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# PERFECT COUPLES OF GRAPHS

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We generalize the concept of perfect graphs in terms of additivity of a functional called graph entropy. The latter is an information theoretic functional on a graph $G$ with a probability distribution $P$ on its vertex set. For any fixed $P$ it is sub-additive with respect to graph union. The entropy of the complete graph equals the sum of those of $G$ and its complement $\overline{G}$ iff $G$ is perfect. We generalize this recent result to characterize all the cases when the sub-additivity of graph entropy holds with equality.

## 1. Introduction

Although the notion of graph entropy has its roots in information theory, it was proved to be closely related to some classical and frequently studied graph theoretic concepts. For example, it provides an equivalent definition for a graph to be perfect and it can also be applied to obtain lower bounds in graph covering problems.

A celebrated result of Lovász [14] states that a graph $G$ is perfect if and only if so is its complement $\overline{G}$. Quite recently it was proved in [3] that, in fact, 'perfectness' means that a certain type of sub-additive functionals becomes additive for the union of $G$ and $\overline{G}$. Along this line, as a generalization of perfectness we introduce a similar quantitative relation between edge-disjoint graphs. In our interpretation a graph $G$ is perfect if it is perfect with respect to the complete graph, i.e., the union of $G$ and $\overline{G}$ — or, in other words, $G$ and $\overline{G}$ form a 'perfect couple'.

The main result of our paper (Theorem 1) is a structural characterization of perfect couples. We prove that it is necessary and sufficient for those pairs of graphs to satisfy two requirements. One of them is concerning induced subgraphs that form a complete graph in the union. The other assumption is formulated in terms of a specific equivalence relation for edges of graphs, introduced by Gallai [4] as an important tool in the characterization of comparability graphs. It states that no equivalence class of edges should be cut into two by the couple.

A weakening of perfectness in a natural way leads to the concept of 'normal couples' of graphs. Some of their properties are described in Section 4; it remains an open problem, however, to give their characterization analogously to perfect couples.

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## Definitions and related results

Graph entropy was introduced in [7] as an information theoretic functional $H(G,P)$, where $G$ is a graph and $P$ is a probability distribution on its vertex set $V(G)$. It will be more convenient for us to use an alternative definition which, in [3], was shown to be equivalent to the original one. First we need the notion of the vertex packing polytope.

**Definition.** The *vertex packing polytope* $VP(G)$ of a graph is the convex hull of the characteristic vectors of the independent sets of $G$ (cf. [5] and [3]).

**Definition.** Let $G$ be a graph on the vertex set $V(G) = \{1,\dots,n\}$ and let $P = (p_1,\dots,p_n)$ be a probability distribution on $V(G)$ (i.e., $p_1+\ldots+p_n = 1$, $p_i \geq 0$ for $1 \leq i \leq n$). The entropy of $G$ with respect to $P$ is defined as

$$H(G,P) = \min_{\underline{a}\in VP(G)} -\sum_{i=1}^{n} p_i \log a_i. \tag{1.1}$$

Here (as well as throughout the paper) log means logarithm to the base 2.

Extending the results of [10], it was observed in [9] that graph entropy is subadditive. This means that if $F$ and $G$ are two graphs on the same vertex set $V$ and $F \cup G$ denotes the graph on $V$ having edge set $E(F\cup G) = E(F) \cup E(G)$, then for any fixed $P$

$$H(F\cup G,P) \leq H(F,P) + H(G,P). \tag{1.2}$$

This inequality can be applied e.g. to obtain lower bounds for the minimum number of graphs from a given family needed to cover a certain graph not in the family, cf. [9], [12].

Let $\overline{G}$ denote the complement of $G$. The special case $F = \overline{G}$ of the above inequality was already discussed in [10] in a purely information-theoretic context and the condition of equality in this special case was treated in [10], [11] and [3].

We speak about equality in the *weak sense* if there exists a nowhere vanishing probability distribution $P$ yielding equality in (1.2).

Equality in the *strong sense* means that equality holds in (1.2) for every $P$.

To state the results of the above cited papers we need the following notions.

**Definition.** A graph $G$ is called *normal* if its vertex set has two coverings, one with independent sets and one with cliques, such that every independent set of the first covering has a nonempty intersection with every clique of the second covering.

**Definition.** A graph $G$ is *perfect* if for any induced subgraph $G'$ of $G$ $\chi(G') = \omega(G')$, where $\chi(A)$ is the chromatic number, $\omega(A)$ is the clique number of a graph $A$.

In [8] it is proved that every perfect graph is normal while there exist normal graphs which are not perfect. An example is $C_9$, the cycle on nine points. For more on perfect graphs cf. Lovász [13].

**Definition.** A graph $G$ is called *weakly splitting* if for $G$ and $F = \overline{G}$ equality holds in the weak sense in (1.2), i.e., there exists a nowhere vanishing $P$ such that

$$H(G,P) + H(\overline{G},P) = H(P) \tag{1.3}$$

where $H(P) = -\sum p_i \log p_i$ is the Shannon entropy of $P$. In fact, (1.3) requires additivity for $G$ and $\overline{G}$ because $H(P)$ is equal to the entropy of the complete graph with respect to $P$. (This can easily be checked from the definition.) Furthermore $G$ is called *strongly splitting* if (1.3) holds for every $P$.

The easier task of characterization of weakly splitting graphs was already achieved in [10] and [11] cf. also [3]. This is stated as

**Theorem A.** *A graph is weakly splitting if and only if it is normal.*

Körner and Marton proved in [11] that bipartite graphs are strongly splitting. Their conjecture concerning complete characterization of strongly splitting graphs was proved in [3]. This result is

**Theorem B.** *A graph is strongly splitting if and only if it is perfect.*

In this paper we give necessary and sufficient conditions for equality in (1.2) in the general case, i.e., when $F$ and $G$ are not necessarily complementary graphs of each other.

In view of Theorem B we can look at equality in (1.2) in the strong sense as a definition of a more general notion of perfectness, as mentioned at the end of [11]. The usual notion of perfectness can be considered as a relationship between $G$ and $\overline{G}$, namely that they satisfy (1.3) for every $P$. This idea leads to the following

**Definition.** Two graphs $F$ and $G$ on the same vertex set are said to form a *perfect couple* if for every $P$

$$H(F,P) + H(G,P) = H(F \cup G, P). \tag{1.4}$$

**Remark.** This definition implicitly contains the restriction that $F$ and $G$ are edge-disjoint graphs. Were $F$ and $G$ not edge-disjoint, then for a $P$ which is positive on the two endpoints of a common edge and zero everywhere else, we had $H(F,P) = H(G,P) = H(F \cup G, P) > 0$, so that (1.4) could not hold.

In section 3 we give a necessary and sufficient condition for two graphs to form a perfect couple. Section 4 is devoted to the case of equality in (1.2) in the weak sense; it deals with a generalization of normality, similar to that of perfectness. The next section contains some technical lemmas for the proofs.

## 2. The Substitution Lemma

The next lemmas are useful to determine the entropy of graphs resulting from simple operations.

**Definition.** (cf. [4], [6]): A set $U \subseteq V(T)$ is said to be *autonomous* in $T$ if every vertex in $V(T) \setminus U$ is either adjacent in $T$ to every vertex of $U$ or to none of them.

The following observation is straightforward from the definition of graph entropy.

**Contraction Lemma.** *Let $S$ be an autonomous independent set of the graph $G$. Let the graph $G'$ be obtained from $G$ when replacing $S$ by a single vertex $s$ connected* *with exactly those vertices in $V(G) \setminus S$ which were connected to the vertices of $S$. Having a probability distribution $P$ on $V(G)$, we define $P'$ on $V(G')$ by*

$$P'(x) = \begin{cases} P(x) & \text{if } x \in V(G) \setminus S \\ \sum_{y \in S} P(y) & \text{if } x = s \end{cases}$$

*Then*

$$H(G', P') = H(G, P).$$

For $U \subseteq V$ and a probability distribution $P$ on $V$ let $P(U)$ denote the total probability of vertices in $U$ and let $P_U$ denote the normalized distribution on $U$, i.e., for every $v \in U$

$$P_U(v) = \frac{P(v)}{P(U)}.$$

For $U \subseteq V$ let $G(U)$ denote the graph induced by $U$ in $G$.

**Substitution Lemma.** *Let $U \subseteq V(T) = T$ be autonomous in $T$. Set $F = T(U)$ and let $G$ be the graph defined by $V(G) = V$ and $E(G) = E(T) \setminus E(F)$. Then for every $P$*

$$H(G, P) + P(U) \cdot H(F, P_U) = H(T, P). \tag{2.1}$$

**Proof.** Let us denote the family of the maximal independent sets of a graph $B$ by $S(B)$.

Since $U$ is an independent set in $G$, one can observe that $S(G)$ is the disjoint union of $S_0(G)$ and $S_1(G)$ where

$$S_0(G) = \{Y \in S(G), \quad Y \cap U = \emptyset\} \quad \text{and}$$
$$S_1(G) = \{Y \in S(G), \quad U \subseteq Y\}.$$

Furthermore

$$S(T) = S_0(G) \cup \{(Y \setminus U) \cup Z, \quad Y \in S_1(G), \quad Z \in S(F)\}. \tag{2.2}$$

Consider the $\underline{a}_0 \in VP(T)$ achieving the minimum in (1.1). Let the characteristic vector of $I_j \in S(T)$ have coefficient $\alpha_0(I_j)$ in the convex combination giving $\underline{a}_0$. (It is easily seen that only maximal independent sets can have positive coefficients.) Now we define a convex combination of the characteristic vectors of the sets in $S(F)$ and $S(G)$, respectively.

Let the coefficient for $Y_k \in S(G)$ be

$$\beta(Y_k) = \sum_{Y_k \supseteq I_j} \alpha_0(I_j). \tag{2.3}$$

It is a straightforward consequence of (2.2) that

$$\sum_{Y_k \in S(G)} \beta(Y_k) = \sum_{I_j \in S(T)} \alpha_0(I_j) = 1 \tag{2.4}$$

as it is required.

Let the coefficients for $Z_k \in S(F)$ be

$$\gamma(Z_k) = \sum_{Z_k \subseteq I_j} \alpha_0(I_j) \Big/ \sum_{I_\ell \cap U \neq \emptyset} \alpha_0(I_\ell). \tag{2.5}$$

Again by (2.2) $I_\ell \cap U \neq \emptyset$ is equivalent to $I_\ell \notin S_0(G)$ and so

$$\sum_{Z_k \in S(F)} \gamma(Z_k) = 1 \tag{2.6}$$

as it is required.

Now

$$H(G,P) + P(U)H(F,P_U) =$$

$$= \min_{\underline{b} \in VP(G)} - \sum_{i \in V} p_i \log b_i + \min_{\underline{c} \in VP(F)} - \sum_{i \in U} p_i \log c_i \leq$$

$$\leq - \sum_{i \in V} p_i \log \Big( \sum_{i \in Y_k} \beta(Y_k) \Big) - P(U) \sum_{i \in U} \frac{p_i}{P(U)} \log \Big( \sum_{i \in Z_k} \gamma(Z_k) \Big) =$$

$$= - \sum_{i \in V} p_i \log \Big( \sum_{i \in Y_k} \sum_{Y_k \supseteq I_j} \alpha_0(I_j) \Big) - \frac{\sum_{i \in U} p_i \log \Big( \sum_{i \in Z_k} \sum_{Z_k \subseteq I_j} \alpha_0(I_j) \Big)}{\Big( \sum_{I_\ell \cap U \neq \emptyset} \alpha_0(I_\ell) \Big)}$$

$$= - \sum_{i \in V \setminus U} p_i \log \Big( \sum_{i \in I_j} \alpha_0(I_j) \Big) - \sum_{i \in U} p_i \log \Big( \sum_{I_j \cap U \neq \emptyset} \alpha_0(I_j) \Big) -$$

$$- \sum_{i \in U} p_i \log \frac{\Big( \sum_{i \in I_j} \alpha_0(I_j) \Big)}{\Big( \sum_{I_\ell \cap U \neq \emptyset} \alpha_0(I_\ell) \Big)} =$$

$$= - \sum_{i \in V \setminus U} p_i \log a_{0i} - \sum_{i \in U} p_i \log a_{0i} = H(T,P). \tag{2.7}$$

The opposite inequality is an easy consequence of the subadditivity of graph entropy. For the sake of completeness we give a direct proof. Let us consider the coefficients $\beta_0(Y_j)$ and $\gamma_0(Z_k)$ achieving the minimizing $\underline{b}_0 \in VP(G)$ and $\underline{c}_0 \in VP(F)$ in $H(G,P)$ and $H(F,P_U)$, respectively. We define the coefficients $\alpha(I_\ell)$ as follows:

$$\begin{aligned} &\alpha(I_\ell) = \beta_0(Y_j) \quad \text{if} \quad I_\ell = Y_j, \quad \text{i.e.} \quad I_\ell \in S_0(G), \\ &\alpha(I_\ell) = \beta_0(Y_j)\gamma_0(Z_k) \quad \text{if} \quad I_\ell = (Y_j \setminus U) \cup Z_k, \quad \text{(cf. (2.2))}. \end{aligned} \tag{2.8}$$

It is easy to see that

$$\sum_{I_\ell \in S(T)} \alpha(I_\ell) = 1 \tag{2.9}$$

as needed.

Now

$$H(T,P) = \min_{\underline{a}\in VP(T)} -\sum_{i\in V} p_i \log a_i \le -\sum_{i\in V} p_i \log\Big(\sum_{i\in I_\ell}\alpha(I_\ell)\Big) =$$

$$= -\sum_{i\in V\setminus U} p_i \log\Big(\sum_{i\in Y_j}\beta_0(Y_j)\Big) - \sum_{i\in U} p_i \log\Big(\sum_{i\in Y_j\cap Z_k}\beta_0(Y_j)\gamma_0(Z_k)\Big) =$$

$$= -\sum_{i\in V\setminus U} p_i \log\Big(\sum_{i\in Y_j}\beta_0(Y_j)\Big) - \sum_{i\in U} p_i \log\Big[\Big(\sum_{U\subseteq Y_j}\beta_0(Y_j)\Big)\Big(\sum_{i\in Z_k}\gamma_0(Z_k)\Big)\Big] =$$

$$= -\sum_{i\in V} p_i \log\Big(\sum_{i\in Y_j}\beta_0(Y_j)\Big) - P(U)\sum_{i\in U}\frac{p_i}{P(U)}\log\Big(\sum_{i\in Z_k}\gamma_0(Z_k)\Big) =$$

$$= H(G,P) + P(U)H(F,P_U). \tag{2.10}$$

By (2.7) and (2.10) the Lemma is proved. ∎

We shall make explicit use of the following simple special case of the Substitution Lemma.

**Corollary C.** [9] *Let the connected components of the graph $G$ be the subgraphs $G_i$. Further set*

$$P_i(v) = P(v)[P(V(G_i))]^{-1}, \quad v \in V(G_i).$$

*Then*

$$H(G,P) = \sum_i P(V(G_i))H(G_i,P_i).$$

## 3. Perfect couples

In this section we prove the following characterization of perfect couples.

**Theorem 1.** *The edge-disjoint graphs $F$ and $G$ on $V = V(F) = V(G)$ form a perfect couple if and only if the following two conditions hold for the graph $T = F \cup G$.*

(i) *for every $U \subseteq V$ with $T(U) = K_{|U|}$ (i.e., for cliques of $T$), $F(U)$ (and consequently $G(U)$, too) should be perfect;*

(ii) *for every $U \subseteq V$, $|U| = 3$ with $T(U) = P_2$ (i.e., for an induced path of length two), either $F(U) = P_2$ or $G(U) = P_2$. This means that the two edges of no $P_2$ can be separated.*

**Proof of necessity.**

The necessity of (i) is a straightforward consequence of Theorem B, obtained concentrating a probability distribution on the vertices of a clique of $T$.

Concentrating a uniform distribution $P$ on the three vertices of a subset $U \subseteq V$ with $T(U) = P_2$, easy calculation shows that if one separates the two edges of $P_2$, then equality does not hold in (1.2). Indeed, the right hand side is 4/3, while

$H(P_2, P) = \log 3 - 2/3$. This proves the necessity of (ii). (We note that the entropy of $P_2$ is always less than the sum of the entropies of its two edges as separate graphs if all the three vertices have positive probabilities. In other words, $P_2$ is not a "separable" graph even in the weak sense; cf. the Remark in Section 4.)

Observe that if two edge-disjoint graphs $F$ and $G$ on the vertex set $V$ do not separate any edge class of $F \cup G$ (i.e., the two edges of no induced $P_2$ of $F \cup G$ are separated) then any three vertices of $V$ induce at least two edges in some of $F$, $G$, and $\overline{F \cup G}$. This remark leads to the following useful concept.

**Definition.** A *Gallai-partition* is a partition of the edges of a complete graph in which no triangle has edges in three distinct partition-classes.

A straightforward and well-known consequence of Gallai's Decomposition Theorem [4] (see also in [6] and [1]) is the following

**Lemma G.** *If the graphs $G_1, G_2, \ldots, G_k$ form a Gallai-partition of the edges of the complete graph on vertex set $V$, then at most two of those graphs $G_i$ are connected and meet all vertices of $V$*

(For more about Gallai-partitions cf. Cameron–Edmonds–Lovász [2] and Cameron–Edmonds [1].)

Now we are ready to complete the proof of Theorem 1.

**Proof of sufficiency.**

The proof goes by induction on the number of vertices. Assume that for graphs on less than $|V|$ vertices the theorem is true. For $|V| \leq 3$ this is trivial. Moreover, if $E(F) = \emptyset$ or $E(G) = \emptyset$, then we have nothing to prove.

Consider a graph $T = F \cup G$ and the three graphs $F$, $G$ and $\overline{T}$. Assume that the two conditions of Theorem 1 hold for $F$, $G$ and $T$. By Lemma $G$ at least one of $F$, $G$, $\overline{T}$ does not span connectedly the whole vertex set $V$. First we consider the case when at least one of $F$ and $G$ has this property.

We can suppose that it is $F$ which does not span $V$ connectedly. Let the connected components of $F$ be $F_1, F_2, \ldots, F_k$.

Let $T_i$ and $G_i$ denote the graphs induced by $V(F_i)$ in $T$ and $G$, respectively. If $A$ and $B$ are two graphs with $V(A) \supseteq V(B)$ then $A - B$ will mean the graph on $V(A)$ with $E(A - B) = E(A) \setminus E(B)$.

First we show that

$$H(F_1', P) + H(T - F_1, P) = H(T, P) \tag{3.1}$$

where $F_1'$ means the graph $F_1$ on the vertex set $V$, i.e., the vertices in $V \setminus V(F_1)$ are added to $F_1$ as isolated points.

It is an easy consequence of condition (ii) of Theorem 1 that $V(F_1)$ must be an autonomous set of $T$. Then by the Substitution Lemma

$$P(V(F_1))H(T_1, P_{V(F_1)}) + H(T - T_1, P) = H(T, P). \tag{3.2}$$

Since $|V(F_1)| < |V|$ and the conditions of the theorem are such that if they hold on $V$ then they must hold on $V(F_1)$, too, from the induction hypothesis we have

$$H(F_1, P_{V(F_1)}) + H(G_1, P_{V(F_1)}) = H(T_1, P_{V(F_1)}). \tag{3.3}$$

Because of the autonomous property of $V(F_1)$ we also have

$$P(V(F_1))H(G_1, P_{V(F_1)}) + H(T - T_1, P) = H(T - F_1, P). \tag{3.4}$$

Now subtracting (3.4) from (3.2) and using (3.3), we obtain

$$P(V(F_1))H(F_1, P_{V(F_1)}) + H(T - F_1, P) = H(T, P). \tag{3.5}$$

According to Corollary C this is equivalent to (3.1). Now repeating this argument for $F_2, F_3, \dots, F_k$, putting $T - \bigcup_{i=1}^{j-1} F_i$ into the role of $T$ above when dealing with $F_j$, we finally get

$$\sum_{i=1}^{k} P(V(F_1))H(F_i, P_{V(F_i)}) + H\Big(T - \bigcup_{i=1}^{k} F_i, P\Big) = H(T, P).$$

By Corollary C, this is just

$$H(F, P) + H(G, P) = H(T, P) \tag{3.6}$$

which proves the theorem whenever $F$ or $G$ is disconnected.

Suppose now that $\overline{T}$ does not span $V$ connectedly. If $\overline{T}$ has no edges then sufficiency follows by Theorem B. Otherwise let $\overline{T}_1$ be a non-trivial connected component of $\overline{T}$ and let $F_1$, $G_1$, $T_1$ denote the induced subgraphs of $F$, $G$, $T$, respectively, on $V(\overline{T}_1)$. We know that $2 \le |V(\overline{T}_1)| < |V|$.

As a consequence of condition (ii), $V(\overline{T}_1)$ is autonomous both in $F$ and $G$ and then, of course, in $T$ too. Thus by the Substitution Lemma,

$$\begin{aligned}
&P(V(\overline{T}_1))H(T_1, P_{V(\overline{T}_1)}) + H(T - T_1, P) = H(T, P)\\
&P(V(\overline{T}_1))H(F_1, P_{V(\overline{T}_1)}) + H(F - F_1, P) = H(F, P)\\
&P(V(\overline{T}_1))H(G_1, P_{V(\overline{T}_1)}) + H(G - G_1, P) = H(G, P).
\end{aligned} \tag{3.7}$$

Observe that $V(\overline{T}_1)$ is an autonomous independent set in each of the graphs $T - T_1$, $F - F_1$, $G - G_1$. Hence, substituting it by a single vertex $s$ in the way described in the Contraction Lemma, the entropy of the three new graphs obtained on $(V \setminus V(\overline{T}_1)) \cup \{s\}$ is equal to the entropy of $T - T_1$, $F - F_1$, $G - G_1$, respectively. This substitution clearly keeps conditions (i) and (ii) and since we have obtained graphs on a smaller vertex set (for $V(\overline{T}_1) \ge 2$), the additivity of graph entropy holds by the induction hypothesis. This implies

$$H(F - F_1, P) + H(G - G_1, P) = H(T - T_1, P). \tag{3.8}$$

Moreover, the induction hypothesis can be used for $V(\overline{T}_1)$, too, since $|V(\overline{T}_1)| < |V|$; whence;

$$H(F_1, P_{V(\overline{T}_1)}) + H(G_1, P_{V(\overline{T}_1)}) = H(T_1, P_{V(\overline{T}_1)}). \tag{3.9}$$

Now from (3.7)–(3.9) we have (3.6) which proves the theorem for disconnected $\overline{T}$. ∎

Theorem 1 gives a characterization of edge partitions of complete graphs for which graph entropy is additive. More precisely we state this as

**Corollary 1.** *Colour the edges of the complete graph on $n$ vertices with $k$ colours and let $G_i$ denote the graph consisting of edges of colour $i$. Then*

$$H(P) = \sum_{i=1}^{k} H(G_i, P) \tag{3.10}$$

*for every $P$ on $V = \{1, 2, \ldots, n\}$ if and only if each $G_i$ is perfect and there is no three-coloured triangle.*

**Proof.** Each $G_i$ should be perfect by Theorem B, since each $G_i$ is strongly splitting by (3.10). Observe that (3.10) holds if and only if for $F_j = \bigcup_{i=1}^{j} G_i$

$$H(F_{j+1}, P) = H(F_j, P) + H(G_{j+1}, P)$$

for every $P$ and every $j = 1, \ldots, k-1$. This means that $F_j$ and $G_{j+1}$ should form a perfect couple for $j = 1, \ldots, k-1$. Applying Theorem 1 to these couples of graphs the statement follows. ∎

The following theorem shows that our general notion of perfectness is also a hereditary property in some sense.

**Theorem 2.** *Let $F$ and $G$ form a perfect couple. Then $T = F \cup G$ is perfect if and only if both $F$ and $G$ are perfect.*

**Proof.** To prove the 'if' part, let $K_n$ be the complete graph on $n$ vertices. The result of Cameron, Edmonds and Lovász [2] states that for the edge-disjoint graphs $L_1$, $L_2$, $L_3$ with $\bigcup_{i=1}^{3} L_i = K_n$ the perfectness of $L_1$ and $L_2$ implies the perfectness of $L_3$ if there is no triangle in $K_n$ having an edge in each $L_i$. Using this for $F = L_1$, $G = L_2$, $\overline{T} = L_3$, Theorem 1 implies that once $F$ and $G$ form a perfect couple then the perfectness of $F$ and $G$ implies the perfectness of $\overline{T}$. Then by Lovász' Perfect Graph Theorem [14] $T$ is perfect.

On the other hand, once $T$ is perfect, Theorem B implies that for every $P$

$$H(P) = H(\overline{T}, P) + H(T, P). \tag{3.11}$$

Since $F$ and $G$ are a perfect couple,

$$H(T, P) = H(F, P) + H(G, P). \tag{3.12}$$

The previous two equalities imply

$$H(P) = H(\overline{T}, P) + H(F, P) + H(G, P), \tag{3.13}$$

i.e., we have additivity of graph entropy in the strong sense for the triple $F$, $G$, $\overline{T}$. Then, because of the sub-additivity of graph entropy Theorem B implies that each of the graphs involved must be perfect. ∎

**Remark.** Because of the above reasons one could conjecture that the union of the two graphs forming a perfect couple should always be perfect. However, this is not true since imperfect graphs can have autonomous independent sets. Substituting any graph on such an independent set we get a couterexample. The simplest one is

$$V = \{1,2,3,4,5,6\}$$
$$E(G) = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\},\{6,5\},\{6,2\}\}$$

while

$$E(F) = \{\{1,6\}\}.$$

It follows from the Substitution Lemma that $F$ and $G$ form a perfect couple although $G$ is imperfect. On the other hand, $F \cup G$ is also imperfect.

**Remark.** (*Converse of the Cameron–Edmonds–Lovász theorem*) Cameron and Edmonds [1] have proved a "partial converse" of the result in [2], saying that all so-called *prime-$\Lambda$ subgraphs* of a perfect graph are perfect. (A prime-$\Lambda$ subgraph is a subgraph formed by the edges of an edge-class in Gallai's sense; a *$\Lambda$-subgraph* is the union of prime-$\Lambda$ subgraphs.) Their theorem is a "partial converse" because writing $\Lambda$-subgraphs instead of prime-$\Lambda$ subgraphs, the above statement would not be true anymore, since any (possibly imperfect) subgraph of the complete graph is a $\Lambda$-subgraph. Our Theorem 2 (combined with Theorem 1) shows that essentially this is the only exceptional case, i.e. if a $\Lambda$-subgraph of a perfect graph is not perfect then it must induce an imperfect graph on some clique. Indeed, if $F \cup G$ is perfect but $F$ or $G$ is not, then $F$ and $G$ do not form a perfect couple. Assuming that $F$ and $G$ are $\Lambda$-subgraphs of $F \cup G$, we obtain that the second condition in Theorem 1 is satisfied, therefore the first one must be violated.

**Remark.** (*Recursive construction of perfect couples*) One of the referees pointed out that Theorem 1 can be presented in a different way, building up perfect couples recursively as follows. There are two kinds of "elementary perfect couples": (1) complementary pairs of perfect graphs, and (2) an arbitrary graph with the empty graph as its pair. As the referee observed, the proof of Theorem 1 also yields that all perfect couples can be obtained by a sequence of substitutions, starting with an elementary perfect couple and substituting some elementary perfect couple in each step.

## 4. Normal couples

As a weakening of the concept of perfect couples, we introduce normal couples as follows.

**Definition.** Two graphs $F$ and $G$ on the same vertex set are said to form a normal couple if there exists a nowhere vanishing probability distribution $P$ such that

$$H(F,P) + H(G,P) = H(F \cup G, P) \tag{4.1}$$

First we give a necessary condition for $F$ and $G$ to be a normal couple.

**Lemma 3.** *If two graphs $F$ and $G$ on the same vertex set $V$ form a normal couple then there exist two coverings $L(F)$ and $L(G)$ of $V$ with maximal independent sets of $F$ and $G$, respectively, such that for every $Z \in L(F)$ and $Y \in L(G)$, $Y \cap Z \in S(F \cup G)$ i.e. $Y \cap Z$ is a maximal independent set in $F \cup G$.*

**Proof.** Assume that (4.1) holds for $F$ and $G$ and for some (positive) distribution $P$. Consider the coefficients $\beta(Y)$ and $\gamma(Z)$ of $\underline{b} \in VP(G)$ and $\underline{c} \in VP(F)$ for which the entropies $H(G,P)$ and $H(F,P)$, respectively, are attained. Setting

$$\alpha(I) = \sum_{I=Y\cap Z} \beta(Y)\gamma(Z)$$

and

$$a_i = \sum_{\substack{i\in I \\ I=Y\cap Z \text{ for} \\ \text{some } Y,Z}} \alpha(I)$$

it is clear that $\underline{a} \in VP(F \cup G)$. Now we obtain

$$\begin{aligned} H(G,P) + H(F,P) &= -\sum_{i\in V} p_i \log b_i - \sum_{i\in V} p_i \log c_i \\ &= -\sum_{i\in V} p_i \log \sum_{\substack{Y\in L(G)\\ Z\in L(F)\\ i\in Y\cap Z}} (\beta(Y)\gamma(Z)) \\ &= -\sum_{i\in V} p_i \log a_i \\ &\geq H(F\cup G). \end{aligned}$$

If $\underline{a}$ were not a maximal vertex of $VP(F \cup G)$ then it would not be a minimizing vector in (1.1) and therefore we could not have equality above. Thus, since (4.1) holds, $\underline{a}$ must be maximal and, by the definition of $\underline{a}$, this proves that the sets $Y \cap Z$ are maximal independent sets in $F \cup G$. ∎

It has been shown in [8] that every perfect graph is normal. Combining our Theorem 1 and Lemma 3 we immediately obtain the following stronger statement.

**Corollary 3.** *If for two edge-disjoint graphs $F$ and $G$ on a vertex set $V$ the two conditions of Theorem 1 hold then there exists coverings $L(F)$ and $L(G)$ with the properties described in Lemma 3.*

**Remark.** It is easy to check that if $V = \{x, y, z\}$, $F = (V, \{x, y\})$, and $G = (V, \{x, z\})$ then $F$ and $G$ do not satisfy the condition of Lemma 3, i.e., there are no coverings $L(F)$ and $L(G)$ with the required properties. Hence, by Lemma 3, $F$ and $G$ do not form a normal couple. This fact explains condition (ii) of Theorem 1 in a somewhat larger extent.

Unfortunately, the condition given in Lemma 3 is necessary but not sufficient. An example when coverings $L(F)$, $L(G)$ with the required property exist while $F$ and $G$ do not form a normal couple is when $F$ and $G$ both are paths of length 2 and their union is the path of length 4. To see that this is a counterexample indeed, one can use e.g. Proposition 5 below.

It follows from the results of [3] that for every maximal $\underline{a} \in VP(G)$ there exists a probability distribution $P$ on $V$ for which $\underline{a}$ is a minimizing vector in (1.1). So, from the proof of Lemma 3 it is clear that if the vector $\underline{a}$ can be chosen maximal then (4.1) holds for some $P$. Such an extra condition can help in checking particular cases; in general, however, one would like to have pure "graph theoretic" conditions like in Theorem 1. At present we do not have an analogous theorem for normal couples. Still, a result corresponding to Corollary 1 is easy to find as follows.

**Proposition 4.** *Let $G_1, \dots, G_k$ be graphs on the same vertex set $V$, and let their union be the complete graph on $V$. There exists a nowhere vanishing probability distribution $P$ on $V$ with*

$$H(P) = \sum_{i=1}^{k} H(G_i, P) \tag{4.2}$$

*if and only if there exist coverings $L_1, \dots, L_k$ of $V$, each $L_i$ consisting of independent sets of $G_i$, such that any $Y^{(1)} \in L_1, \dots, Y^{(k)} \in L_k$ have a non-empty intersection.*

**Proof.** Assume that those $L_i$ with the required intersection property exist. Take an arbitrary convex combination $\beta_i$ of the sets in $L_i$ for every $i$, with all coefficients positive. Define the probability $P(v)$ of vertex $v$ as

$$P(v) = \prod_{i=1}^{k} \Bigg( \sum_{v \in Y_j \in L_i} \beta_i(Y_j) \Bigg) := \prod_{i=1}^{k} b_i(v). \tag{4.2}$$

By the condition for the $L_i$'s, this is indeed a (nowhere vanishing) probability distribution. Then

$$\begin{aligned} H(P) &= - \sum_{v \in V} P(v) \log P(v) \\ &= - \sum_{v \in V} \Bigg( \sum_{i=1}^{k} P(v) \log b_i(v) \Bigg) \\ &= - \sum_{i=1}^{k} \sum_{v \in V} P(v) \log b_i(v) \\ &\geq \sum_{i=1}^{k} H(G_i, P). \end{aligned} \tag{4.3}$$

Here the inequality follows by the observation that $\underline{b}_i \in VP(G_i)$. By the subadditivity of graph entropy, however, equality must hold in (4.3).

The necessity of the condition follows similarly as that in Lemma 3. ∎

In the special case when $F \cup G$ is a perfect graph, already Proposition 4 and Theorem B give us the necessary and sufficient condition for $F$ and $G$ to be a normal couple.

**Proposition 5.** *If $T = F \cup G$ is a perfect graph then $F$ and $G$ form a normal couple if and only if the condition of Proposition 4 is satisfied for $G_1 = F$, $G_2 = G$, $G_3 = \overline{T}$ (and $k = 3$).*

**Proof.** By Theorem B and the perfectness of $T$,

$$H(T,P)+H(\overline{T},P)=H(P)$$

holds for every $P$. Thus, for any $P$,

$$H(F,P)+H(G,P)=H(F\cup G,P) \tag{4.4}$$

implies

$$H(F,P)+H(G,P)+H(\overline{T},P)=H(P), \tag{4.5}$$

and Proposition 4 can be applied if (4.4) holds. On the other hand, (4.5) also implies (4.4). ∎

One could guess that the union graph of a normal couple might always be perfect, and so Proposition 5 would solve the problem completely. This is far from being true, however: for instance, partitioning the edges of the pentagon into two classes, including paths of length two and three, respectively, the two (perfect) graphs obtained form a normal couple.

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