Oriented list colouring of undirected graphs

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Abstract

We introduce the choice number analogon of the oriented chromatic number of Sopena. It is shown that its minimum over all orientations of a graph can be arbitrarily much larger than the choice number of the underlying undirected graph. Investigating related problems we also look at the minimum number of transitive triangles needed to cover all tournaments on n vertices. The choosability analogon of the dichromatic number of Erdős and Neumann-Lara is also considered briefly.

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1 Introduction

Coloring the vertices of a graph so that no adjacent vertices receive the same color gives rise to the notion of chromatic number, one of the central notions of graph theory. Similar concepts for oriented or directed graphs have also been defined. (We make the following usual distinction between oriented and directed graphs: while a directed graph may have edges in both directions between the same pair of vertices, this is not allowed for an oriented graph.) The chromatic number of oriented graphs was introduced recently by Sopena [11]. It is defined as follows. A legal coloring of an oriented graph is an assignment of colors to the vertices in such a way that adjacent vertices receive different colors and all edges between any two color classes go in the same direction. The oriented chromatic number $\chi^{\rightarrow}(G)$ is the minimal number of colors that is enough to legally color the oriented graph G. (In other words, it is the size of the smallest tournament T for which a homomorphism from G to T is possible.)

Another recently well investigated graph parameter is $ch(G)$, the choice number of a(n undirected) graph G. (Cf. [1], [6].) Assign a list of k colors to each vertex of G and find a proper coloring of its vertices from the corresponding lists. The smallest k for which this is always possible (whatever the actual lists are) is the choice number of G.

In this paper we introduce the oriented choice number of an oriented graph in the obvious way (see details in next section) and the oriented choice number $och(G)$ of an *undirected graph* as its minimum over all orientations of the edges of G. We will investigate the relation of $och(G)$ to other parameters of G. In particular, we will show that the gap between $och(G)$ and $ch(G)$ can be arbitrarily large while $och(G)$ is both upper and lower bounded by functions of $\tau(G)$, the minimal number of points needed to cover all edges.

Another notion of chromatic number for directed graphs, the so-called dichromatic number, is defined by Neumann-Lara, and the dichromatic number of an undirected graph by Erdös and Neumann-Lara, cf. [9], [2], [3]. In the last section we investigate the list coloring version of this concept. (A third chromatic number type invariant of directed graphs is given by Fachini and Körner [5]. This is of different nature, however, and we do not investigate it in this paper.) The paper is organized as follows. The next section is devoted to the investigation of $och(G)$ and its relation to other graph parameters. Section 3 deals with miscellaneous problems connected to $och(G)$. In particular we consider the question how many transitive triangles are needed to cover every tournament on n points. Section 4 deals with the list coloring analogon of the dichromatic number.

2 The oriented choice number

Recall that a proper coloring of a graph G is an assignment of colors to its nodes such that no adjacent vertices receive the same color. The minimum number of colors needed for a proper coloring is the chromatic number $\chi(G)$ of G. A graph G is said to be k-choosable if for every possible assignment of lists of size k of colors to its vertices there is a proper coloring of G where every vertex is assigned one of the colors from its list. For an oriented graph Sopena [11] defines a proper coloring to be an assignment of colors to the nodes in such a way that in addition to the requirement that adjacent vertices get different colors we also have the following. It is not allowed that we have two (oriented) edges in such a way that the tail of one of them is colored the same color as the head of the other one while at the same time the tail of the second receives the same color as the head of the first. Now an oriented graph F is said to be k-choosable if for every possible assignment of lists of size k to its vertices there is a proper coloring of F where every vertex is assigned one of the colors of its list. The minimum k for which the oriented graph F is k-choosable will be denoted by $ch^{\rightarrow}(F)$. Our main concern will be the following graph parameter.

Definition 1 *The oriented choice number $och(G)$ of the undirected graph G is the smallest k for which there exists an orientation of the edges of G resulting in a k-choosable oriented graph.*

In other words, $och(G)$ is the minimum of $ch^{\rightarrow}(\hat{G})$ for all possible oriented versions $\hat{G}$ of G.

At first glance it is not clear whether $och(G)$ is diferent from $ch(G)$. Indeed, if we would define the oriented chromatic number of an undirected graph in a similar manner (i.e., as the minimum value of the oriented chromatic number of the different orientations), we would only get a more complicated definition of the chromatic number itself.

First we give a simple example showing that $och(G) \neq ch(G)$ in general. Consider the chordless cycle C_6 on six vertices. Let the vertices be $v_1, ..., v_6$ in their cyclic order (that is $\{v_i, v_{i+1}\}$ is an edge for every i where addition is intended modulo 6.) It is easy to see that $ch(C_6) = 2$. We show $och(C_6) > 2$. First observe that if we do not orient the edges in an alternating way then an oriented path of length 2 appears that already has oriented chromatic number 3 and $ch^{\rightarrow}(G)$ is always at least as large as the oriented chromatic number of the same (oriented) graph. (This can be seen by assigning identical lists to the vertices.) So the only way $ch^{\rightarrow}(F)$ could be less than 3 for an oriented version F of C_6 is if every vertex of F is either a source or a sink. There is only one such orientation up to isomorphism so w.l.o.g. we may assume that the edges of F are $(v_1, v_2), (v_3, v_2), (v_3, v_4), (v_5, v_4), (v_5, v_6), (v_1, v_6)$. Now assign the following two-element lists to these vertices:

$$L(v_1) = L(v_4) = \{1, 2\},\ L(v_2) = L(v_5) = \{2, 3\},\ L(v_3) = L(v_6) = \{1, 3\}$$

where $L(x)$ stands for the list assigned to vertex x.

Claim One cannot find a proper coloring of the vertices of the above oriented graph F from the above lists.

Proof. Indeed, if we color v_1 by 2 we are forced to color v_2 by 3, v_3 by 1 and v_4 by 2. Then coloring v_5 by neither of 2 or 3 is allowed. This means we should start coloring v_1

by 1 but then v_6, v_5 and v_4 are forced to be colored by 3, 2 and 1, respectively, and now v_3 cannot be colored properly. □

This example already shows that $och(G) = ch(G)$ does not hold in general. In fact, much more is true.

Theorem 1 *For every positive integer k there is an integer $g(k)$ such that if a graph G contains at least $g(k)$ independent edges then $och(G) > k$.*

Before proving Theorem 1 let us formulate the following immediate

Corollary 1 *For every positive integer k there exists a graph G such that $ch(G) = 2$ and $och(G) > k$.*

Proof. Let G be the graph consisting of $g(k)$ independent edges, where $g(k)$ is the function in Theorem 1. This G has $ch(G) = 2$ and its oriented choice number is larger than k by Theorem 1. □

We remark that if we want our graph to be connected then a path of $2g(k)$ vertices would also do in place of the graph in the proof of Corollary 1.

To prove Theorem 1 we recall the following result from [8] (cf. Theorem 10. on page 15) attributed to Erdös-Moser, and Stearns.

Theorem EMS: For every n there exists a function $F(n)$ such that every tournament on at least $F(n)$ vertices contains a transitive subtournament on n vertices. For the smallest possible such $F(n)$ one has $2^{\frac{n-1}{2}} \leq F(n) \leq 2^n$.

Proof of Theorem 1.
Fix some positive integer k. Consider a set of $F(2k)$ colors where $F(n)$ is the function in the above mentioned result about tournaments. The idea of the proof is that we will make it impossible that any tournament defined on $F(2k)$ colors as vertices would be consistent with a legal coloring from our lists. Take a graph G with $m = 2^{F(2k)}$ independent edges and fix any orientation of the graph. Let the independent edges be $e_1, \ldots, e_m$. Consider all the possible $2^{F(2k)}$ tournaments on the $F(2k)$ colors, let them be called $T_1, \ldots, T_m$. Now we give the lists assigned to those vertices of our graph that are the endpoints of the above mentioned independent edges. For giving the lists of the endpoints of e_i look at T_i and find a transitive subtournament of $2k$ points in it. Let the colors corresponding to this subtournament be $c_1, \ldots, c_{2k}$ indexed according to their ordering in T_i. This means that $(c_j, c_r) \in E(T_i)$ iff $j < r$. Denote the two endpoints of e_i by v_{i1} and v_{i2} and assume e_i is oriented towards the latter. Assign lists $L(v_{i1}) = \{c_{k+1}, c_{k+2}, \ldots, c_{2k}\}$ and $L(v_{i2}) = \{c_1, c_2, \ldots, c_k\}$ to v_{i1} and v_{i2}, respectively. Observe that any legal coloring of the endpoints of e_i will make it impossible that the coloring is consistent with (a homomorphism onto) T_i. Since every possible tournament on our colors is represented by some T_i this proves that no valid coloring exists. Thus $och(G) > k$. □

Remark: The proof above does not really use that the T_i's contain large transitive subtournaments, only a bipartite subgraph of these is needed. Thus some smaller number replacing $F(2k)$ could be used in the argument.

Using the monotonicity of $och(G)$ (i.e., the fact that adding new edges to a graph $och(G)$ cannot decrease) Theorem 1 shows that $och(G)$ is linked to the matching number $\nu(G)$ of G in the sense that a lower bound on $och(G)$ can be formulated in terms of $\nu(G)$. The following observation shows that these two quantities are related, indeed, in the sense that an upper bound on $och(G)$ can also be given in terms of $\nu(G)$. Let $\tau(G)$ denote the minimum number of vertices covering all edges of G. It is obvious that $2\nu(G) \geq \tau(G)$ since the endpoints of edges in a maximum size matching cover every edge. Thus the following observation gives $och(G) \leq 2\nu(G) + 1$.

Proposition 1 *For every graph G one has $och(G) \leq \tau(G) + 1$.*

Proof. Consider G and a minimal set T of points covering all the edges. Orient all edges between a point in T and a point in $V(G) - T$ towards T and orient the rest of the edges arbitrarily. If we assign lists of $\tau(G) + 1$ colors to every node then we can color every element of T first then delete the already used colors from every list in $V(G) - T$ and color the elements of $V(G) - T$ from the remaining lists. Since $|T| = \tau(G)$ (so at least one element remained on every list after coloring T) and $V(G) - T$ must be an independent set (otherwise some edge would not be covered) this is certainly possible thus proving the statement. □

Remark: There are cases when $och(G) = \tau(G) + 1$ indeed. This happens for complete graphs and also for complete bipartite graphs with color classes of size $t = \tau(G)$ and t^t where already $ch(G) = \tau(G) + 1$. The latter can be seen by assigning disjoint t-element lists to the vertices of the smaller color class and all their transversals as lists to the vertices in the larger color class.

Theorem 1 is formulated in such a way that the existence of $g(k)$ is emphasized. It should be clear, however, that $g(k)$ is a well-defined function, namely, its value is the least integer m for which we have $och(G) = k$ if $G = mK_2$, i.e., the union of m independent edges. The proof of Theorem 1 and the statement of Proposition 1 give bounds on $g(k)$ but to determine the actual value seems to be an intriguing problem. We say a bit more about it later.

3 Covering tournaments

Let $C(k)$ mean the class of graphs G with $och(G) = k$ and let its superclass $C(k, r)$ be defined as follows. A graph G belongs to $C(k, r)$ if there exists an orientation of G such that for any assignment of k-element lists of colors *out of not more than r colors* a proper oriented coloring from the lists is possible.

Clearly $C(k,r) \subseteq C(k,r+1)$ and $C(k) = \cap_{r=k}^{\infty} C(k,r)$. It is also clear that $C(k,k)$ is identical to the class of k-chromatic graphs.

In the following we take a closer look on $C(2)$. A complete characterization of graphs belonging to this class seems rather tedious therefore we do not intend to give a complete characterization, but some more interesting remarks instead. It is clear that any graph in $C(2)$ should be bipartite, in fact, even its choice number should be 2. A characterization of 2-choosable graphs is given by Erdős, Rubin, and Taylor [4] as follows. They denote by $\Theta_{k,l,m}$ the graph consisting of three vertex disjoint paths of length k, l, m, respectively, between the same two distinguished endpoints. Their theorem states that deleting successively all degree one points of a 2-choosable graph the remaining part of each component must be (at most) a $\Theta_{2,2,2m}$, where m is a positive integer.
If a graph G has $och(G) = 2$ than it must be achieved by an orientation containing no two-edge oriented path. This means that for some 2-coloring of such a (necessarily 2-chromatic) graph all edges are oriented from one color class to the other. Thus for a bipartite graph with a given bipartition there is essentially only one orientation to be investigated when we want to decide whether its oriented choice number is 2 or not. We will call this orientation the *alternating* orientation. (Cf. the example of C_6 in the previous section.)

We believe that $C(2) = C(2,4)$. (Note that it is not clear whether there exists for every k some finite r_k such that $C(k) = C(k, r_k)$.) One way to see the fact that $C(2) \neq C(2,3)$ is to realize that any graph consisting of independent edges only is in $C(2,3)$ but, for many edges, not in $C(2,4)$. We prove this next.

Proposition 2 *Let F be a graph consisting of k independent edges. Then $F \in C(2,3)$. On the other hand, there exists k_0 for which $k \geq k_0$ implies $F \notin C(2,4)$.*

Proof. We prove the first statement first. Assume we have three colors available, call them a, b, c. Consider the cyclic tournament K given by $a \to b \to c \to a$ on these colors. Now take the edges of F one by one. (Notice that the orientation is considered already fixed when the lists are assigned to the vertices.) If an edge has identical lists on its two endpoints then the two colors occuring in these lists can be put on the vertices in such a way that their order given by the orientation of the edge is consistent with K. If, on the other hand, the two lists are different then put a color on one of the endpoints that is not present in the other list. Now since K is cyclic, one of the colors in the other list can be used to color the other node properly. Since all edges of F can be colored this way, the first statement is proven.

Now we prove the second statement. Let T_0 denote the tournament on the available colors consistent with the proper coloring we are to find. Observe that the endpoints of an edge with list $\{a, b\}$ at the tail and list $\{b, c\}$ at the head cannot be properly colored if and only if the induced subtournament of T_0 on colors a, b, c is a transitive triangle with c being its source and a its sink. This means that an edge of the above type ensures that T_0 cannot contain this transitive triangle. Since any tournament on four points contains some transitive triangle, this way we can exclude every tournament from being a candidate for T_0 if we have enough edges to exclude at least one transitive triangle for each of them. This proves the second statement. □

The k_0 in the above theorem need not be very large, in fact its optimal value is at most 12. We will explain this in what follows. First we introduce a new notion inspired by the foregoing.

Definition 2 *Let T be a tournament on n labelled vertices. We say that a transitive triangle H covers T if it is on three vertices of T and those three vertices induce H in T.*

Remark Note that Definition 2 requires more that H be a subtournament. It is important that the vertices of H are labelled and it occures in T on the triplet of vertices labelled same.

It seems to be a question of independent interest what is the minimum number of transitive triangles that cover every possible tournament on n points. More precisely we mean the following. Let Q be a set of transitive triangles on the vertices $1, 2, \ldots, n$ such that every possible tournament on $\{1, \ldots, n\}$ contains at least one element of Q. Let $q(n)$ denote the minimum possible size of such a set Q. Our question is the value of $q(n)$. It is clear that $q(n)$ is not meaningful (or equals to infinity) for $n = 3$ since there are tournaments on three points not containing a transitive triangle. But $q(n)$ is well defined for all $n \geq 4$.

Two easy observations are as follows.
1) $q(n) \geq 9$ for all $n \geq 4$.
Reason: The number of n-vertex tournaments is exactly eight times the number of those any transitive triangle can cover. On the other hand, the two sets of tournaments covered by two different transitive triangles is disjoint iff the two triangles are on the same three vertices. Since the arcs in a given triple of vertices can be oriented cyclically one cannot cover all n-point tournaments by using triangles on only one triple of the points.

2) $q(n + 1) \leq q(n)$ for every n.
Reason: If we cover all tournaments on $\{1, \ldots, n\}$ then all tournaments on $\{1, \ldots, n+1\}$ are also covered.

It is somewhat frustrating that we do not know the exact value of $q(n)$ for every n. We beleive it is 12 for all $n \geq 4$ but could not prove it for $n > 5$.

Proposition 3 $q(4) = q(5) = 12$.

Proof. By the second observation above it is enough to prove $q(4) \leq 12$ and $q(5) \geq 12$.

$q(4) \leq 12$: Consider all transitive triangles on $V_4 = \{1, 2, 3, 4\}$ in which 1 appears as a source or a sink. The number of such triangles is 12. Since either the outdegree or the indegree of 1 is at least two in any tournament on V_4 this system of transitive triangles covers all possible tournaments on V_4.

$q(5) \geq 12$: Consider those tournaments on $V_5 = \{1, 2, 3, 4, 5\}$ that are the union of two oriented Hamiltonian cycles. The number of such tournaments equals the number of cyclic permutations of $1, \ldots, 5$, that is 24. One can easily check that any transitive triangle appears in at most (in fact, exactly) two of the above tournaments, thus at least 12 transitive triangles are needed to cover all of them. □

Proposition 3 shows that the k_0 of Proposition 2 need not be more than 12. (It does not necessarily give the optimum because of the possibility of edges with non-intersecting 2-element lists at their endpoints. More complicated tournament covering questions can be defined if we want to give an equivalent translation of the problem of determining the function $g(k)$ of the previous section. These problems, however seem to be more complicated than relevant therefore we do not discuss them.)

Let us consider the class of $C(2, 3)$ once more. We sketch a way to characterize this class. We will call an edge an edge of type (xy, zt) if its tail is assigned the list of colors $\{x, y\}$ while its head is assigned the list of colors $\{z, t\}$. Considering $C(2, 3)$ we have 3 colors available, let us call them a, b and c. Let G be a bipartite graph not in $C(2, 3)$. Consider (one of) its alternating orientation(s) and an assignment of two element lists of the available colors to its vertices such that no proper coloring exists from these lists. (Such an assignment should exist if $G \notin C(2, 3)$.) In the following we refer to our oriented graph with this assignment of color lists simply as G (or sometimes the assigned G to avoid confusion).

Claim: G must have an edge of each of the following six types: $(ab, ac), (ac, ab), (ab, bc), (bc, ab), (ac, bc), (bc, ac)$.

Proof. By our assumptions there is no tournament on $U = \{a, b, c\}$ to which G could be homomorphically mapped (in a way consistent also with the list assignment). Consider a transitive triangle on U let x be its source and z its sink and y its middle point where x, y, z is some permutation of the elements of U. One can easily check that the only obstacle of a homomorpism to this tournament can be an edge of type (yz, xy). This applies to all the six possible transitive triangles on U thus the six edges above must be present. □

Observe that once we have a bipartite graph with alternating orientation, then each of its points is either a source or a sink. Thus the Claim above implies that the assigned G must have at least six points: a source and a sink for each of the three possible lists ab, ac, bc. In fact, we have the six edges given in the Claim and each of these six type points appear as an endpoint of two of them. These same type points can be identified in an arbitrary manner and then the resulting six edge graph investigated whether or not it is still possible to find a legal homomorphism of it onto a cyclically oriented triangle on the three colors. If, for example, we make all the the possible identifications we arrive to C_6, our first example with $och(G) > ch(G)$. If fewer identifications are made then we may need additional edges to exclude homomorphisms to the cyclic triangular tournaments of 3 colors. A three edge path with edges of type $(xy, yz), (xz, yz), (xz, xy)$ excludes a cyclic

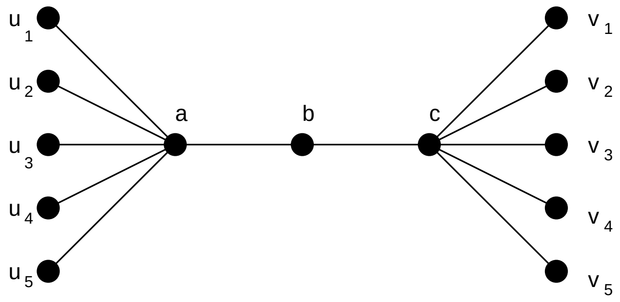

Figure 1.

triangle and exchanging directions the other cyclic triangle can be excluded. Thus the two paths of length three into which the C_6 above can be broken exclude all three vertex tournaments on the colors. If a cyclic triangle is not excluded by such a three edge path it can also be excluded by two connecting edges of type (xy, xz) and (xy, xy). Thus here we need some additional edge (to the previous six ones) which will have the same list on both of its endpoints. To exclude both cyclic triangles we either need two such two or three edge configurations or two can be glued together at a degree three point. (Thus we can exclude both cyclic triangles on a three edge star, for example.) This means that we have six edges to exclude the six transitive triangles and at most two additional edges (in appropriate locations) are needed to exclude the two cyclic triangles. Thus deciding whether a graph is in $C(2,3)$ needs only to decide whether certain configarations of at most eight edges appear in the graph or not. This is easy to check in polynomial time, say.

Finally in this section we give an example of a connected graph that belongs to $C(2,3)$ but not to $C(2,4)$.
Example Let T be the tree defined by $V(T) = \{a, b, c, u_1, \ldots u_{m_1}, v_1, \ldots v_{m_2}\}$ with $m_i \geq 3$ and $E(T) = \{\{a,b\}, \{b,c\}, \{a, u_i\}, \{c, v_j\}\}$ where $i = 1, \ldots, m_1, j = 1, \ldots, m_2$. (In short: T is the union of two stars having a non-central vertex in common.) See Figure 1.
Consider the alternating orientation of T determined by (a, b) being an edge. We assign the following 2-element lists of colors x, y, z, t to the vertices. $L(a) = \{x, y\}, L(b) = \{x, z\}, L(c) = \{z, t\}$,$L(u_1) = \{x, t\}, L(u_2) = \{y, z\}, L(u_3) = \{y, t\}$,$L(v_1) = \{x, t\}, L(v_2) = \{y, z\}, L(v_3) = \{y, t\}$. Simple case checking shows that no proper oriented coloring from these lists is possible. On the other hand, if we have only three colors, than T can always be colored properly since only a and c are sources of the considered oriented version of T and we have seen that a graph not in $C(2,3)$ must have at least three sources.
We remark that there are infinitely many graphs in $C(2)$ having arbitrarily many sources and sinks in their alternating orientations. Such an infinite family of graphs can be constructed by putting arbitrarily large stars on two vertices of a $K_{2,3}$ that are not in the same color class. The reason of these graphs belonging to $C(2)$ is this. The given graphs

have choice number 2 and one can easily check that any coloring that is proper as an ordinary coloring will also be legal as an oriented coloring for these graphs.

4 Dichoice number

In the previous sections we investigated the choice number analogon of the oriented chromatic number defined in [11]. In this section we do similarly with a different chromatic number type invariant of a directed graph defined by Neumann-Lara [9] and (its undirected version by) Erdős and Neumann-Lara, cf. [2] and also [3].
Let D be a directed graph. Its dichromatic number $d(D)$ is the smallest integer k for which the vertex set of D can be partitioned into k classes such that none of the classes contains an oriented cycle. For an undirected graph G its dichromatic number $d(G)$ is defined as the maximum of the dichromatic numbers of all oriented versions of G. Here we allow the orientation of each edge only in one direction; observe however that the dichromatic number of a symmetrically directed graph (where each edge is present in both directions) is equal to the chromatic number of the underlying undirected graph. It follows that $d(G) \leq \chi(G)$ and it is remarkable that it is not known whether the gap between the two invariants can be arbitrarily large. More precisely, even that is not known whether there exists some k_0 such that $\chi(G) > k_0$ would imply $d(G) > 2$, cf. [2].

Here we introduce the choice number analogon $dch(G)$ of the undirected graph G and call it the dichoice number of G.

Definition 3 *Let G be an undirected graph. Its dichoice number $dch(G)$ is the smallest integer k satisfying the following conditions. For any assignment of k-element lists of colors for the vertices and any orientation of the edges of G there exists a coloring of the vertices from their lists such that no color class contains an oriented cycle.*

It is clear that $dch(G)$ is upper bounded by $ch(G)$. For any tree T we have $dch(G) = 1$ so the dichoice number can even be smaller than the chromatic number. The main result in this section is that we show the existence of bipartite graphs with arbitrarily large dichoice number. Our construction is an extension of that of Erdős, Rubin, and Taylor [4] by which they showed the existence of bipartite graphs with arbitrarily large choice number.

Theorem 2 *For any positive integer k there exists graph G with $\chi(G) = 2$ and $dch(G) > k$.*

Proof. Let $m = 2k\binom{2k-1}{k}$ and G be the complete bipartite graph $K_{m,m}$. We show that this G satisfies the requirements in the statement, that is its dichoice number is larger than k. To this end we give an orientation of its edges that has the property that we can assign k-element lists to the vertices for which no proper "dicoloring" exists from the lists. Let the color classes of G be called A and B. To define the orientation partition

both A and B into $\binom{2k-1}{k}$ parts of size $2k$ each. Within each such class label the vertices with $1, \ldots, 2k$. Now orient those edges of G from A to B that have different labels on their two endpoints and orient the rest from B to A.
Observe that vertices having the same label induce complete bipartite graphs isomorphic to $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$. For these subgraphs of G we use the idea of Erdős, Rubin, and Taylor. Consider all possible k-element lists formed from a given set of $2k-1$ colors and assign each of these lists to exactly two vertices of each of the above mentioned subgraphs (i.e., those induced by vertices of identical labels), one in A and one in B, respectively.
Consider any coloring from the above lists. We show that there exists some color class containing an oriented cycle. By the argument in [4] for every i the graph induced by vertices labelled i must contain an edge whose two endpoints are colored by the same color. (This is because each $k-1$ element subset of the $2k-1$ colors are missing at some of these vertices thus at least k colors are used altogether both in A and B. But since there are only $2k-1$ colors they cannot all be different, thus some point in A got the same color as another one in B and they are adjacent.) Consider the edges with the same color on their endpoints for every i. This means $2k$ edges. Since we have only $2k-1$ colors there are two of these edges that have the same color on their altogether four endpoints. These two edges are oriented from B to A since they both connect identically labelled vertices. These labels are different for the two edges, so the two other edges present among the four endpoints are oriented from A to B. This means that these vertices induce an oriented 4-cycle and the proof is complete. □

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