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Discrete Mathematics 185 (1998) 279–285

DISCRETE MATHEMATICS

Note

Intersecting set systems and graphic matroids

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Received 14 November 1996; revised 28 July 1997; accepted 11 August 1997

Abstract

Two simple proofs are given to an earlier partial result about an extremal set theoretic conjecture of Chung, Frankl, Graham, Shearer and Faudree, Schelp, Sós, respectively. The statement is slightly strengthened within a matroid theoretic framework. The first proof relies on results from matroid theory, while the second is based on an explicit construction providing an elementary proof. © 1998 Elsevier Science B.V. All rights reserved

Keywords: Intersection theorems; Graphic matroids

1. Introduction

Our starting point is the following conjecture due to Chung, Frankl, Graham, and Shearer [2], also conjectured by Faudree et al. [3].

Let $[n]$ denote the set $\{1,2,\ldots,n\}$ and let $\binom{[n]}{t}$ denote the collection of all t-subsets of $[n]$.

Definition 1. Let $\mathscr{B}\subset\binom{[n]}{t}$. We say that a family $\mathscr{F}\subset 2^{[n]}$ is *intersecting over* $\mathscr{B}$ if for every F,G in $\mathscr{F}$ there exists a $B\in\mathscr{B}$ such that $B\subseteq F\cap G$. The maximum size of an intersecting family over $\mathscr{B}$ is denoted by $v(\mathscr{B})$.

It is clear that $v(\mathscr{B})\geqslant 2^{n-t}$, since this is the number of sets in $2^{[n]}$ containing a fixed t-set.

Conjecture 2 (Chung et al. [2] and Faudree et al. [3]). *Let $\mathscr{B}=\mathscr{B}_n(X)$ be the collection of cyclic translates of a t-subset $X=\{a_1,a_2,\ldots,a_t\}$ of $[n]$, i.e., the collection of*

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[1] Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) Nos. 4264, T016386 and F023442.

PII S0012-365X(97)00218-5

t-sets of form $X+i=\{a_1+i, a_2+i, \dots, a_t+i\}$, *where the addition is intended modulo* n. *Then*

$$v(\mathscr{B}_n(X))=2^{n-t}.$$

Graham has offered \$100 for a proof of this conjecture [5]. It was shown to be true for arbitrary X when $t=1,2$, furthermore for the particular (consecutive) set $X=\{1,2,\dots,t\}$ with arbitrary t by Chung et al. [2], also by Faudree et al. [3] and later by Griggs and Walker, too [6]. The case $t=3$ and X arbitrary was also proven, see [4].

Griggs and Walker [6] used the observation that $v(\mathscr{B})$ is equal to the conjectured value provided there exists a suitable partition of $2^{[n]}$ into 2^{n-t} classes such that intersection of two subsets from the same class does not contain any member of $\mathscr{B}$. They call such a partition class an *anti-cluster for* $\mathscr{B}$. They show that a partition of $2^{[n]}$ into 2^{n-t} anti-clusters for $\mathscr{B}$ exists provided a $t\times n$ $(0,1)$-matrix exists with the property that for every $B\in\mathscr{B}$ the columns indexed by the elements of B are linearly independent over $GF(2)$. That is, though they did not formulate this way, they proved the following theorem.

Theorem 3 (Griggs and Walker [6]). *Let* $\mathscr{B}$ *be a collection of t-subsets of* $[n]$. *If a* binary matroid $\mathscr{M}$ *of rank t exists such that every member of* $\mathscr{B}$ *is a base of* $\mathscr{M}$, *then* $v(\mathscr{B})=2^{n-t}$.

Griggs and Walker proved that for the *consecutive* case a suitable (0,1)-matrix exists, hence by Theorem 3 the conjecture holds. They also observed that for arbitrary X there exists a suitable matrix for the *ordinary* (i.e., not cyclic) translates of X.

In the present note we show that if $\mathscr{B}$ is the collection of cyclic translates of the consecutive t-set, then there exists a *graphic matroid* $\mathscr{M}$, such that all members of $\mathscr{B}$ are bases of $\mathscr{M}$. This strengthens Griggs and Walker's result by the well known fact that every graphic matroid is binary, but not all binary matroids are graphic.

We will assume some familiarity with matroid theory throughout the paper. For definitions and basic theorems one may consult Welsh's book [8].

2. The constructions

In this section we present two constructions of graphic matroids so that cyclic translates of a consecutive t-subset X of $[n]$ are all included amongst the bases.

The first construction remains implicit by using existing results from matroid theory. The second one is explicit and completely elementary. We start by formalizing the statement.

Theorem 4. *Let* $\mathscr{B}=\mathscr{B}_n(X)$ *be the collection of cyclic translates of* $X=\{1,2,\dots,t\}$. *Then there exists a* graphic matroid $\mathscr{M}$ *such that all members of* $\mathscr{B}$ *are bases of* $\mathscr{M}$.

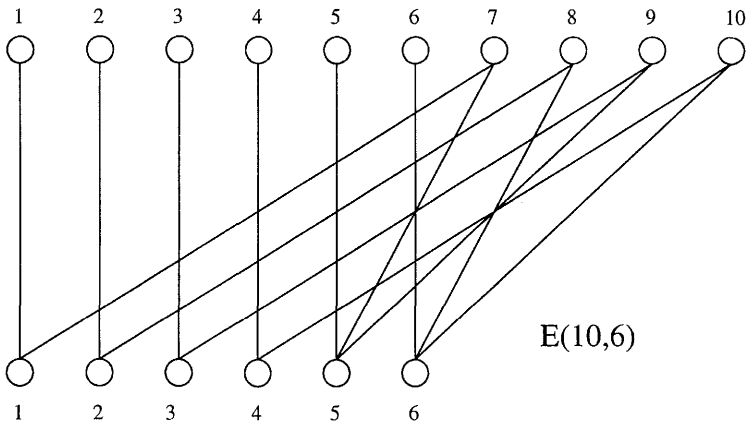

Fig. 1.

Our first proof is based on the following theorem of Edmonds (unpublished, see [1]) and a theorem of Sousa and Welsh [7].

Theorem 5 (Edmonds [1]). *A transversal matroid is binary iff it possesses a presentation by a forest.*

Theorem 6 (Sousa and Welsh [7]). *A transversal matroid is binary iff it is graphic.*

The following definition is also needed.

Definition 7. Let n and t be natural numbers with $t \leqslant n$. The *Euclidean bipartite graph* $E(n,t)$ with parameters n and t is defined recursively. Let $n = st + r$, where $0 \leqslant r < t$. $E(n,t)$ has color classes $|A| = n$ and $|B| = t$, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_t\}$. The edge set of $E(n,t)$ consists of the matchings $\{(b_1, a_{it+1}), (b_2, a_{it+2}), \ldots, (b_t, a_{(i+1)t})\}$ for $i = 0, 1, \ldots, s-1$, together with the edge set of $E(t,r)$ put on the vertex set $B \cup \{a_{st+1}, a_{st+2}, \ldots, a_{st+r}\}$. If $r = 0$, then $E(n,t)$ consists only of the matchings.

The name of $E(n,t)$ is explained by the construction imitating the Euclidean Algorithm on n and t. $E(10,6)$ is shown in Fig. 1.

The two most important properties of $E(n,t)$ are collected in the next two lemmas.

Lemma 8. *$E(n,t)$ is a forest.*

Proof. We use induction. By removing the degree one vertices $\{a_1, a_2, \ldots, a_{st}\}$ and the edges incident to them, the remaining graph is $E(t,r)$ (or a set of independent vertices), and by induction that is a forest, so $E(n,t)$ is a forest, as well. □

Lemma 9. *Let T be a cyclically consecutive t-subset of A, i.e., $T = \{a_i, a_{i+1}, \ldots, a_{i+t-1}\}$ where the additions are taken modulo n. Then T can be matched into B.*

Proof. The statement is true for $i \leqslant (s-1)t+1$, by definition. If $(s-1)t+1 < i \leqslant n+1-t$, then by the construction the edges $\{(b_1, a_{st+1}), (b_2, a_{st+2}), \dots, (b_r, a_n)\}$ are all contained in $E(n,t)$, so a matching of T onto B can easily be constructed.

If $n+1-t < i$, then T can be written for some $1 \leqslant j < T$ in the form $T = \{a_1, a_2, \dots, a_j, a_{n-t+j+1}, \dots, a_n\}$. We use induction on $n+t$ to conclude, applying the induction hypothesis to $E(t,r)$, that every modulo t cyclically consecutive r subset of B can be matched to $\{a_{st+1}, a_{st+2}, \dots, a_n\}$. In other words, $\{a_{st+1}, a_{st+2}, \dots, a_n\}$ can be matched to any cyclically consecutive r-set of B. $\{a_1, a_2, \dots, a_j\}$ are of degree one, thus they have to be matched to $\{b_1, \dots, b_j\}$. If $t-j > r$, then $\{a_{n-t+j+1}, \dots, a_{n-r}\}$ are matched onto $\{b_{j+r+1}, \dots, b_t\}$ by the construction of $E(n,t)$, and by the induction hypothesis, the last r vertices of A can be matched onto the consecutive set $\{b_{j+1}, \dots, b_{j+r}\}$.

On the other hand, if $r \geqslant t-j$, the only thing to show is that the last $t-j$ points of A can be matched onto the last $t-j$ vertices of B. The induction hypothesis on $E(t,r)$ yields that $\{a_{st+1}, a_{st+2}, \dots, a_n\}$ can be matched onto the cyclically consecutive r set $\{b_1, \dots, b_{r-t+j}, b_{j+1}, \dots, b_t\}$. However, none of $\{a_{n-t+j+1}, \dots, a_n\}$ is connected to any of $\{b_1, \dots, b_{r-t+j}\}$ (recall that $b_1, b_2, \dots, b_r$ are of degree *one* in $E(t,r)$), thus the matching above must match the last $t-j$ points of A onto the last $t-j$ vertices of B. □

First Proof of Theorem 4. We claim that the transversal matroid defined by $E(n,t)$ satisfies the requirements. Indeed, all cyclic translates of $X = \{1, 2, \dots, t\}$ are bases by Lemma 9. Since $E(n,t)$ is a forest by Lemma 8, Theorems 5 and 6 imply that its transversal matroid is graphic. □

Now we give a second proof of Theorem 4. This proof explicitly constructs a graph defining the graphic matroid we need. The construction is completely elementary and easy to check. Its main merit, we think, is the finding of the graph.

Second Proof of Theorem 4. Let $n = at + k$ with $0 \leqslant k < t$. We construct a simple graph G of $t+1$ vertices and $t+k$ edges, numbered $1, 2, \dots, t, t+1, \dots, t+k$. We color the elements of $[n]$ with $1, 2, \dots, t, t+1, \dots, t+k$. The collection of elements of color i will correspond to a set of parallel edges in place of the edge numbered i in our graph thus defining the graphic matroid.

Let the color of an element $j \in [n]$ be its remainder modulo t, except for those divisible by t where we write t instead of 0. This rule is for $1 \leqslant j \leqslant at$, the remaining k elements of $[n]$ will be colored by $t+1, t+2, \dots, t+k$, respectively. It is immediate that any two elements that are contained in some consecutive t-block receive distinct colors.

Now we construct our graph G. Let g.c.d.$(k,t) = d$.

Let the vertices of G be $v_0, v_1, \dots, v_t$ that form a path. The edges of the path are numbered with numbers from $[t]$. Denoting the number of edge $v_i v_j$ by $c(v_i v_j)$, we start with $c(v_0 v_1) = d$, then increase the number by k mod(t) as long as it does not

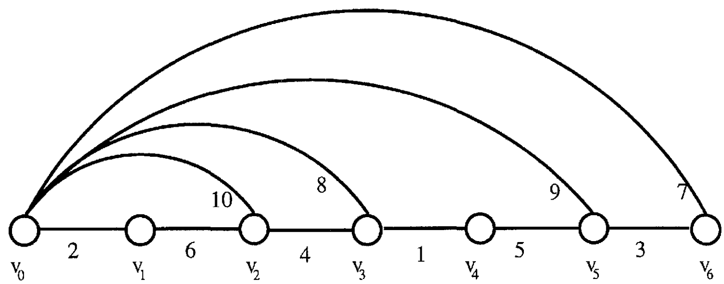

Fig. 2.

repeat. Then we continue with $d-1$, and at the next repetition with $d-2$ and so on. This results in the following pattern:

$$d, d+k, d+2k, \ldots, d-k,\ d-1, d-1+k, \ldots,\ d-2, \ldots,\ \ldots 1,$$
$$1+k, \ldots, 1+t-k,$$

where the numbers are mod(t).

The vertex v_i is called an *overflow point* if $i=t$, or $i \leqslant t$ and $c(v_{i-1}v_i) > c(v_i v_{i+1})$. The remaining k edges of G, that is those not contained in the previous path, will be all of the form $v_0 v_i$ where v_i is an overflow point. Now we have to label these new edges by the numbers $t+1, t+2, \ldots, t+k$. The number of edge $v_0 v_i$ is defined as follows. If $c(v_{i-1}v_i) = \alpha$, then $t+k \geqslant c(v_0 v_i) + k > t$. Now define $c(v_0 v_i)$ to be $k+\alpha$. By the previous inequality this value is indeed in the required range of numbers. All "overflow edges" receive pairwise different numbers, because the edges of the path receive mutually distinct numbers. There are k overflow points, because they occur exactly after the edges numbered $t-k+1, t-k+2, \ldots, t$.

Thus we have proved the following:

Claim 1. *There are k overflow points and the edges connecting them to v_0 all get different numbers listing $t+1, t+2, \ldots, t+k$.*

We show an example for G when $n=10$ and $t=6$ in Fig. 2.

By this point we have already given our construction completely. What is left to show that it works. The graph has $t+1$ vertices, and it is connected, so any spanning tree (a basis of its cycle matroid) consists of t edges (elements). First observe that any consecutive cyclic translate U of X corresponds to the edges of the path according to the numbering, provided U does not contain any of $at+1, \ldots, at+k$. The coloring of those t-subsets that contain some of these elements matches some consecutive t-set of the following sequence that we call the *joining sequence*:

$$1, 2, \ldots, t, t+1, \ldots, t+k, 1, 2, \ldots, t.$$

Now we have to show that consecutive t-subsets of the joining-sequence also correspond to cycle-free subgraphs of G. The following observation will be useful. Let

$c(v_0v_i)=t+s$, $c(v_{i-1}v_i)=\alpha$ and $c(v_iv_{i+1})=\beta$. Then $t+s=\alpha+k$ and $\beta=\alpha+k-t$ or $\beta=\alpha+k-t-1$. This implies the following:

Claim 2. *The α on the right part of the joining sequence cannot be in a consecutive t-block of the joining sequence together with $t+s$. Similarly, the β on the left part cannot be in the same consecutive t-block as $t+s$.*

We say that a subset Z of the joining sequence *covers* a set of edges of G if the set of numbers of the edges is a subset of Z. It is immediate that any cycle of G contains at least one overflow edge v_0v_i. Suppose, that a consecutive t-block covers a cycle of G. If the cycle contains at least two overflow edges, say v_0v_i and v_0v_j, with $i<j$, then according to Claim 2, the consecutive t-block must contain $c(v_iv_{i+1})=\beta$ from the *right part* of the joining sequence, while $c(v_{j-1}v_j)=\alpha$ from the *left part.* Thus, there are adjacent edges on the v_iv_j arc of the cycle, such that the first one's number is contained from the right, and the next one's from the left part. (This already excludes the case $j=i+1$.) However, their difference is k or $k-1 \bmod t$, therefore their distance in the joining sequence is at least t, so they cannot be in a consecutive t-block. If the cycle contains only one overflow edge v_0v_i of number $t+s$, then by Claim 2 the consecutive t-block must contain $\alpha=c(v_{i-1}v_i)$ on the *left part* of the joining sequence. If there is an edge on the v_0v_i arc of the cycle whose number is contained from the right part of the joining sequence, then there is an adjacent pair of edges so that their number is contained from different parts of the joining sequence in the same order as before that leads to a contradiction in the same way, as above. On the other hand, if all edge numbers of arc v_0v_i are from the left part of the joining sequence, then we show there must be some $r\leqslant s$ among them. This will be enough for a contradiction since such an r from the left part cannot be together with $t+s$ in a consecutive t-sequence. By the definition of the numbering of overflow edges $t+s$ is congruent $\alpha \bmod d$ and so by k and t being multiples of d we must have some number r on the v_0v_i arc that is congruent $s \bmod d$ for which $r\leqslant s$ and we are done. Thus, we have proved:

Claim 3. *Edges of a cycle of G cannot be covered by a consecutive t-block of the joining sequence.*

This last Claim proves that our construction has the required property. □

3. Conclusion

Two constructions were given for the same problem. It is natural to ask whether they result in the same matroid or not. It is not too hard to see that, in general, they are different. Indeed, for the $n=10$, $t=6$ case the first construction gives a matroid, which contains *four* cycles of size three ($\{1,5,7\},\{2,6,8\},\{3,5,9\}$ and $\{4,6,10\}$), while the second construction contains *three* cycles of size three ($\{2,6,10\}, \{4,8,10\}$ and $\{3,7,9\}$).

On the other hand, by a theorem of Bondy [1], the graphic matroid of the second construction is also transversal.

One would hope that explicit constructions may help to prove some new special cases of Conjecture 2. Unfortunately, we did not succeed in this so far. A problem to be handled is that for the non-consecutive case the resulting matroid cannot always be graphic. Indeed, already for $t=3$ and $n=7$ one may need Fano's matroid (cf. also [4]), which is an excluded minor for graphic matroids.

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