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A Sperner-Type Theorem and Qualitative Independence*

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We determine the asymptotics of the largest family $\{C_i\}_{i=1}^{M}$ of subsets of an n-set with the property that for some bipartitions $C_i = A_i \cup B_i$ of the C_i's none of the inclusions $A_i \subset C_j$, $B_i \subset C_j$ occurs. Our construction implies a new lower bound on the size of qualitatively independent partition systems in the Rényi sense.

INTRODUCTION

This paper is in three parts. In the first part we state and solve a combinatorial problem in extremal set theory. The problem, described in the abstract, is a new variation on the familiar Sperner theme [1]. In the second part, we shall deal with some immediate implications of and problems raised by our result well within the framework of extremal set theory. In particular, we will improve on the Poljak–Tuza lower bound [2] on the number of qualitatively independent 3-partitions of an n-set. Our new bound is an incidental by-product of our original result and is certainly sub-optimal. In the last part of the paper we will put the foregoing into a broader perspective using an analogy of the present problems and results to a well-established class of zero-error capacity problems in information theory [3]. We will show that this analogy is the source of a wealth of problems and conjectures.

Although in proving our main result, we could rely on [3] for the non-existence part, we prefer to avoid an explicit use of information theory at that point. Logarithms and exponents are to the base 2.

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1. A Sperner-Type Theorem

The celebrated Sperner theorem [1] states that if $\{A_i\}_{i=1}^m$ is a family of subsets of an n-set with the property that $A_i \subset A_j$ never holds for $i \neq j$, then

$$m \leqslant \binom{n}{\lfloor n/2 \rfloor}.$$

Moreover, we have the more precise LYM inequality, cf. Berge [4], asserting that for a system $\{A_i\}_{i=1}^m$ as above,

$$\sum_{i=1}^m \binom{n}{|A_i|}^{-1} \leqslant 1.$$

Obviously, both inequalities are tight (and the second implies the first one). Our main result is about systems of subsets of an n-set satisfying a similar but stronger condition. For other variations on the Sperner theme, we refer the reader to Berge [4] and Engel and Gronau [5].

Theorem 1. *Let $M(n)$ denote the cardinality of the largest family $\{C_i\}_{i=1}^{M(n)}$ of subsets of an n-element set with the property that for an appropriate bipartition of every C_i*

$$C_i = A_i \cup B_i, \qquad A_i \cap B_i = \varnothing,$$

one has

$$A_i \not\subset C_j,\ B_i \not\subset C_j \qquad \text{for} \quad j \neq i.$$

Then

$$\lim_{n \to \infty} [M(n)]^{1/n} = \frac{1+\sqrt{5}}{2}.$$

Proof. We claim first that

$$\limsup_{n \to \infty} \frac{1}{n} \log M(n) \leqslant \log \frac{1+\sqrt{5}}{2}. \tag{1}$$

To see this, let $\{\hat{C}_i\}_{i=1}^{\hat{M}}$ be any system as above with $\hat{C}_i \subset X$, $|X| = n$, $\hat{M} = M(n)$, and let the bipartitions of the $\hat{C}_i$'s be

$$\hat{C}_i = \hat{A}_i \cup \hat{B}_i, \qquad \hat{A}_i \cap \hat{B}_i = \varnothing.$$

For every i, let us associate to $\hat{C}_i$ the pair of integers $(|\hat{A}_i|, |\hat{B}_i|)$. As this pair can take at most $(n+1)^2$ different values, there clearly is a subsystem $\{C_i\}_{i=1}^M$ satisfying the previous conditions for which

$$M \geqslant (n+1)^{-2} \hat{M} \tag{2}$$

and, for some a and b,

$$|A_i| = a, \qquad |B_i| = b, \qquad \text{for every } i = 1, 2, ..., M. \tag{3}$$

Let us set $D_i = X - C_i$. We can see that the pairs (A_i, D_i), $i = 1, 2, ..., M$ satisfy the conditions of Bollobás' inequality [6], i.e.,

$$A_i \cap D_j = \varnothing \qquad \text{if and only if } i = j.$$

The same is true for the pairs (B_i, D_i). Hence, by Bollobás' theorem [6]

$$\sum_{i=1}^{M} \binom{|A_i| + |D_i|}{|A_i|}^{-1} \leqslant 1$$

and

$$\sum_{i=1}^{M} \binom{|B_i| + |D_i|}{|B_i|}^{-1} \leqslant 1.$$

Using (3) we conclude that

$$M \leqslant \min\left[\binom{n-b}{a}, \binom{n-a}{b}\right].$$

An elementary inequality shows (cf. [7, Lemma 1.2.3]) that

$$\binom{n}{t} \leqslant 2^{nh(t/n)},$$

where $h(x) = -x \log x - (1-x) \log (1-x)$ is the binary entropy function. Using the last two inequalities and (2) we conclude that

$$\limsup_{n \to \infty} \frac{1}{n} \log M(n) \leqslant \max_{\alpha, \beta} \min\left[(1-\alpha)\, h\left(\frac{\beta}{1-\alpha}\right), (1-\beta)\, h\left(\frac{\alpha}{1-\beta}\right)\right].$$

Applying Jensen's inequality to the concave function h,

$$\begin{aligned} &\min\left[(1-\alpha)\, h\left(\frac{\beta}{1-\alpha}\right), (1-\beta)\, h\left(\frac{\alpha}{1-\beta}\right)\right] \\ &\qquad \leqslant \frac{1}{2}\left[(1-\alpha)\, h\left(\frac{\beta}{1-\alpha}\right) + (1-\beta)\, h\left(\frac{\alpha}{1-\beta}\right)\right] \leqslant (1-\gamma)\, h\left(\frac{\gamma}{1-\gamma}\right) \end{aligned}$$

for $\gamma = (\alpha + \beta)/2$. Combining this with the previous inequality we see that

$$\limsup_{n \to \infty} \frac{1}{n} \log M(n) \leqslant \max_{\gamma} (1-\gamma)\, h\left(\frac{\gamma}{1-\gamma}\right).$$

Thus, in order to establish (1) it remains to prove that

$$\max_{\gamma} (1-\gamma)\, h\left(\frac{\gamma}{1-\gamma}\right) = \log \frac{1+\sqrt{5}}{2}. \tag{4}$$

Now, once again using the asymptotics of the binomial coefficient as featured, e.g., in [7, Lemma 1.2.3], i.e.,

$$(n+1)^{-2}\, 2^{nh(t/n)} \leqslant \binom{n}{t} \leqslant 2^{nh(t/n)},$$

we can write

$$\begin{aligned}\max_{\gamma} (1-\gamma)\, h\left(\frac{\gamma}{1-\gamma}\right) &= \lim_{n\to\infty} \frac{1}{n} \log \max_{k} \binom{n-k}{k}\\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{k=0}^{n} \binom{n-k}{k}.\end{aligned}$$

Thus, in order to prove (4) it will be sufficient to see that

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{k=0}^{n} \binom{n-k}{k} = \log \frac{1+\sqrt{5}}{2}. \tag{5}$$

This, however, is clear, since $\sum_{k=0}^{n} \binom{n-k}{k}$ is the number of n-length binary sequences with no consecutive 1's. The latter is known to be equal to the $(n+2)$th Fibonacci number, f_{n+2}. (We use the notation $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$.) This proves (5).

To complete the proof of the theorem, we will give a construction of $M(n)$ sets $C_i \subset X$, $|X| = n$ with the required properties. We shall represent the sets C_i along with their bipartitions

$$C_i = A_i \cup B_i, \qquad A_i \cap B_i = \varnothing$$

by ternary sequences of length n. To this end, let us define

$$X = \{1, 2, ..., n\}$$

and to the sequence $\mathbf{x} \in \{0, 1, 2\}^n$ let correspond the sets A, B, $C = A \cup B$ by the rule

$$\begin{aligned} A &= A(\mathbf{x}) = \{i, x_i = 1\}\\ B &= B(\mathbf{x}) = \{j, x_j = 2\}\\ C &= C(\mathbf{x}) = \{k, x_k \neq 0\}.\end{aligned} \tag{6}$$

Now, let $\mathscr{C}_n^{**} \subset \{0, 1, 2\}^n$ be the set of those ternary sequences $\mathbf{x} = (x_1, x_2, ..., x_n)$ for which

(i) $x_n \neq 1, x_1 \neq 2$

(ii) $x_i = 1$ iff $x_{i+1} = 2$ for $i = 1, ..., n-1$.

Clearly,

$$|\mathscr{C}_n^{**}| = f_{n+1}, \tag{7}$$

where f_{n+1} is the $(n+1)$th Fibonacci number, since $|C^{**}|$ is equal to the number of $(n-1)$-length binary sequences with no consecutive 1's. To every $\mathbf{x} \in \{0, 1, 2\}^n$ let us make the corresponding pair of integers $(|A(\mathbf{x})|, |B(\mathbf{x})|)$ defined through (6). Then, as in the first part of the proof, there is a subset $\mathscr{C}_n^* \subset \mathscr{C}_n^{**}$ with

$$|\mathscr{C}_n^*| \geqslant (n+1)^{-2} |\mathscr{C}_n^{**}|, \tag{8}$$

for which $(|A(\mathbf{x})|, |B(\mathbf{x})|)$ is the same pair of integers for every element $\mathbf{x} \in \mathscr{C}_n^*$. Finally, we define

$$\lambda(\mathbf{x}) = \sum_{i : x_i = 1} i;$$

in other words, $\lambda(\mathbf{x})$ is the sum of those coordinate indices for which $\mathbf{x}$ has a one in the corresponding coordinate. Obviously, $\lambda(\mathbf{x})$ is an integer between 0 and $n^2/2$, say. Hence there is a subset $\mathscr{C}_n \subset \mathscr{C}_n^*$ with

$$|\mathscr{C}_n| > n^{-2} |\mathscr{C}_n^*|$$

on which $\lambda(\mathbf{x})$ is constant. Combining this with (7) and (8) we obtain

$$|\mathscr{C}_n| > (n+1)^{-4} f_{n+1},$$

whence

$$\liminf_{n \to \infty} \frac{1}{n} \log |\mathscr{C}_n| \geqslant \log \frac{1+\sqrt{5}}{2}. \tag{9}$$

The sets $C(\mathbf{x})$, $\mathbf{x} \in \mathscr{C}_n$, along with their bipartitions defined as in (6) are easily shown to satisfy the conditions of the theorem. In fact, what we have to prove amounts in the sequence language to establishing that for every $\mathbf{x} \neq \mathbf{x}'$ in $\mathscr{C}_n$ there is a coordinate i with

$$x_i = 1, \qquad x_i' = 0 \tag{10}$$

and a coordinate j with

$$x_j = 2, \qquad x_j' = 0. \tag{11}$$

First, we claim that to any distinct $\mathbf{x}$ and $\mathbf{x}'$ in $\mathscr{C}_n^{**}$ there is a coordinate i in which one has a 1 and the other has a 0. This is true since if $\mathbf{x}$ and $\mathbf{x}'$ are different then there is a leftmost coordinate in which they differ and that coordinate cannot feature a 2; otherwise the two sequences would disagree in the preceding one. If in this coordinate i we have $x_i = 1$ and $x_i' = 0$ then we are done. If, however, $x_i = 0$ and $x_i' = 1$, then we have to exploit the fact that $\mathbf{x}$ and $\mathbf{x}'$ both are in $\mathscr{C}_n$, as well. Let us suppose that the 1 appearing in the ith coordinate in $\mathbf{x}'$ is the kth 1 from the left in the sequence. Then, for every $j \leqslant k$ the jth 1 of $\mathbf{x}'$ appears before (or simultaneously with) the jth 1 of $\mathbf{x}$. But the same cannot happen for all the later 1's of $\mathbf{x}'$ for $\mathbf{x}$ and $\mathbf{x}'$ have the same number of 1's and in order for the two sequences to have the same value of λ, at one point $\mathbf{x}$ must have a 1 appearing strictly before the corresponding 1 in $\mathbf{x}'$. Suppose this first happens in the lth coordinate. Then, clearly, $x_l' = 0$. This establishes (10). Relation (11) can be verified in the same manner, changing "1" to "2," "left" to "right," and noticing that the function corresponding to λ for the positions of the 2's is constant on $\mathscr{C}_n$ by definition. Thus $\mathscr{C}_n$ gives the desired construction.

It is important to note that (1) is a straightforward simple consequence of the result in [3] and could have been established in a self-contained manner using elementary information theory. However, we have opted to avoid that technique in this part of the paper. The connection of the present problems to information theory and to our problem in [3] in particular will be discussed in the last part of this article.

Several questions arise naturally in connection with Theorem 1. We do not know the answer to any of them.

Problem 1. What is the exact form of the function $M(n)$ and which are the extremal configurations; i.e., what do the families of sets of maximum cardinality with respect to the conditions of Theorem 1 look like?

A more subtle question is related to

Problem 2. It can be seen that a family of sets $\{C_i\}_{i=1}^M$, $C_i \subset X$, $|X| = n$ with

$$C_i = A_i \cup B_i, \qquad A_i \cap B_i = \varnothing, \qquad A_i \not\subset C_j, \qquad B_i \not\subset C_j, \qquad i \neq j$$

satisfies the LYM-type inequality

$$\sum_{i=1}^{M} \binom{n-|A_i|}{|B_i|}^{-1} + \binom{n-|B_i|}{|A_i|}^{-1} - \binom{n}{|A_i|+|B_i|}^{-1} \leqslant 1.$$

Is this inequality ever tight?

Perhaps the most challenging open question is

Problem 3. Let $N(n)$ denote the cardinality of the largest family $\{D_i\}_{i=1}^{N(n)}$ of subsets of an n-element set with the property that for every i and an appropriate tripartition

$$D_i = A_i \cup B_i \cup C_i, \qquad A_i \cap B_i = A_i \cap C_i = B_i \cap C_i = \varnothing,$$

one has

$$A_i \not\subset D_j, \qquad B_i \not\subset D_j, \qquad C_i \not\subset D_j \qquad \text{for} \quad j \neq i.$$

Determine

$$\lim_{n \to \infty} [N(n)]^{1/n}.$$

Our ideas do not extend to this problem. We can show that the logarithm of the limit is at most

$$\max_{\lambda} (1-2\lambda)\, h\left(\frac{\lambda}{1-2\lambda}\right),$$

but we have no reason to believe this bound to be tight. Maybe, this problem is once again a manifestation of the "magic quality of twoness," cf. Schrijver [8].

2. Qualitative Independence

Let X be a set of n elements and let $P = (P_1, P_2, ..., P_t)$ and $P' = (P'_1, P'_2, ..., P'_t)$ be two partitions of X into t disjoint classes, i.e.,

$$X = \bigcup_{i=1}^{t} P_i, \qquad P_i \cap P_j = \varnothing \qquad \text{if} \quad i \neq j$$

and similarly for P'. Following Rényi [9], the partitions P and P' are qualitatively independent if

$$P_i \cap P'_j \neq \varnothing \qquad \text{for every } i \text{ and } j.$$

This definition has a simple probabilistic meaning. In fact, two partitions can be generated by two independent random variables if and only if the partitions are qualitatively independent. Let $N(n, t)$ be the largest possible size of a family of t-partitions (partitions into at most t disjoint classes) of an n-set under the condition that any two different partitions in the family are qualitatively independent. Qualitative independence has an extended literature. The question of determining $N(n, t)$ goes back to the book [9] of Alfréd Rényi. Apparently no good technique exists to construct large partition systems with the above property. To highlight this circumstance, we improve on a recent result of Poljak and Tuza [2] for the case $t=3$. In [2], Theorem 1 and Theorem 4 state

THEOREM PT.

$$\frac{1}{2}\binom{\frac{n}{3}}{\frac{n}{6}} \leqslant N(n, 3) \leqslant \frac{1}{2}\binom{\frac{2}{3}n}{\frac{n}{3}}.$$

Asymptotically, this amounts to

COROLLARY PT.

$$\limsup_{n\to\infty} \frac{1}{n} \log N(n, 3) \leqslant \frac{2}{3} \tag{12}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log N(n, 3) \geqslant \frac{1}{3}. \tag{13}$$

We are not able to bridge this huge gap. In the last section we shall comment on the upper bound that can be obtained also under much weaker assumptions by standard information theory. (Poljak and Tuza use Bollobás' inequality.) However, our main concern is with the lower bound (13). Just as the upper bound, it features a deceivingly "nice" number. We claim that

THEOREM 2.

$$\liminf_{n\to\infty} \frac{1}{n} \log N(n, 3) \geqslant 0.409.$$

Proof.

We want to construct a large number of 3-partitions of an n-set such that any 2 of them are qualitatively independent. We will represent a 3-partition of $X=\{1, 2, ..., n\}$ by a ternary sequence $\mathbf{x} \in \{0, 1, 2\}^n$ in the obvious manner. This means that a particular partition might have six diffeent representations but this will not cause any problem. The partitions represented by

$\mathbf{x}$ and $\mathbf{x}'$ are qualitatively independent if and only if for any two (not necessarily distinct) elements $a \in \{0, 1, 2\}$, $b \in \{0, 1, 2\}$ there is a coordinate $i = i(a, b) = i(a, b, \mathbf{x}, \mathbf{x}')$ such that

$$x_i = a, \qquad x_i' = b.$$

Let us fix some $\alpha \in (0, 1)$. By Theorem 1, we can construct a set $\mathscr{C}_{\alpha n}$ of elements of $\{0, 1, 2\}^{\alpha n}$ such that for any two of them, $\mathbf{y}$ and $\mathbf{y}'$, say, and for the choices

$$\begin{aligned} a &= 0, \qquad b = 1 \\ a &= 1, \qquad b = 0 \\ a &= 0, \qquad b = 2 \\ a &= 2, \qquad b = 0, \end{aligned} \tag{14}$$

we have a coordinate $i = i(a, b)$ with

$$y_i = a, \qquad y_i' = b.$$

The size of $\mathscr{C}_{\alpha n}$ satisfies

$$\liminf_{n \to \infty} \frac{1}{n} \log |\mathscr{C}_{\alpha n}| \geqslant \alpha \log \frac{1 + \sqrt{5}}{2}. \tag{15}$$

On the other hand, by Sperner's theorem, we can construct a subset $\mathscr{D}_{(1-\alpha)n}$ of $\{1, 2\}^{(1-\alpha)n}$ such that for every $\mathbf{z}, \mathbf{z}' \in \mathscr{D}_{(1-\alpha)n}$ and the two choices

$$\begin{aligned} a &= 1, \qquad b = 2, \\ a &= 2, \qquad b = 1, \end{aligned} \tag{16}$$

there is a coordinate $i = i(a, b)$ with

$$z_i = a, \qquad z_i' = b.$$

The size of $\mathscr{D}_{(1-\alpha)n}$ satisfies

$$\liminf_{n \to \infty} \frac{1}{n} \log |\mathscr{D}_{(1-\alpha)n}| \geqslant 1 - \alpha. \tag{17}$$

Now, choose α in such a way that

$$\alpha \log \frac{1 + \sqrt{5}}{2} = 1 - \alpha.$$

This gives

$$\alpha = [\log(1+\sqrt{5})]^{-1}$$

and thus $1-\alpha \sim 0.409\ldots$. Hence, for this α, by (15) and (17) we can construct sets

$$\hat{\mathscr{C}}_n \subset \mathscr{C}_{\alpha n}, \qquad \hat{\mathscr{D}}_n \subset \mathscr{D}_{(1-\alpha)n}$$

such that

$$|\hat{\mathscr{C}}_n| = |\hat{\mathscr{D}}_n|$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log |\hat{\mathscr{C}}_n| \geqslant 0.409\ldots.$$

Let $\varphi : \hat{\mathscr{C}}_n \to \hat{\mathscr{D}}_n$ be an arbitrary one-to-one correspondence between the elements of these two sets. Using this correspondence, we shall now construct a set $\mathscr{A}_m \subset \{0, 1, 2\}^m$ as requested. Choose $m = n+3$. For every elementary $\mathbf{y} \in \hat{\mathscr{C}}_n$ we shall define a sequence $\mathbf{x} \in \{0, 1, 2\}^m$ as follows: We set

$$x_1 \cdots x_{\alpha n} = \mathbf{y}, \qquad x_{\alpha n+1} \cdots x_n = \varphi(\mathbf{y}), \qquad x_{n+1} = 0, \qquad x_{n+2} = 1, \qquad x_{n+3} = 2.$$

Let us denote the sequence $\mathbf{x} \in \{0, 1, 2\}^m$ so obtained by $\mathbf{x}(\mathbf{y})$. Set

$$\mathscr{A}_m = \{\mathbf{x}(\mathbf{y}), \mathbf{y} \in \hat{\mathscr{C}}_n\}.$$

Clearly, the set $\mathscr{A}_m \subset \{0, 1, 2\}^m$ satisfies the requirements stated at the beginning of the proof. In fact, if (a, b) is as in (14), then the corresponding i is $i \in [1, \alpha n]$, if (a, b) is as in (16), then i satisfies $\alpha n + 1 \leqslant i \leqslant n$, and finally, if $a = b$, then $i \in [n+1, n+3]$.

Further, we have

$$\liminf_{n\to\infty} \frac{1}{m} \log |\mathscr{A}_m| = 1 - \alpha = 0.409\ldots.$$

Although Theorem 2 improves on the result of Poljak and Tuza, it is quite clear that our lower bound on $N(n, 3)$ cannot be tight. Those believing in "nice results" might conjecture that the upper bound is tight. Unfortunately, we have nothing else to say.

3. Symmetric Versions of Zero-Error Capacities

Theorem 1 is a by-product of our interest in a new class of problems in information theory. In this last part of our paper, we will try to explain this connection. No new results will be proved here. Yet, the interested reader will find a whole range of new combinatorial problems along with some possible ways of further improving on the lower bound to $N(n, 3)$.

Sperner's problem [1] is a good, and in fact, the simplest example of what we have in mind. Take the primitive fact that there are 2^n different binary sequences of length n. We want to "symmetrize" the concept of "different." Two binary sequences, $\mathbf{x}$ and $\mathbf{x}'$ of length n are different if for the ordered pair (0, 1) there is a coordinate i such that either

$$(x_i, x_i') = (0, 1) \qquad \text{or} \qquad (x_i', x_i) = (0, 1).$$

The corresponding symmetrized condition is this: two binary sequences of length n, $\mathbf{x}$ and $\mathbf{x}'$, are symmetrically different if for the ordered pair (0, 1) there is a coordinate i such that

$$(x_i, x_i') = (0, 1).$$

From our present point of view, the main content of Sperner's theorem [1] is that the largest subset of $\{0, 1\}^n$, any two elements of which are symmetrically different, has a cardinality c_n for which

$$\lim_{n \to \infty} \frac{1}{n} \log c_n = 1,$$

or else the symmetrization of the condition to be "different" does not affect the asymptotics of the largest subset satisfying the condition. In order to let the reader understand how information theory enters the picture as a unifying force behind our scattered problems, we would like to stress that both the problem of our Theorem 1 and the qualitative independence problem of Theorem 2 are symmetrized versions of different instances of the same information theory problem introduced in [3]. We will state this problem in its full generality, but we will only hint at its information-theoretic interpretation. The (hopefully interested) reader might consult [3] for more details.

Let us be given a finite set $\mathscr{B}$ and a family $\mathscr{G}$ of graphs with vertex set $\mathscr{B}$, each. We will say that the subset $\mathscr{C} \subset \mathscr{B}^n$ is $\mathscr{G}$-separated if for every pair $\mathbf{x} \in \mathscr{C}$, $\mathbf{x}' \in \mathscr{C}$ and *every* graph $G \in \mathscr{G}$ there is a coordinate i such that

$$(x_i, x_i') \in E(G).$$

Here $E(G)$ is the set of edges of G. Let us denote by $N(n, \mathscr{G})$ the largest cardinality of any subset $\mathscr{C} \subset \mathscr{C}$ that is $\mathscr{G}$-separated. In [3], we have treated the asymptotics

$$\lim_{n \to \infty} \sup \frac{1}{n} \log N(n, \mathscr{G}) \tag{18}$$

at great length, derived a general upper bound, and determined the limit in some interesting special cases.

The limit (18) is called the zero-error capacity of the compound channel $\mathscr{G}$. Shannon's famous graph capacity problem amounts to finding it in a special case. In fact, according to Shannon, if $\mathscr{G}$ consists of a single graph G, then $N(n, G)$ is the largest cardinality of a code of length n that can be used for error-free communication over a noisy channel the error pattern of which is described by G. In such a description, the vertices of G stand for the input letters of the channel, and two vertices are connected by an edge if the corresponding input letters cannot result in the same output with positive probability. Thus $N(n, \mathscr{G})$ is the natural generalization of this concept to the case when the probabilistic description (and hence the graph) of the channel is not known. Rather, the users of the channel are informed that the channel is an unknown member of a given class. Every member of this class is represented by a graph in $\mathscr{G}$. Here we introduce the "symmetric version" of this problem.

DEFINITION. Let us be given a finite set $\mathscr{B}$ and a family $\mathscr{G}$ of graphs with vertex set $\mathscr{B}$, each. We will say that the set $\mathscr{C} \subset \mathscr{B}^n$ is symmetrically $\mathscr{G}$-separated if for every pair $\mathbf{x} \in \mathscr{C}$, $\mathbf{x}' \in \mathscr{C}$ and every graph $G \in \mathscr{G}$ there are two coordinates, i and j, such that for the ordered pairs

$$(x_i, x_i') = (x_j', x_j) \in E(G),$$

where $E(G)$ is the set of edges of G. We insist that the equality is between the ordered pairs of vertices. Let us denote by $N_\sigma(n, \mathscr{G})$ the largest cardinality of any subset $\mathscr{C} \subset \mathscr{B}^n$ such that $\mathscr{C}$ is symmetrically $\mathscr{G}$-separated.

Problem 4. What is

$$\limsup_{n \to \infty} \frac{1}{n} \log N_\sigma(n, \mathscr{G})?$$

Problem 5. When does the lim sup of Problem 4 equal that in (18)? Is this true for every class of graphs $\mathscr{G}$?

We do not dare to conjecture it is. At any rate, Theorem 1 is an example when the two coincide. It is quite obvious that

Remark

$$\limsup_{n \to \infty} \frac{1}{n} \log N_\sigma(n, \mathscr{G}) \leqslant \limsup_{n \to \infty} \frac{1}{n} \log N(n, \mathscr{G}). \tag{19}$$

Using this trivial observation, the upper bound in Theorem 1 follows immediately from our Lemma 1 in [3]. The same is true for (12) in

Corollary PT. These upper bounds turn out to be trivial from an information-theoretic point of view. Our inability to treat $N_\sigma(n, \mathscr{G})$ separately from $N(n, \mathscr{G})$ at least in the asymptotic sense is the more disconcerting.

We will complete this paper by discussing at some length the non-symmetric analogues of the problems treated herein. In all these cases we have $\mathscr{B} = \{0, 1, 2\}$. For the problem of Theorem 1, $\mathscr{G}$ consists of two single-edge graphs on the ternary vertex set $\mathscr{B}$ and the fact that

$$\lim_{n \to \infty} \frac{1}{n} \log N(n, \mathscr{G}) = \log \frac{1+\sqrt{5}}{2}$$

is part of Proposition 3 in [3]. Hence our Theorem 1 can be interpreted as saying that in this particular case we have equality in (19).

Much less is known about the problem of Theorem 2, qualitative independence. The non-symmetric analogue of this is obtained by setting $\mathscr{B} = \{0, 1, 2\}$ and considering $\mathscr{G}$ as the family consisting of the three different single-edge graphs on $\mathscr{C}$. It is stated at the end of [3] that in this case

$$0.61 \leqslant \lim_{n \to \infty} \frac{1}{n} \log N(n, \mathscr{G}) \leqslant \frac{2}{3}.$$

Hence, if by some means we could prove that also in this case (19) is tight, this would lead to a considerable improvement on our bound of Theorem 2. Another possibility of such an improvement could be obtained by looking at the non-symmetric problems with a known solution.

Once again, set $\mathscr{B} = \{0, 1, 2\}$ and let $\mathscr{G}$ consist of the three different graphs on $\mathscr{B}$ that have two edges each. As a special case of Proposition 2 in [3] we obtain

$$\lim_{n \to \infty} \frac{1}{n} \log N(n, \mathscr{G}) = h\left(\frac{1}{3}\right).$$

If we could prove that (19) is tight in this case, this would imply that

$$\liminf_{n \to \infty} \frac{1}{n} \log N(n, 3) \geqslant \frac{1}{2} h\left(\frac{1}{3}\right) \sim 0.459.$$

(Although this needs some explanation, we omit the details.)

Likewise, if $\mathscr{B} = \{0, 1,2\}$ and $\mathscr{G}$ consists of a single graph, the complete graph on three vertices, then, clearly,

$$\lim_{n \to \infty} \frac{1}{n} \log N(n, \mathscr{G}) = \log 3.$$

Were (19) tight in this case, it would mean that

$$\liminf_{n \to \infty} \frac{1}{n} \log N(n, 3) \geqslant \tfrac{1}{3} \log 3 \sim 0.528.$$

Hence the symmetrized versions of zero-error capacities are useful to deal with well-known problems in extremal set theory. We believe, however, that they are also interesting on their own. It is quite astonishing that even in the simple case of a single complete graph these "symmetrical capacities" seem to be intractable by the methods we have at hand.

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Note added in proof. Problem 2 was settled by Ahlswede and Cai. Contrary to our expectation this does not solve Problem 1. Recently Gargano, Körner, and Vaccaro solved Problem 3 showing that the upper bound given was tight. They also proved equality in (12).

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