vdash^* \varphi$ or $p \Vdash^* \psi$
- $p \Vdash^* \exists x \varphi(x)$ iff $\exists d\ p \Vdash^* \varphi(d)$.

Lemma

(1) ***The relation*** $p \Vdash^* \varphi(x_1, \ldots x_n)$ ***is definable in*** M

(2) $\mathcal{M}[\mathcal{G}] \models \varphi(val_{\mathcal{G}}(x_1), \ldots val_{\mathcal{G}}(x_n))$ ***iff*** $\exists p \in \mathcal{G}\ p \Vdash^* \varphi(x_1, \ldots x_n)$

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