Orientations making k-cycles cyclic

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Abstract

We show that the minimum number of orientations of the edges of the n-vertex complete graph having the property that every triangle is made cyclic in at least one of them is $\lceil \log_2(n-1) \rceil$. More generally, we also determine the minimum number of orientations of K_n such that at least one of them orients some specific k-cycles cyclically on every k-element subset of the vertex set. Though only formally related, the questions answered by these results were motivated by an analogous problem of Vera T. Sós concerning triangles and 3-edge-colorings.
Some variants of the problem are also considered.

1 Introduction

How many different sequences of a given length over the alphabet $\{0, 1, 2\}$ can be given so, that for any three (pairwise distinct) sequences there is a position where all of them are different, that is exactly one of them is 0, one is 1, and one is 2? Such a set of sequences we call *trifferent*, see [7]. Equivalently, what is the minimum length t such that there exist M distinct ternary sequences forming a trifferent set of sequences? This is a notoriously hard problem investigated as a special case of the *perfect hashing problem* as defined in [4, 8].

What happens if we want only some specified triples of our sequences being trifferent? Vera T. Sós asked such a question in 1991: How many different 3-colorings of the edges of the complete graph K_n should be given if for each triangle there is at least one of these colorings that makes it completely 3-colored. In [7] an explicit construction is given proving that this minimum number is at most $\lceil \log_2 n \rceil - 1$. The best lower bound we know is $\lceil \log_3(n-1) \rceil$ given by a simple local consideration: the ternary sequences defined by 3-colorings of the $n-1$ edges connected to any fixed vertex should all be different. There seems to be no reason to believe that such a simple lower bound obtained by a local argument would be sharp.

In this note we consider an analogous problem involving orientations in place of colorings. The question here is the minimum number of orientations of the edges of the complete graph K_n such that each triangle becomes cyclically oriented in at least one of them. A simple lower bound can be given here, too, by just looking at orientations locally: If the condition is satisfied then any pair of the $n-1$ edges attached to a fixed vertex should be oriented in opposite directions (when viewing them from the common vertex) in at least one of our orientations. This gives the lower bound $\lceil \log_2(n-1) \rceil$.

Somewhat surprisingly, in this case, this simple lower bound is sharp. In fact, even the more general statement of Theorem 1 below is true.

Definition 1 *Let $n \geq k \geq 3$ and K_n be the complete graph on vertex set $[n] = \{0, 1, \ldots, n-1\}$. A family of orientations of the edges of K_n is called* increasingly k-cycling *if for any k-subset $\{u_1, \ldots, u_k\}$ of the vertex set with $u_1 < u_2 < \ldots < u_k$, there exists an orientation in our family that makes the cycle $u_1 u_2 \ldots u_k u_1$ cyclically oriented.*

A family of orientations is called weakly k-cycling *if for any k-subset $\{w_1, ..., w_k\}$ of $[n]$ there is an orientation making some k-cycle consisting of the vertices $w_1, ..., w_k$ (in an arbitrary order) cyclically oriented.*
The minimum number of orientations in a weakly or increasingly k-cycling family for K_n is denoted by $w(n, k)$ and $t(n, k)$, respectively.

Theorem 1

$$w(n, k) = t(n, k) = \left\lceil \log_2 \frac{n-1}{k-2} \right\rceil . \tag{1}$$

Note that the following corollary immediately follows by setting $k = 3$. (It was proven as a predecessor of the general theorem by the first author in [5].) Note also that $t(n, 3) = w(n, 3)$ is trivial, because all cyclic triangles are "increasingly cyclic". We denote this common value by $t(n)$.

Corollary 1 ([5])
$$t(n) = \lceil \log_2(n-1) \rceil .$$

The next section is devoted to the proof of the above theorem, while in Section 3 we discuss some variants of the problem.

All logarithms are meant to be of base 2.

2 Proof of Theorem 1

2.1 Lower bound

First we prove the easy fact that the right hand side of (1) is a lower bound for $w(n, k)$ which obviously bounds $t(n, k)$ from below.
Assume we have $t' := w(n, k)$ orientations that satisfy the requirements. Fix a vertex u and consider the $n-1$ edges adjacent to u. Define a binary sequence of length t' for each of these edges as follows. Let the i^{th} bit of the sequence belonging to edge $\{u, v\}$ be 0 if the edge is oriented from u to v in the i^{th} orientation. If it is oriented from v to u then the corresponding bit is 1.

We claim that at most $k-2$ edges adjacent to u can get the same t'-length binary sequence this way. Indeed, if there were $k-1$ edges connecting u to $v_1, \dots, v_{k-1}$ and all oriented towards u or all oriented away from u then we could not have a cyclic k-cycle on vertices $u, v_1, \dots, v_{k-1}$. This is because that would need a pair of edges uv_i and uv_j such that one is oriented towards u and the other away from u. Thus we have at least $\left\lceil \frac{n-1}{k-2} \right\rceil$ different binary sequences, so the length of these should be at least $\left\lceil \log \frac{n-1}{k-2} \right\rceil$.

2.2 Upper bound

To prove that the right hand side of (1) is also an upper bound we give a construction.
We may assume that $n = 2^r(k-2) + 1$ for some positive integer r. (Otherwise we simply make the construction for the first such integer above n and then delete the superfluous points.)
Consider the vertices of K_n put around a cycle in the cyclic order $0, 1, \dots n-1, 0$. For $a, b \in \{0, 1, \dots, n-1\}, a \neq b$ let $d(a, b)$ denote $b - a - 1$ if $b > a$ and $n - 2 - (a - b - 1) = n + b - a - 1 = n - 2 - d(b, a)$ if $a > b$. Thus $d(a, b)$ is the number of vertices put on the cycle strictly between a and b when moving from a to b in the clockwise direction. For each ordered pair (i, j) of vertices we define the value $f(i, j) := \left\lfloor \frac{d(i,j)}{k-2} \right\rfloor$. (Note that we

write $f(i,j)$ in place of $f((i,j))$.) For each pair of vertices $i \neq j$ consider the binary form of $f(i,j)$. Since $d(i,j) \leq n-2 < 2^r(k-2)$ the number of binary digits of $f(i,j)$ is at most r. If it is less, than put as many 0's in front of this binary form as are missing for making it an r-digit binary sequence. We denote by $f_s(i,j)$ the sth bit in this extended binary form of $f(i,j)$. We attach an orientation to each bit, this gives $r = \log \frac{n-1}{k-2}$ orientations. We orient the edge $\{i,j\}$ from i to j in the ℓth orientation if $f_\ell(i,j) = 0$. This definition is meaningful since $d(i,j) + d(j,i) = n-2 = 2^r(k-2)-1$ implying $f(i,j) + f(j,i) = 2^r - 1$, thus $f_\ell(i,j) + f_\ell(j,i) = 1$ for all ℓ, that is $f_\ell(i,j) = 0$ holds if and only if $f_\ell(j,i) = 1$.

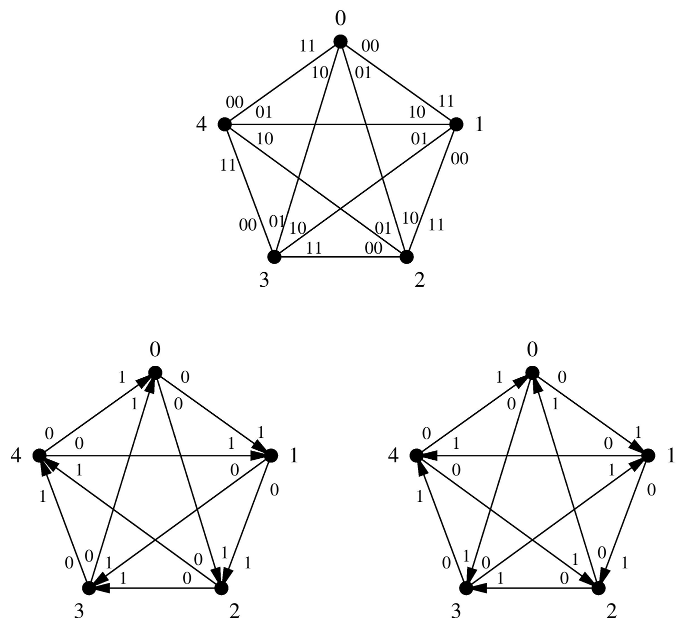

Figure 1: The construction for $n = 5, k = 3$

We have to prove now that this construction works. Let $u_1 < u_2 < \ldots < u_k$ be k distinct vertices. The k-cycle they form is well oriented in our ℓth orientation if $f_\ell(u_i, u_{i+1}) = 0$ for all $1 \leq i \leq k$, where the index $k+1$ is meant to be 1. To show that such an integer $\ell \in \{1, \ldots, r\}$ exists first note that $d(u_1,u_2) + d(u_2,u_3) + \ldots d(u_{k-1},u_k) + d(u_k,u_1) = n - k = 2^r(k-2) - (k-1)$. Thus we have $\sum_{1 \leq i \leq k} f(u_i,u_{i+1}) \leq \frac{\sum_{1 \leq i \leq k} d(u_i,u_{i+1})}{k-2} < 2^r - 1$. If there was no integer ℓ for which all $f_\ell(u_i,u_{i+1}) = 0$, then we would have at least one 1-digit at every position making $\frac{\sum_{1 \leq i < \leq k} d(u_i,u_{i+1})}{k-2} \geq \sum_{h=0}^{r-1} 2^h = 2^r - 1$ contradicting the previous formula. This contradiction proves the statement. $\square$

3 Variants and open problems

There are several ways to generalize or modify the problems treated above. Here we mention two of them and give some preliminary results.

3.1 Orienting simplices

It is customary to define the orientation of a simplex in the following way.

Definition 2 *An ordering of the vertices* $v_1, \ldots, v_r$ *of an* $(r-1)$*-dimensional simplex (a simplex with* r *vertices) determines an orientation of the simplex. Two orderings determine the same orientation if an even permutation of the vertices transforms one to the other. In particular, there are two possible orientations of a simplex, one belonging to odd, the other to even permutations of the indices of vertices. The orientation defined by the permutation* $v_{\sigma(1)} \ldots v_{\sigma(r)}$ *is denoted by* $\langle v_{\sigma(1)} \ldots v_{\sigma(r)} \rangle$. *The opposite orientation is denoted by* $(-1)\langle v_{\sigma(1)} \ldots v_{\sigma(r)} \rangle$.

Example: Let K be a 3-dimensional simplex with vertices $v_1, \ldots, v_4$. The orderings $v_2v_4v_1v_3$ and $v_3v_2v_1v_4$ belong to the same orientation (both 2413 and 3214 are odd permutations), while $v_4v_3v_2v_1$ determines the *opposite* orientation. So $\langle v_4v_3v_2v_1 \rangle = (-1)\langle v_3v_2v_1v_4 \rangle$

As usual, the $(d-1)$-dimensional faces of a d-dimensional simplex are called its *facets*.

Definition 3 *An orientation* $\langle v_1 \ldots v_r \rangle$ *of the simplex with vertices* $v_1 \ldots v_r$ induces *the orientation* $(-1)^i \langle v_1 \ldots v_{i-1}v_{i+1} \ldots v_r \rangle$ *on its facet* $v_1 \ldots v_{i-1}v_{i+1} \ldots v_r$. *Two facets* $v_1 \ldots v_{i-1}v_{i+1} \ldots v_r$ *and* $v_1 \ldots v_{j-1}v_{j+1} \ldots v_r$ *are oriented* consistently *if their orientations can be induced by the same orientation of the original simplex. In other words, orientations* $\langle v_1 \ldots v_{i-1}v_{i+1} \ldots v_r \rangle$ *and* $\langle v_1 \ldots v_{j-1}v_{j+1} \ldots v_r \rangle$ *are consistent if they induce opposite orientations on their common face* $v_1 \ldots v_{i-1}v_{i+1} \ldots v_{j-1}v_{j+1} \ldots v_r$.

Example: Let K be a 3-dimensional simplex with vertices $v_1, \ldots, v_4$. The orientations $\langle v_1v_2v_4 \rangle$ and $\langle v_2v_3v_4 \rangle$ are consistent, because the first induces orientation $(-1)\langle v_2v_4 \rangle$ while the second induces $(-1)^2\langle v_2v_4 \rangle = \langle v_2v_4 \rangle$ on the face v_2v_4 of K. Note that the above two orientations can be the induced orientations of the orientation $(-1)\langle v_1v_2v_3v_4 \rangle$.

Now the question generalizing the one answered by Corollary 1 is the following: What is the minimum number of orientations of the $(r-2)$-dimensional facets of the $(r-1)$-dimensional simplices defined by r-element subsets of $[n]$ if we want that for every such simplex there is at least one orientation where all its facets are oriented consistently with each other?
We will use the notation $t_r(n)$ for the number answering the above question. Note that the function $t(n)$ of Corollary 1 is just $t_3(n)$.
For $r = 4$ we can think about the cyclic orientations of triangles. A 4-subset (3-dimesional simplex) is "satisfied" if the 4 triangles it contains get cyclical orientations so that for all 6 pairs formed by these triangles, the common edge of the pair is traversed in the opposite direction by the given cyclic orientations of these neighboring triangles.

We cannot answer the previous question completely for $r > 3$ and give only some estimates. Let us call an orientation of all $(r-1)$-subsets of $[n]$ a *round*. (That is, it is a function $f : \binom{[n]}{r-1} \to \{+,-\}$.) Our problem is to determine the minimum number of rounds where for each r-subset there is a round where all its $(r-1)$-subsets are oriented consistently.

Proposition 1

$$\lceil \log(n-r+2) \rceil \le t_r(n) \le \left(\log \frac{2^{r-1}}{2^{r-1}-1} \right)^{-1} \log n + o(\log n).$$

Proof. The lower bound is a straightforward generalization of the lower bound in Corollary 1. Fix an $(r-2)$-element subset of $[n]$, we may assume it is $L := \{1, \ldots, r-2\}$. For an orientation of each $(r-1)$-element subset containing L we attach a $+$ or $-$ sign according to whether it induces the orientation $\langle 1, 2, \ldots, r-2 \rangle$ or $(-1)\langle 1, 2, \ldots, r-2 \rangle$ on L. Since for any two $(r-1)$-subsets of $[n]$ containing L we must have a round where one of them gets a $+$ while the other a $-$ sign, we must have at least $\log(n-r+2)$ rounds.

The upper bound is obtained by a simple application of the Lovász Local Lemma (cf. [3, 2]). We will orient the $(r-1)$-subsets of $[n]$ randomly in t rounds. Let A_i be the event that the $(r-1)$-element subsets of the i^{th} r-subset of $[n]$ are not oriented consistently in any of the t rounds. Each $(r-1)$-subset has two orientations, so for a fixed r-subset there are 2^r possible sets of orientations of its $(r-1)$-element subsets out of which 2 satisfies our consistency requirement. So the probability that the requirement is not satisfied for a fixed r-subset in a given round is $\frac{2^{r-1}-1}{2^{r-1}}$. For t independent rounds we get $p := P(A_i) = \left(\frac{2^{r-1}-1}{2^{r-1}}\right)^t$. We want to avoid the events $A_1, \ldots, A_{\binom{n}{r}}$. A pair of events A_j and A_s are mutually independent unless the j^{th} and s^{th} r-subset of $[n]$ have a common $(r-1)$-subset. So in the dependency graph (formed by the events A_i as vertices two being adjacent if they are not mutually independent) the degree of every vertex is $d := r(n-r)$. The Local Lemma says (its original form is enough for our purposes) that if $4pd \le 1$ then all the bad events can be avoided simultanously. So we need $4\left(\frac{2^{r-1}-1}{2^{r-1}}\right)^t r(n-r) \le 1$. This is satisfied if $t \ge \log(4r(n-r)) \left(\log \frac{2^{r-1}}{2^{r-1}-1}\right)^{-1} = \left(\log \frac{2^{r-1}}{2^{r-1}-1}\right)^{-1} \log n + o(\log n)$. Thus $t_r(n)$ is bounded from above by the smallest t satisfying the previous inequality and this proves the claimed upper bound. $\square$

Note that Proposition 1 gives $t_4(n) \le 5.20 \log n$, while for $t_3(n) = \lceil \log(n-1) \rceil$ it gives $2.41 \log n$ as an upper bound.

Remark 1. We note that the lower bound in Proposition 1, though tight for $r = 3$ cannot be tight in general. It is already not so for $r = 4$ and $n = 5$. It is not hard to check then that we cannot make more than two of the five 4-vertex simplices have their facets oriented consistently in one round. Thus at least 3 rounds are needed, while the lower bound is 2. $\lozenge$

3.2 Orienting every k-cycle

In Section 2 we considered only some special k-cycles of complete graphs, namely those that we called increasing. This property was crucial when proving the upper bound in Theorem 1. Here we look at the problem, where all k-cycles are considered.

Fix a $k \geq 3$. Now our question is the following. What is the minimum number of orientations of the edges of K_n needed if we want all the k-cycles being oriented cyclically in at least one of these orientations?

We do not know the exact answer to this question for $k > 3$. Below we show that this problem is related to the problem of k-independent set systems, cf. [1, 6], and obtain some bounds by applying existing results on the latter.

Definition 4 *A family $\mathcal{F}$ of subsets of a t-element set is called k-independent if for every k members $A_1, \ldots, A_k$ of $\mathcal{F}$, membership of an element in any collection of these k subsets has no implication to membership in any of the others. In other words, choosing B_i to be either A_i or its complement, for no such choice is the intersection $\cap_{i=1}^{k} B_i$ empty.*

Kleitman and Spencer [6] investigated k-independent set systems. Denoting by $g(t,k)$ the maximum number of sets in a k-independent family of subsets of a t-element set, they proved that $g(t,k)$ is exponential in t, and in particular, the following upper and lower bounds hold.

Theorem 2 (Kleitman, Spencer [6]) *For fixed $k \geq 3$ and t sufficiently large we have*

$$2^{d_1 t 2^{-k} k^{-1}} \leq g(t,k) \leq 2^{d_2 t 2^{-k}},$$

where d_1 and d_2 are absolute constants.

If we denote by $T(m,k)$ the minimum size of a set for which an m-element k-independent family of its subsets exists, then the above result of Kleitman and Spencer immediately implies

$$c_1 2^k \log m \leq T(m,k) \leq c_2 k 2^k \log m, \tag{2}$$

where c_1 and c_2 are absolute constants (that can be chosen to be the reciprocal of d_2 and d_1, respectively).

Let $W(n,k)$ denote the minimum size of a family of orientations of the edges of K_n satisfying that for every odered k-element subset $(i_1, i_2, \ldots, i_k)$ of the vertex set there is an orientation in our family which orients the cycle $i_1 \ldots i_k i_1$ cyclically (in one or the other direction).

Theorem 3 *For $k \geq 3$ fixed and n sufficiently large*

$$s_1 2^{k/2} \log n \leq W(n,k) \leq s_2 k 2^k \log n,$$

where s_1 and s_2 are absolute constants.

Proof. Let us fix an orientation of the edges of K_n that we will call the *reference orientation* and denote it by R. An arbitrary orientation of K_n can be encoded by an $\binom{n}{2}$-length sequence of 0's and 1's: we attach a 0 or a 1 to every edge depending on whether it is oriented the same way as in R or in the opposite direction. Consider a maximum matching of size $\lfloor n/2 \rfloor$. Taking an arbitrary orientation of $\lfloor k/2 \rfloor$ of its edges, it can be extended to a cyclically oriented cycle of length k. Since this cycle or its reverse must be present at some orientation we give, the following must be true. Every possible $\lfloor k/2 \rfloor$-length $0-1$ sequence $x = (x_1, \ldots, x_{\lfloor k/2 \rfloor})$ or its complementary sequence $\overline{x} = (1 - x_1, \ldots, 1 - x_{\lfloor k/2 \rfloor})$ must appear in at least one of our $0-1$ sequences describing our orientations, as the subsequence belonging to the $\lfloor k/2 \rfloor$ edges we looked at. (It is probably helpful to imagine a $0-1$ matrix where the rows belong to the edges and the colums are indexed by the orientations.) This should hold for every choice of $\lfloor k/2 \rfloor$ edges from our $\lfloor n/2 \rfloor$ matching edges. Now double the length of each of the sequences belonging to the edges (that is the rows) by concatenating its own complementary sequence to each of them. Considering the sequences obtained this way as characteristic vectors of subsets of a set, the size of which is two times the number of orientations we had, they must define an $\lfloor n/2 \rfloor$-element family of $\lfloor k/2 \rfloor$-independent subsets. This implies $c_1 2^{k/2} \log(n/2) \leq 2W(n,k)$ by (2), and thus the lower bound in the statement.

For the upper bound consider a k-independent family of $\binom{n}{2}$ subsets of a $T(\binom{n}{2}, k)$-size basic set. The characteristic vectors of these sets define $T(\binom{n}{2}, k)$ orientations of the edges of our K_n according to the encoding of orientations described above. The k-independence of our subsets implies that all possible orientations of any k edges will appear in one of these orientations of the K_n. This holds in particular to the edges of any k-cycle, so in one of the orientations it will be cyclically oriented. Thus $T(\binom{n}{2}, k)$ orientations are sufficient, i.e., $W(n,k) \leq T(\binom{n}{2}, k)$. Plugging in the upper bound of (2) we obtain the upper bound in the statement. This completes the proof. □

Remark 2. We note that our problem is also related to (i,j)-separating systems investigated in [4]. For given positive integers i and j a family of bipartitions of a set X forms an (i,j)-separating system if for every two disjoint subsets A and B of X with $|A| = i, |B| = j$ the family contains a bipartition for which A is completely in one, and B is completely in the other partition class. Fredman and Komlós [4] gave bounds on the minimum number of bipartitions in an (i,j)-separating system of an m-element set. Their results can also be used to prove the lower bound in Theorem 3 and a somewhat weaker upper bound on $W(n,k)$. ♢

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