

# Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities

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## **Abstract**

We prove that the edges of a self-complementary graph and its complement can be oriented in such a way that they remain isomorphic as digraphs and their union is a transitive tournament. This result is used to explore the relation between the Shannon and Sperner capacity of certain graphs. In particular, using results of Lovász, we show that the maximum Sperner capacity over all orientations of the edges of a vertex-transitive self-complementary graph equals its Shannon capacity.

# 1 Introduction

The Shannon capacity  $C(G)$  of a graph  $G$  was defined by Shannon in [16] (see also [13, 2]). It is easy to determine for many graphs, highly non-trivial but known for some others and not even known for another many. Sperner capacity is a generalization of this notion for directed graphs given by Gargano, Körner, and Vaccaro [8]. The motivation of this generalization was the applicability of the new concept in extremal set theory that was carried out quite successfully in [9] and [10]. In this paper we are dealing with the connections of these values, the Shannon and Sperner capacities of a graph. Let us give the definitions first.

**Definition 1** *Let  $G$  be a directed graph on vertex set  $V$ . The  $t^{\text{th}}$  power of  $G$  is defined to be the directed graph  $G^t$  on vertex set  $V^t = \{\mathbf{x} = (x_1 \dots x_t) : x_i \in V\}$  with edge set*

$$E(G^t) = \{(\mathbf{x}, \mathbf{y}) : \exists i (x_i, y_i) \in E(G)\}.$$

Notice that  $G^t$  may contain edges in both directions between two vertices even if such a pair of edges is not present in  $G$ .

**Definition 2** *For a directed graph  $G$  let  $tr(G)$  denote the size (number of vertices) of the largest transitive tournament that appears as a subgraph of  $G$ . The (logarithmic) Sperner capacity of a digraph  $G$  is*

$$\Sigma(G) = \lim_{t \rightarrow \infty} \frac{1}{t} \log tr(G^t).$$

All logarithms in the paper are on base 2.

The Shannon capacity of an undirected graph  $G$  can be defined as the following special case of Sperner capacity. Let us call a graph symmetrically directed if for each of its edges it also contains the edge going opposite way between the same two endpoints. In what follows we often identify an undirected graph  $G$  with the symmetrically directed graph that has edges (in both directions) between the same endpoints as  $G$  has. This digraph is called the symmetrically directed equivalent of  $G$ . Note, that powers of a symmetrically directed graph are also symmetrically directed, hence they can be considered as undirected graphs, as well.

**Definition 3** *The Shannon capacity  $C(G)$  of  $G$  is the Sperner capacity of its symmetrically directed equivalent.*

Notice that writing undirected edges instead of directed ones in Definition 1 and the clique number  $\omega(G^t)$  in place of  $tr(G^t)$  in Definition 2 we get  $C(G)$  in place of  $\Sigma(G)$ . We also remark that originally the definition of  $\Sigma(G)$  was formulated in a different (but equivalent) way. The above definition already appears, e.g., in [6].

It should be clear from the definitions that the Sperner capacity of a digraph is always bounded from above by the Shannon capacity of the underlying undirected graph. The two values are not the same in general. It was first shown in [5] (cf. also [3] for a short and different proof) that the Sperner capacity of a cyclically oriented triangle is  $1 (= \log 2)$  while the Shannon capacity of  $K_n$  is  $\log n$  in general, i.e.,  $\log 3$  for a triangle.

For an undirected graph  $G$  let

$$D(G) = \max_{\hat{G}} \Sigma(\hat{G}),$$

where  $\hat{G}$  stands for an oriented version of  $G$ , i.e., the maximum is taken over all oriented graphs  $\hat{G}$  containing exactly one oriented edge for each edge of  $G$ . Clearly  $D(G) \leq C(G)$  holds. Our main concern is the question whether this inequality is an equality or not.

## 2 $D(G)$ versus $C(G)$

It was already proved by Shannon that  $C(G)$  satisfies  $\log \omega(G) \leq C(G) \leq \log \chi(G)$  where  $\chi(G)$  is the chromatic number of the graph  $G$ . (In fact, he proved more, namely, though in different terms, that the logarithm of the fractional chromatic number of  $G$  is also an upper bound for  $C(G)$ . For further details of this, see [12]. We also remark that Shannon and many authors following him used a complementary language, i.e., defined  $C(\bar{G})$  as we defined  $C(G)$ . The two approaches are equivalent, our reason to break the tradition is our need to orient edges that would become non-edges in the original language.) It is easy to observe that  $tr(G^t) \geq (tr(G))^t$ , thus  $\Sigma(G) \geq \log tr(G)$  always holds. (Shannon's  $\log \omega(G) \leq C(G)$  is a special case of this.) This inequality is not an equality in general even among tournaments as was shown recently by Alon [1]. On the other hand, a clique of an undirected graph can always be oriented in an acyclic way, thus giving an induced transitive tournament. Therefore the previous inequalities imply that if  $\chi(G) = \omega(G)$  for  $G$  then  $D(G) = C(G)$  also holds. In particular, this happens for all perfect graphs.

The smallest graph for which Shannon was not able to determine the value of its capacity is  $C_5$ , the chordless cycle on five points. He observed that  $\omega(C_5^2) \geq 5 > (\omega(C_5))^2 = 4$  implying  $C(C_5) \geq \frac{1}{2} \log 5$ . The theorem that this lower bound is sharp was proven by Lovász in his celebrated paper [13]. Lovász proved there the more general result that any vertex-transitive self-complementary graph on  $n$  points has Shannon capacity  $\frac{1}{2} \log n$ .

It was already observed in [7] (see Proposition 4 of quoted paper) that  $C_5$  has an orientation for which  $C_5^2$  contains an acyclically oriented clique on five points. This implies  $D(C_5) \geq \frac{1}{2} \log 5$  and thus by Lovász' result  $D(C_5) = C(C_5)$ . It is worth mentioning that this orientation of  $C_5$  is unique (up to isomorphism); for all other orientations the Sperner capacity is strictly smaller. This can be shown using the methods of [5] and [3], as it was shown to us by Rob Calderbank [4].

The main result of this paper is a generalization of the above observation, showing that for any self-complementary graph  $G$  on  $n$  points the edges can be oriented in such a way that  $G^2$  contains a transitively oriented clique of size  $n$ . By the above mentioned result of Lovász this will immediately imply  $D(G) = C(G)$  for all vertex-transitive self-complementary graphs. Another implication, due to the work of Alon and Orlitsky [2], is that  $D(G)$  can be exponentially larger than  $\log \omega(G)$ .

The core of our result is a theorem about self-complementary graphs that we think to be interesting in itself. This is the topic of the next section.

### 3 Self-complementary graphs

A graph  $G = (V, E)$  is self-complementary if there exists a complementing permutation of the elements of  $V$ . That is, there exists a bijection  $\tau: V \mapsto V$  with the property that  $\forall v \neq w \in V \{v, w\} \in E \iff \{\tau(v), \tau(w)\} \notin E$ . A characterization of self-complementary graphs can be found in [14] or [15], cf. also [11].

Let  $G$  be a self-complementary graph with complementing permutation  $\tau$ . Then the set  $\{(v, \tau(v)) : v \in V\}$ , or equivalently, the set  $\{(\tau(v), v) : v \in V\}$  of pairs (two-length sequences) induces a clique of size  $|V| = n$  in  $G^2$ . Using these two-length sequences as building blocks, we can find cliques of size  $n^{\frac{t}{2}}$  in  $G^t$  for every even  $t$ . This shows  $C(G) \geq \frac{1}{2} \log n$  and by Lovász' result this is sharp if  $G$  has the additional property of being vertex-transitive. If we could orient the edges of  $G$  in such a way that for the so obtained oriented graph  $\hat{G}$  at least one of the cliques of the above type in  $\hat{G}^2$  would become a transitive tournament then we would have  $D(G) \geq \frac{1}{2} \log n$  and thus  $D(G) = C(G)$  for vertex-transitive self-complementary graphs. Therefore we seek for such orientations. The following theorem will imply that this kind of orientation always exists. The proof also gives a construction.

If  $\rho$  is a linear order of the elements of the set  $\{1, \dots, n\}$  then  $\rho(x)$  denotes the element standing on the  $x^{\text{th}}$  position in this linear order. Thus  $\rho^{-1}(i)$  is the position where element  $i$  can be found. We say that  $j$  is to the right of  $i$  (according to  $\rho$ ) iff  $\rho^{-1}(i) < \rho^{-1}(j)$ .

**Theorem 1** *Let  $G = (V(G), E)$  be a self-complementary graph on  $V(G) = \{1, 2, \dots, n\}$  with complementing permutation  $\tau$ . Then there exists a linear order  $\sigma$  on  $V(G)$  such that if  $\{i, j\} \notin E$  and  $\sigma^{-1}(i) < \sigma^{-1}(j)$ , then  $\sigma^{-1}(\tau^{-1}(i)) < \sigma^{-1}(\tau^{-1}(j))$ .*

**Proof.** The essential part of the argument concerns the case when  $\tau$  contains only one cycle, the remaining cases can be reduced to this easily. So assume first that  $\tau$  consists of only one cycle. By results in [14], [15] this implies that  $n$  should be even, but this will not be exploited here.

We may assume without loss of generality that  $\tau = (123 \dots n)$  and that  $\{1, 2\}$  is an edge of  $G$ . An algorithm will be given, that starting from the identity order, successively

rearranges the terms thus generating the linear order  $\sigma$ . We give the formal description of this algorithm and explain it afterwards.

**let**  $\sigma_0 = \text{id}, k = 0,$

**general:**

Let  $\sigma_k(n) = m$  and  $\sigma_k^{-1}(m+1) = i,$

where  $m+1$  is 1 in case  $m = n$ , that is in the first step.

$A = \{r: i < r \text{ and } \{\sigma_k(r), \sigma_k(n)\} \in E\}$

**if**  $A = \emptyset$  **then goto** 'end'

$j = \min A$

$\sigma_{k+1}(s) = \sigma_k(s)$  if  $s < j$ ,  $\sigma_{k+1}(j) = m$  and  $\sigma_{k+1}(s) = \sigma_k(s-1)$  for  $j < s \leq n$

$k := k + 1$

**goto** 'general'

**end:**

$\sigma = \sigma_k$

**stop**

That is, in the general step when  $m$  stands on the last (rightmost) position in  $\sigma_k$ , we check whether  $m$  has a neighbour to the right of  $m+1$ . If there is one, then  $m$  is inserted just in front of its leftmost neighbour which is to the right of  $m+1$ . This step is repeated until, finally, all neighbours of the currently last element  $m$  are to the left of the (previously inserted) element  $m+1$ . Since the number of elements to the right of the previously inserted element is decreasing at every step, the algorithm surely terminates. (As an example, see the graph on Figure 1. In the first run of 'general' 8 is inserted in front of its leftmost neighbour, which is 3, resulting in the  $\sigma_1$ -sequence 1,2,8,3,4,5,6,7. In the next run 7 is inserted in front of 4, its leftmost neighbour to the right of 8 in  $\sigma_1$ . Finally, 6 is inserted in front of 5, thus the resulting  $\sigma$ -sequence is 1,2,8,3,7,4,6,5 for this graph.)

We have to prove that the linear order we obtain satisfies the requirements. Assume that  $\sigma(n) = m$ , then  $\sigma$  can be viewed as a merge of the two sequences  $1, 2, \dots, m$  and  $n, n-1, \dots, m+1$ . Furthermore, if  $\sigma^{-1}(m+1) = i$ , then  $\{\sigma(j), m\} \notin E$  for  $i < j < n$ , in other words,  $m$  has no neighbour between  $m+1$  and itself. We have to prove that if  $\{i, j\} \notin E$  and  $j$  is to the right of  $i$ , then  $\tau^{-1}(j)$  is also to the right of  $\tau^{-1}(i)$ . Note, that  $\tau^{-1}(b) = b-1$  for  $1 < b$  and  $\tau^{-1}(1) = n$ . The elements  $m+1, m+2, \dots, n$  moved by the algorithm are called *inserted*, while  $1, 2, \dots, m$  are called *original*. Let  $\{i, j\} \notin E$  and  $j$  be to the right of  $i$ . Four cases are distinguished according to which of  $i$  and  $j$  is inserted.

*Case 1.* Both  $i, j$  are original. Then  $i-1$  and  $j-1$  are also original, provided  $1 < i$ . In this case the order of  $i-1$  and  $j-1$  is the same as the order of  $i$  and  $j$ . If  $i = 1$ , then  $\tau^{-1}(1) = n$  and  $\{\tau^{-1}(1), \tau^{-1}(j)\} \in E$ , thus  $n$  is put to the left of  $j-1$ .

*Case 2.*  $i$  is inserted and  $j$  is original. Now  $i-1$  is to the right of  $i$ . Since  $\{i, j\} \notin E$   $i$  could not be inserted just in front of  $j$ , so  $j-1$  is also to the right of  $i$ . However,  $\{i-1, j-1\} \in E$ , that is  $i-1$  must be inserted, otherwise  $i-1$  would have remained as  $\sigma(n)$  and then it must not have any connection to the right of  $i$ . Furthermore it had to be inserted somewhere before  $j-1$ , thus  $\tau^{-1}(j)$  is also to the right of  $\tau^{-1}(i)$ .

*Case 3.*  $i$  is original and  $j$  is inserted. If  $i = 1$ , then  $\tau^{-1}(i) = n$  and  $\{n, j-1\} \in E$ , thus

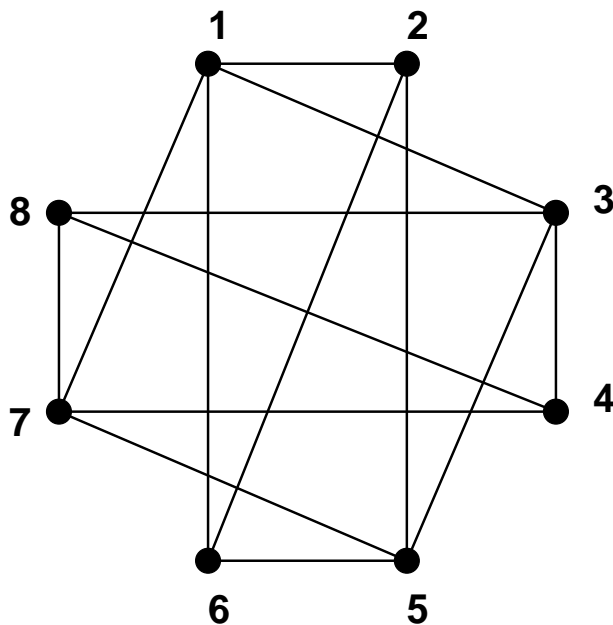


Figure 1.

$n$  is inserted before  $j - 1$ , i.e.,  $\tau^{-1}(j)$  is also to the right of  $\tau^{-1}(i)$ . Otherwise,  $i - 1$  is to the left of  $i$  and  $j - 1$  is to the right of  $j$ .

*Case 4.* Both  $i, j$  are inserted. In this case the larger of  $i$  and  $j$  is to the left of the other, thus  $i > j$ . Also, either both  $i - 1$  and  $j - 1$  are also inserted or one of them is inserted and the other is the rightmost element in the linear order obtained. In either case, the larger of the elements  $i - 1$  and  $j - 1$  is also to the left of the other.

This completes the proof for unicyclic  $\tau$ .

If  $\tau$  is not unicyclic then let the number of cycles in  $\tau$  be  $d$ . For  $d = 1$  the theorem is proved by the foregoing. If  $\tau$  decomposes into more than one cycle, then the subgraphs induced by the vertices in each individual cycle are all self-complementary, thus the above argument can be applied to them one by one. The resulting partial order on  $V(G)$ , which is the union of  $d$  total orders, can be extended to one total order by putting the cycles in order. Thus,  $\sigma^{-1}(i) < \sigma^{-1}(j)$  if  $i$  is in a cycle of  $\tau$  put before the cycle of  $j$ , or if  $i$  and  $j$  are in the same cycle of  $\tau$  then this is their order given by the algorithm applied to that cycle alone. Let  $\{i, j\} \notin E$  and  $\sigma^{-1}(i) < \sigma^{-1}(j)$ . If  $i, j$  are contained in the same cycle of  $\tau$ , then  $\sigma^{-1}(\tau^{-1}(i)) < \sigma^{-1}(\tau^{-1}(j))$  holds by the first part of the proof. On the other hand, if  $i$  and  $j$  are in different cycles, then one can use that  $i$  and  $\tau^{-1}(i)$ , furthermore  $j$  and  $\tau^{-1}(j)$  are in the same cycles, respectively, so their order according to  $\sigma$  is the same.  $\square$

Note that, taking the left-to-right ordering according to  $\sigma$ , each edge of  $G$  is mapped by  $\tau$  to an edge of  $\bar{G}$  of the same orientation. The union of  $G$  and  $\bar{G}$  is the transitive

tournament given by the order  $(\sigma(1), \sigma(2), \dots, \sigma(n))$ .

## 4 Consequences for Sperner capacity

An immediate implication of Theorem 1 is a lower bound on the Sperner capacity of appropriately oriented self-complementary graphs.

**Corollary 1** *If  $G$  is a self-complementary graph on  $n$  vertices then  $D(G) \geq \frac{1}{2} \log n$ .*

**Proof.** Let the vertex set of  $G$  be  $V = \{1, \dots, n\}$  and a complementing permutation of these vertices be  $\tau$ . Let  $\sigma$  be the linear order on  $V$  satisfying the requirements of Theorem 1 and orient the edges of  $G$  according to  $\sigma$ , that is, the edge  $\{i, j\}$  is oriented from  $i$  towards  $j$  iff  $\sigma^{-1}(i) < \sigma^{-1}(j)$ . The resulting oriented graph is denoted by  $\hat{G}$ . Consider the subset  $U$  of the vertices of  $\hat{G}^2$  defined by

$$U = \{(i, \tau^{-1}(i)) : i = 1, \dots, n\}.$$

By the properties of  $\sigma$  if  $\{i, j\} \notin E(G)$  then  $(\tau^{-1}(i), \tau^{-1}(j)) \in E(\hat{G})$  iff  $\sigma^{-1}(i) < \sigma^{-1}(j)$ . Thus  $U$  induces a transitive tournament in  $\hat{G}^2$ , because each edge is oriented according to the  $\sigma$ -order of the first coordinate of the vertices. Therefore every even power  $\hat{G}^{2k}$  of  $\hat{G}$  contains a transitive tournament of size  $|U|^k = n^k$  implying  $\Sigma(\hat{G}) \geq \frac{1}{2} \log n$ . Since  $\Sigma(\hat{G})$  is a lower bound on  $D(G)$ , this proves  $D(G) \geq \frac{1}{2} \log n$ .  $\square$

We remark that the above given lower bound is not tight for all self-complementary graphs. It is easy to give, for example, self-complementary graphs on 8 vertices with clique number 4. If such a maximum clique of size 4 is oriented transitively in this graph then the Sperner capacity of the resulting oriented graph is at least  $\log 4 > \frac{1}{2} \log 8$ . (Since  $\log 3 > \frac{1}{2} \log 8$  the same argument applies also for the graph on Figure 1.) If, however, our graph is not only self complementary but also vertex-transitive, then by Lovász' results the above bound is tight. The next theorem formulates this statement.

**Theorem 2** *For a vertex-transitive self-complementary graph  $G$ , the value of  $D(G)$  equals the Shannon capacity of  $G$ .*

**Proof.** Lovász proved in [13] that for a vertex-transitive self-complementary graph  $G$  on  $n$  vertices  $C(G) \leq \frac{1}{2} \log n$ . This combined with Corollary 1 and the fact that  $D(G) \leq C(G)$  implies the statement.  $\square$

A consequence of Corollary 1 is that the Sperner capacity of a graph can be exponentially larger than the value implied by its clique number. This will follow by using the proof of the analogous result for Shannon capacity by Alon and Orlitsky [2]. First we quote a lemma of theirs (see as Lemma 3 of [2] on page 1282). We remark that [2] uses the complementary language.

**Lemma AO:** For every integer  $n$  that is divisible by 4, there exists a self-complementary graph  $G$  on  $n$  vertices with  $\omega(G) < 2 \lceil \log n \rceil$ .



**Corollary 2** *For every integer  $n$  divisible by 4, there exists a graph  $G$  on  $n$  vertices such that  $D(G) > 2^{\log(\omega(G)-2)-2}$ .*

**Proof.** Let  $n$  be an integer divisible by 4 and  $G$  be the graph constructed by Alon and Orlicsky proving Lemma AO. Since this graph is self-complementary, Corollary 1 implies  $D(G) \geq \frac{1}{2} \log n > 2^{\log(\omega(G)-2)-2}$ .  $\square$

We remark that the graphs in the proof of Lemma AO all have complementing permutations containing cycles of length four only. Thus, the proof of Corollary 2 does not require the full generality of Theorem 1.

## 5 Further remarks

According to the relation of  $D(G)$  and  $C(G)$ , we can distinguish among the following three classes of (undirected) graphs. The first class consists of those graphs every oriented version of which has its Sperner capacity equal to the Shannon capacity of the graph. The second class contains the graphs  $G$  for which this is not true but still  $D(G) = C(G)$  holds. The third class is the class of those graphs for which  $D(G) < C(G)$ .

Since every graph containing at least one directed edge has its Sperner capacity at least 1, all graphs with Shannon capacity 1 belong to the first class. Using Shannon's observation that  $\log \omega(G) \leq C(G) \leq \log \chi(G)$  one knows that all bipartite graphs have this property. The same chain of inequalities imply that all graphs with  $\chi(G) = \omega(G)$  belong to one of the first two classes. This can be seen by orienting a largest clique transitively. Clearly, the possibility of such orientations shows  $D(G) = C(G)$  for any graph  $G$  having  $C(G) = \log \omega(G)$  even if the chromatic number and the clique number of  $G$  are different. Examples of such graphs are the complements of Kneser graphs of appropriate parameters, as it is shown by Theorem 13 of Lovász [13].

It is not clear whether a graph  $G$  with  $C(G) > 1$  can belong to the first class. There is no graph identified to belong to the third class and it is not clear at all whether class three is empty or not. (This question is also mentioned in [6].) The main novelty of this paper is that many graphs that have a gap between their capacity values and the logarithm of their clique number also satisfy the  $D(G) = C(G)$  equality. This may support the guess that perhaps this equality always holds but we have too little evidence to state this as a conjecture.

Finally, let us express our feeling that Theorem 1, though motivated completely by the capacity questions exposed here, might have rather different applications, too.

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