

# GRAPH–DIFFERENT PERMUTATIONS

**János Körner**

`korner@di.uniroma1.it`

”La Sapienza” University of Rome  
ITALY

**Claudia Malvenuto**

`claudia@di.uniroma1.it`

”La Sapienza” University of Rome  
ITALY

**Gábor Simonyi\***

`simonyi@renyi.hu`

Rényi Institute of Mathematics, Budapest  
HUNGARY

## Abstract

We strengthen and put in a broader perspective previous results of the first two authors on colliding permutations. The key to the present approach is a new non-asymptotic invariant for graphs.

---

\*Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) Nos. T046376, AT048826, and NK62321.

# 1 Introduction

In [4] the first two authors began to investigate the following mathematical puzzle. Call two permutations of  $[n] := \{1, \dots, n\}$  *colliding* if, represented by linear orderings of  $[n]$ , they put two consecutive elements of  $[n]$  somewhere in the same position. For the maximum cardinality  $\rho(n)$  of a set of pairwise colliding permutations of  $[n]$  the following conjecture was formulated.

**Conjecture 1** ([4]) *For every  $n \in \mathbb{N}$*

$$\rho(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

It was proved that the right hand side expression above is actually an upper bound for  $\rho(n)$ , while the best lower bound given in [4] was a somewhat deceiving

$$35^{n/7-O(1)} \leq \rho(n). \tag{1}$$

The initial motivation for the present paper was to improve on the above lower bound. For this purpose we will put the original problem in a broader perspective leading to a new graph invariant that we believe to be interesting on its own. For brevity's sake let us call a graph *natural* if its vertex set is a finite subset of  $\mathbb{N}$ , the set of all positive integers and if the graph is simple (without loops and multiple edges). An infinite permutation of  $\mathbb{N}$  is simply a linear ordering of all the elements of  $\mathbb{N}$ . (Instead of *infinite permutations of  $\mathbb{N}$*  we will often say simply *infinite permutations* in the sequel.) For an arbitrary natural graph  $G = (V(G), E(G))$  we will call the infinite permutations  $\pi = (\pi(1), \pi(2), \dots, \pi(n), \dots)$  and  $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n), \dots)$  *G-different* if there is at least one  $i \in \mathbb{N}$  for which

$$\{\pi(i), \sigma(i)\} \in E(G).$$

(We will use the same expression for a pair of finite sequences if at some coordinate they contain the two endpoints of an edge of  $G$ .) Let  $\kappa(G)$  be the maximum cardinality of a set of infinite permutations any two elements of which are  $G$ -different. (It is easy to see that the finiteness of  $G$  implies that this number is finite as well, see Lemma 1 below.) Clearly, the value of  $\kappa$  is equal for isomorphic natural graphs. In this paper we will analyze this quantity for some elementary graphs and will apply some of the results to improve on the earlier estimates on  $\rho(n)$ . We have been able to determine the value of  $\kappa(G)$  only for some very small or simply structured graphs  $G$ . Thus, to further simplify matters, we ask questions about the extremal values of  $\kappa$  for graphs with a fixed number of edges (and, eventually, vertices). We define

$$K(\ell) = \max\{\kappa(G) ; |E(G)| = \ell\} \tag{2}$$

and

$$k(\ell) = \min\{\kappa(G) ; |E(G)| = \ell\} \quad (3)$$

as well as

$$K(v, \ell) = \max\{\kappa(G) ; |V(G)| = v, |E(G)| = \ell\} \quad (4)$$

We conjecture

**Conjecture 2** *For every  $\ell \in \mathbb{N}$*

$$K(\ell) = 3^\ell.$$

In fact, we will show that  $K(\ell)$  lies between  $3^\ell$  and  $4^\ell$  for every natural number  $\ell$ . We will also see that  $k(\ell)$  is linear in  $\ell$ .

As we will explain, the values of  $\kappa(P_r)$ , where  $P_r$  is the  $r$ -vertex path, are relevant when investigating colliding permutations. Giving a lower bound on  $\kappa(P_4)$  the lower bound of (1) will be improved to  $10^{n/4-O(1)}$ .

Also, we will discuss the following conjecture and its relation to Conjecture 1:

**Conjecture 3** *For every even  $v \in \mathbb{N}$*

$$K(v, v-1) = \binom{v+1}{\lfloor \frac{v+1}{2} \rfloor}.$$

The concept of graph-different sequences from a fixed alphabet goes back to Shannon's classical paper on zero-error capacity [7]. This fundamental work has inspired much of information theory ever since while in combinatorics it led Claude Berge to define the intriguing class of perfect graphs, see [1], cf. also [2]. As the reader knows, Berge's conjectures about the structure of perfect graphs (cf. [2]) have had a tremendous impact on the evolution of combinatorics and are by now important and deep theorems at the center of the field. As explained in the survey [5], a large body of problems in extremal combinatorics can be treated as zero-error problems in information theory. For the relationship of the present problems to zero-error information theory we refer to [4].

## 2 Natural graphs and infinite permutations

Let  $G$  be a natural graph and let again  $\kappa(G)$  be the maximum cardinality of a set of infinite permutations any two elements of which are  $G$ -different, provided that this number is finite. It is easy to see that this is always the case. Let  $\chi(G)$  denote the chromatic number of graph  $G$ .

**Lemma 1** *For every natural graph*

$$\kappa(G) \leq (\chi(G))^{|V(G)|}$$

*holds.*

**Proof.** Let us consider a proper coloring  $c : V(G) \rightarrow \{1, \dots, \chi(G)\}$  of  $G$ . Let us write  $v = |V(G)|$  and denote by  $W = [\chi(G)]^{\mathbb{N}}$  the set of infinite sequences over the alphabet  $\{1, \dots, \chi(G)\}$ . Let  $\mu$  be the uniform probability measure on  $W$ . Let us consider a set  $C$  of pairwise  $G$ -different permutations. We assign to any  $\pi \in C$  the set  $W(\pi)$  of all those sequences of  $W$  that for all  $u \in V(G)$  have the element  $c(u)$  in the position where  $\pi$  contains  $u$ . By our hypothesis on  $C$ , the sets  $W(\pi)$  are pairwise disjoint for the different elements  $\pi \in C$  whence,

$$1 = \mu(W) \geq \sum_{\pi \in C} \mu(W(\pi)) = \sum_{\pi \in C} \chi(G)^{-v} = |C| \chi(G)^{-v}.$$

□

In the rest of this section we first investigate  $K(\ell)$  and  $k(\ell)$ . Subsequently our new lower bound on  $\rho(n)$  will be proved.

Let us denote by  $S(G)$  the set of non-isolated vertices of the graph  $G$ . We introduce a graph transformation that increases the value of  $\kappa$ .

**Proposition 1** *Let  $F$  and  $G$  be two graphs with  $G$  obtained from  $F$  upon deleting an arbitrary edge in  $E(F)$  followed by the addition of two new vertices to  $V(F)$  so that the latter form an additional edge in  $G$ . Then*

$$\kappa(F) \leq \kappa(G).$$

**Proof.** Let us consider the  $m = \kappa(F)$  pairwise  $F$ -different infinite permutations of an arbitrary optimal configuration for  $F$ . Let  $t$  be large enough for the initial prefixes of length  $t$  of these infinite sequences to be pairwise  $F$ -different and let  $q$  be the largest integer appearing in their coordinates. By the finiteness of  $\kappa(F)$  such  $t$  and  $q$  exist. Without restricting generality we can suppose that the new edge of  $G$  is  $\{c, d\}$  with both  $c$  and  $d$  being strictly larger than  $q$ . We also suppose that the edge we will delete is  $\{a, b\} \in E(F)$ . Let us now suffix to each of our sequences a new sequence of the same length  $t$  where the suffix to a sequence  $x_1 x_2 \dots x_t$  is obtained from it by substituting every  $a$  with  $c$  and every  $b$  with  $d$  while the remaining coordinates are defined in an arbitrary manner but in a way that the coordinates of the overall sequence of length  $2t$  be all different. Clearly, the  $m$  new sequences of length  $2t$  are  $G$ -different. The rest is obvious, since we can complete the new sequences to yield infinite permutations any way we like. □

A straightforward consequence of the previous proposition is the following.

**Corollary 1**  $K(\ell) = \kappa(\ell K_2)$ . □

Thus we know that  $K(\ell)$  is achieved by  $\ell$  independent edges. It seems equally interesting to determine which graphs achieve  $k(\ell)$ . At first glance one might think that  $S(F) \subseteq S(G)$  implies  $\kappa(F) \leq \kappa(G)$ , but this is false. In particular, complete graphs do not have minimum  $\kappa$  among graphs with the same number of edges. Yet, determining their  $\kappa$  value seems an interesting problem. As we will see, the right guess for what graphs

achieve  $k(\ell)$  turn out to be stars, at least for  $\ell$  not too small. Below we will study the value of  $\kappa$  for complete graphs, stars, and paths. In particular, path graphs will take us back to the original puzzle about colliding permutations.

**Proposition 2** *For the complete graph  $K_n$  on  $n$  vertices*

$$\frac{(n+1)!}{2} \leq \kappa(K_n).$$

**Proof.** Consider the set of even permutations of  $[n+1]$  and suppose  $V(K_n) = [n]$ . One can observe that these permutations are  $K_n$ -different. Indeed, if two arbitrary permutations of  $[n+1]$  are not  $K_n$ -different, then they differ only in positions in which for some fixed  $i \in [n]$  one has  $n+1$  and the other has  $i$ . Thus any of these two permutations can be obtained from the other by exchanging the positions of  $n+1$  and the corresponding  $i$ . But then the two permutations have different parity and in particular they cannot both be even. In particular, the undesired relation does not occur between even permutations and this gives us  $\frac{(n+1)!}{2}$  permutations of  $[n+1]$  that are  $K_n$ -different. Next extend each of these permutations to infinite ones by suffixing the remaining natural numbers in an arbitrary order.  $\square$

**Proposition 3** *For the graph of  $\ell$  independent edges we have*

$$3^\ell \leq \kappa(\ell K_2) \leq 4^\ell.$$

**Proof.** Notice that the graph  $\ell K_2$  has chromatic number two and its number of vertices is  $2\ell$ , whence our upper bound follows by Lemma 1.

To prove the lower bound, let us denote the edge set of our graph by  $E(\ell K_2) = \{\{1, 2\}, \{3, 4\}, \dots, \{2\ell - 1, 2\ell\}\}$ . Consider the set of cyclic permutations  $C_1 = \{(12\star), (2\star 1), (\star 12)\}$  and for every  $1 < i \leq \ell$  the sets  $C_i$  obtained from  $C_1$  by replacing 1 with  $2i - 1$  and 2 with  $2i$ . It is clear that for every  $i \in [\ell]$  any two of the three ministrings in  $C_i$  "differ" in the edge  $\{2i - 1, 2i\}$  of our graph  $\ell K_2$ , meaning that they have somewhere in the same position the two different endpoints of this edge. But this means that the  $3^\ell$  strings in their cartesian product

$$C = \times_{i=1}^{\ell} C_i$$

are pairwise  $\ell K_2$ -different as requested. Replacing the symbol  $\star$  in our strings in an arbitrary order with the different numbers from  $[3\ell] - [2\ell]$  we obtain  $3^\ell$  permutations of  $[3\ell]$  that continue to be pairwise  $\ell K_2$ -different. The extension to infinite permutations is as always.  $\square$

The only infinite class of graphs for which we are able to completely determine  $\kappa$  are stars, i.e., the complete bipartite graphs  $K_{1,r}$ . We have

**Proposition 4** *For every  $r$*

$$\kappa(K_{1,r}) = 2r + 1.$$

**Proof.** By Lemma 1 we know that  $\kappa(K_{1,r}) < \infty$ . Let us denote its value by  $m$ . Let us consider the vertices of  $K_{1,r}$  to be the elements of  $[r+1]$  and let 1 be the "central" vertex of degree  $r$ . It is obvious that in a set of  $m$  sequences (infinite permutations) achieving the maximum we are looking for all the sequences must have the central vertex 1 in a different position. Let us consider our  $m$  sequences as vertices of a directed graph  $T$  in which  $(a, b) \in E(T)$  if the sequence corresponding to  $a$  has a  $j \in \{2, \dots, r+1\}$  in the same position where the 1 of the sequence corresponding to  $b$  is placed. Then, by definition the directed graph  $T$  must contain a tournament, implying that

$$|E(T)| \geq \binom{m}{2}.$$

On the other hand, every  $a \in V(T)$  has at most  $r$  outgoing edges. This means that

$$|E(T)| \leq mr.$$

Comparing the last two inequalities we get

$$m \leq 2r + 1.$$

To prove a matching lower bound, consider the following set of permutations of  $[2r+1]$ . For every  $i \in [2r+1]$  let us define the coordinates of the  $i^{\text{th}}$  sequence  $x_1(i)x_2(i) \dots x_{2r+1}(i)$  by  $x_i(i) = 1$  and, in general,  $x_{i+j}(i) = j+1$  for any  $0 \leq j \leq r$  where all the coordinate indices are considered modulo  $2r+1$ . The remaining coordinates are defined in an arbitrary manner so that the resulting sequences define permutations of  $[2r+1]$ . It is easily seen that this is a valid construction. In fact, observe that for any of our sequences the "useful" symbols, those of  $[r+1]$ , corresponding to the vertices of the star graph, occupy  $r+1$  "cyclically" consecutive coordinates, forming cyclical intervals. Since  $2(r+1) > 2r+1$ , these intervals are pairwise intersecting, and thus for any two of them there must be a coordinate in the intersection for which the "left end" of one of the intervals is contained in the other. The resulting permutations can be considered as prefixes of infinite permutations in the usual obvious way.  $\square$

Now we are ready to return to the problem of determining  $k(\ell)$ , at least for large enough  $\ell$ . The following easy lemma will be needed.

**Lemma 2** *If a finite graph  $F$  contains vertex disjoint subgraphs  $F_1, \dots, F_s$ , then*

$$\kappa(F) \geq \prod_{i=1}^s \kappa(F_i).$$

**Proof.** The proof is a straightforward generalization of the construction given in the proof of Proposition 3. Let  $\hat{C}_i$  be a set of  $\kappa(F_i)$  infinite sequences that are obtained from  $\kappa(F_i)$  pairwise  $F_i$ -different permutations of  $\mathbb{N}$  by substituting all natural numbers

$i \notin V(F_i)$  by a  $\star$ . As the sequences in  $\hat{C}_i$  contain only a finite number of elements different from  $\star$ , we can take some finite initial segment of all these sequences that already contain all elements of  $V(F_i)$ . Let  $C_i$  be the set of these finite sequences. Now consider the set

$$C := \times_{i=1}^s C_i$$

of finite sequences that each contain all vertices in  $\bigcup_{i=1}^s V(F_i)$  exactly once and finitely many  $\star$ 's. These sequences are also pairwise  $F_i$ -different for some  $F_i$ , thus they are pairwise  $F$ -different. By construction, their number is  $\prod_{i=1}^s \kappa(F_i)$ . Extending them to infinite permutations of  $\mathbb{N}$  the statement is proved.  $\square$

It is straightforward from the previous lemma that if  $F + G$  denotes the vertex disjoint union of graphs  $F$  and  $G$  then  $\kappa(F + G) \geq \kappa(F)\kappa(G)$ . We do not know any example for strict inequality here. If equality was always true that would immediately imply Conjecture 2.

Now we use Lemma 2 to prove our main result on  $k(\ell)$ .

**Proposition 5** *Let  $G$  be a natural graph with  $n := |S(G)| > 20$  and  $|E(G)| = \ell$ . Then*

$$\kappa(G) \geq 2\ell + 1.$$

*The value of  $k(\ell)$  is achieved by the graph  $K_{1,\ell}$  whenever  $\ell > 150$ .*

**Proof.** Let  $G$  be a graph as in the statement and let  $\nu = \nu(G)$  denote the size of a largest matching in  $G$ .

First assume that  $\nu \geq n/4$ . Then by Proposition 3 and the obvious monotonicity of  $\kappa$  we have

$$\kappa(G) \geq \kappa(\nu K_2) \geq 3^\nu.$$

Since  $G$  is simple, we have  $\ell \leq \binom{n}{2}$ , thus  $k(\ell) \leq \kappa\left(K_{1,\binom{n}{2}}\right) = n(n-1) + 1$ . So in this case (when  $\nu \geq n/4$ ) it is enough to prove that

$$3^{\lceil n/4 \rceil} \geq n(n-1) + 1$$

holds. This is true if  $n > 20$ .

Next assume that  $3 \leq \nu < n/4$ . Consider a largest matching of  $G$  consisting of edges  $\{u_{2i-1}, u_{2i}\}$  with  $i = 1, \dots, \nu$ . The set  $U := \{u_1, \dots, u_{2\nu}\}$  covers all edges of  $G$  thus  $\ell \leq \binom{2\nu}{2} + 2\nu(n - 2\nu)$ . So we have

$$k(\ell) \leq \kappa(K_{1,\ell}) \leq 2 \left[ \binom{2\nu}{2} + 2\nu(n - 2\nu) \right] + 1$$

in this case. On the other hand, for each vertex  $a \in S(G) \setminus U$  there is an edge  $\{a, u_i\}$  for some  $i$ . We also know that if  $a$  and  $b$  are two distinct vertices in  $S(G) \setminus U$  and one of them is connected to  $u_{2j-1}$  (resp.  $u_{2j}$ ) for some  $j$ , then the other one cannot be connected to  $u_{2j}$

(resp.  $u_{2j-1}$ ) since otherwise replacing the matching edge  $\{u_{2j-1}, u_{2j}\}$  with the other two edges of the path formed by the vertices  $a, u_{2j-1}, u_{2j}, b$  would result in a larger matching, a contradiction. Choosing an edge for each  $a \in S(G) \setminus U$  that connects it to a vertex in  $U$ , we can form vertex disjoint star subgraphs  $K_{1,\ell_1}, \dots, K_{1,\ell_\nu}$  of  $G$ , where  $\ell_i \geq 1$  for all  $i$  and  $\sum_{i=1}^{\nu} \ell_i = n - \nu$ . Then by Lemma 2 and Proposition 4 we have  $\kappa(G) \geq \prod_{i=1}^{\nu} (2\ell_i + 1)$ . The latter product is minimal (with respect to the conditions on the  $\ell_i$ 's) if one  $\ell_i$ , say  $\ell_1$ , equals to  $n - 2\nu + 1$  and  $\ell_2 = \dots = \ell_\nu = 1$ . Thus it is enough to prove that

$$3^{\nu-1}(2(n - 2\nu + 1) + 1) > 2 \left[ \binom{2\nu}{2} + 2\nu(n - 2\nu) \right] + 1$$

as the left hand side is a lower bound on  $\kappa(G)$  while the right hand side is an upper bound on  $k(\ell)$ . The latter inequality would be implied by

$$3^{\nu-1}(n - 2\nu + 1) > \binom{2\nu}{2} + 2\nu(n - 2\nu) = \nu(2n - 2\nu - 1)$$

which, in turn, is equivalent to

$$\frac{3^{\nu-1}}{\nu} > \frac{2n - 2\nu - 1}{n - 2\nu + 1}$$

The left hand side of this last inequality is at least 3 if  $\nu \geq 3$ , while the right hand side is strictly less than 3 for  $\nu \leq n/4$ .

The only case not yet covered is that of  $\nu < 3$ . For  $\nu = 1$  there is nothing to prove since then  $G$  itself is a star. If  $\nu = 2$ , then let a largest matching be formed by the two edges  $\{u_1, u_2\}$  and  $\{u_3, u_4\}$ , while once again let  $U$  denote the union of their vertices. Let  $a_1, \dots, a_{n-4}$  be the rest of the non-isolated vertices of  $G$  and note that  $n - 4 > 16$ . Assume some  $a_i$  is connected to both  $u_1$  and  $u_2$  yielding a triangle. Then no  $a_j, j \neq i$  can be connected to either of  $u_1$  or  $u_2$ , otherwise we could form a larger matching. For similar reasons, if any  $a_j$  is connected to  $u_3$  then no  $a_s, s \neq j$  can be connected to  $u_4$ . (If some  $a_i$  forms a triangle with  $u_1, u_2$  and some  $a_j$  with  $u_3$  and  $u_4$ , then the remaining vertices  $a_s$  must be isolated implying  $n \leq 6$ , a contradiction.) Thus if  $a_i$  is connected to both  $u_1$  and  $u_2$ , then the rest of the  $a_j$ 's form a star centered at either  $u_3$  or  $u_4$ . Thus in this case, using again Lemma 2, Propositions 2 and 4 imply  $\kappa(G) \geq 12[2(n - 4) + 1] = 24n - 84$ . The foregoing also implies  $\ell \leq n + 4$ , thus  $k(\ell) \leq 2n + 9 < 24n - 84$ , whenever  $n \geq 5$ . Clearly, the situation is similar if we exchange the role of the two matching edges.

Assuming that no triangle is formed, we can again attach each vertex in  $S(G) \setminus U$  to one of the edges  $\{u_1, u_2\}$  and  $\{u_3, u_4\}$ , whichever it is connected to. Two vertex disjoint stars can be formed this way establishing the lower bound  $\kappa(G) \geq 3(2(n - 3) + 1) = 6n - 15$ . For the number of edges we now get  $\ell \leq 6 + 2(n - 4) = 2n - 2$  since the graph induces at most 6 edges on  $U$ . Thus we have  $k(\ell) \leq 4n - 3$  which is less than  $6n - 15$  if  $n > 6$ . This completes the proof of the first statement.

If a simple graph has at most 20 vertices then its number of edges is at most 190, so the second statement immediately follows from the first one if  $\ell > 190$ . If the graph contains



a  $K_6$  subgraph, then by Proposition 2 we have  $\kappa(G) \geq 7!/2 > 381 = \kappa(K_{1,190}) \geq k(\ell)$  if  $\ell \leq 190$ . Thus we may assume  $K_6 \not\subseteq G$  and this implies by Turán's theorem that  $\ell \leq 160$  if  $n \leq 20$ . But  $\kappa(K_{1,160}) = 321 \leq 6!/2$ , so if the conclusion is not true, we may also assume that  $G$  has no  $K_5$  subgraph. Applying Turán's theorem again, this gives  $\ell \leq 150$  for  $n \leq 20$ . Thus the statement is true whenever  $\ell > 150$ .  $\square$

**Remark 1.** We are quite convinced that the statement of Proposition 5 holds without any restriction on  $n$  or  $\ell$ . Some improvement on our threshold on  $\ell$  is easy to obtain. It seems to us, however, that proving the statement in full generality either leads to tedious case checkings or needs some new ideas.  $\diamond$

The problem of determining  $\kappa$  seems interesting in itself, moreover, it helps to obtain better bounds for the original question on colliding permutations. To explain this, we introduce a notion connecting the two questions. Let  $\kappa(G, n)$  be the maximum number of pairwise  $G$ -different permutations of  $[n]$ . Clearly,

$$\kappa(G) = \sup_n \kappa(G, n). \quad (5)$$

Notice that by the finiteness of  $\kappa(G)$  the supremum above is always attained, so we could write maximum instead. Further, for the graph  $P_r$ , the path on  $r$  vertices, we have the following.

**Lemma 3** *For every  $n > m > r$  the function  $\rho$  satisfies the recursion*

$$\rho(n) \geq \kappa(P_r, m)\rho(n - r).$$

**Proof.** We will call two arbitrary sequences of integers colliding if they have the same length and if somewhere in the same position they feature integers differing by 1. By the definition of  $\kappa(P_r, m)$  we can construct this many sequences of length  $m$  such that in each of them every vertex of  $P_r$  appears exactly once, the other positions are occupied by the "dummy" symbol  $\star$  and moreover these sequences are pairwise  $P_r$ -different. The latter implies that these sequences are pairwise colliding. Furthermore, we have, also by definition,  $\rho(n - r)$  permutations of  $[n - r]$  that are pairwise colliding. Let us "shift" these permutations by adding  $r$  to all of their coordinates. The new set of permutations of the set  $r + [n - r] = [r + 1, n]$  maintains the property that its elements are pairwise colliding. Next we execute our basic operation of "substituting" the permutations of the second set into those coordinates of any sequence  $\mathbf{x}$  from the first set where the sequence  $\mathbf{x}$  has a star. More precisely, consider any sequence  $\mathbf{x} = x_1x_2 \dots x_m$  from our first set and let  $S(\mathbf{x}) \in \binom{[m]}{m-r}$  be the set of those coordinates which are occupied by stars. Let further  $\mathbf{y} = y_1y_2 \dots y_{n-r}$  be an arbitrary sequence from our second set, i.e., a permutation of  $[r + 1, n]$ . The sequence  $\mathbf{z} = \mathbf{y} \rightarrow \mathbf{x}$  is a sequence of length  $n$  in which the first  $m$  coordinates are defined in the following manner. We have the equality  $z_i = x_i$ , if  $i \leq m$  and  $i \notin S(\mathbf{x})$ . Suppose further that  $S(\mathbf{x}) = \{j_1, j_2, \dots, j_{m-r}\}$ . In the  $j_k^{\text{th}}$  position we replace the symbol  $\star$  by  $y_k$ . (For  $i > m$  we set  $z_i = y_{i-r}$ .) Clearly, the resulting

sequence is a permutation of  $[n]$ . Further, the so obtained  $\kappa(P_r, m)\rho(n - r)$  permutations are pairwise colliding.  $\square$

Observe next the following equality.

**Lemma 4**

$$\kappa(P_4, 5) = 10.$$

**Remark 2.** The existence of 10 permutations of  $\{1, \dots, 5\}$  with the requested properties is implicit in [4] since the construction of the 35 colliding permutations of  $\{1, \dots, 7\}$  in that paper does contain such a set in some appropriate projection of its coordinates. Still, we prefer to give a simpler direct proof here. The argument for the reverse inequality is the same as that proving  $\rho(n) \leq \binom{n}{\lfloor n/2 \rfloor}$  in [4].  $\diamond$

**Proof.** Let us consider the 10 permutations of  $\{1, \dots, 5\}$  obtainable by considering the cyclic configurations of  $(1, 2)$ ,  $(4, 3)$  and the single element 5. We indeed have 10 different permutations by "cutting" in all the 5 possible ways both of the two cyclic configurations 3 building blocks can define. (So these are 12435, 24351, 43512, 35124, 51243 and similarly the five cyclic shifts of the sequence 43125.) Let us further consider the graph  $P_4$  (or, in fact,  $P_4 + K_1$ ) with vertex set  $\{1, \dots, 5\}$  and with edge set  $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ . In other words, consecutive numbers are adjacent vertices but 5 is isolated. It is easy to check that the 10 sequences above are  $P_4$ -different for the natural graph we defined. One can verify this by hand, yet let us give a more structured argument.

The statement is true for two of our permutations if any of the two blocks with two elements are featured in intersecting positions in the two respective permutations, or if the two different blocks of length two are completely overlapping at least once. We claim that one of these two things will always happen. In fact, suppose to the contrary to have two different cyclic configurations, say red and blue so that the only possible intersections of their blocks are intersections in one element between a  $(1, 2)$  and a  $(4, 3)$  of different colors. This implies that the four cyclic intervals of length two are contained in a cycle of length 5 with only two points covered twice. But this is impossible as their total length is 8 and  $8-5$  is strictly larger than 2.

To see that 10 is an upper bound it is enough to observe that the two even elements of  $\{1, \dots, 5\}$  cannot be placed in the same two positions in two permutations belonging to a set of  $P_4$ -different permutations of  $\{1, \dots, 5\}$ .  $\square$

The above construction gives the following improved lower bound for the exponential asymptotics of  $\rho(n)$ .

**Proposition 6**

$$\lim_{n \rightarrow \infty} \rho^{\frac{1}{n}}(n) \geq 10^{\frac{1}{4}}$$

**Proof.** A simple combination of our two preceding lemmas implies

$$\rho(n) \geq 10\rho(n - 4).$$

An iterated application of this inequality gives the desired result. □

To close this section, let us take another look at Lemma 4. We believe that in fact  $\kappa(P_4) = \kappa(P_4, 5) = 10$  and more generally,

$$\kappa(P_v) = \kappa(P_v, v + 1) = \binom{v + 1}{\lfloor \frac{v+1}{2} \rfloor}$$

for even values of  $v$ . The original conjecture (see Conjecture 1) for  $\rho(v)$  would be an immediate consequence of this conjecture. To see this, suppose first that  $v$  is even. Then  $\rho(v + 1) = \kappa(P_{v+1}, v + 1) \geq \kappa(P_v, v + 1)$  and this would imply Conjecture 1 for odd values of  $n$  right away. Now, since for even  $n$

$$\rho(n) = \kappa(P_n, n) \geq 2\kappa(P_{n-2}, n - 1)$$

and likewise,

$$\binom{n}{\frac{n}{2}} = 2 \binom{n - 1}{\lfloor \frac{n-1}{2} \rfloor}$$

the last two relations would lead us to settle the conjecture for  $n$  even. (The inequality above follows by putting an  $n$  to the end of each sequence in an optimal construction for  $\kappa(P_{n-2}, n - 1)$  and then double each sequence by considering also its variant one obtains by exchanging in it  $n - 1$  and  $n$ .)

We also believe that

$$K(v, v - 1) = \kappa(P_v).$$

As a combination of the two conjectures above we arrive at Conjecture 3 mentioned in the introduction.

## 3 Related problems

### 3.1 A graph covering problem

We show that determining  $K(\ell)$  is equivalent to a graph covering problem introduced below. The following standard definition is needed.

**Definition 1** *The (undirected) line graph  $L(D)$  of the directed graph  $D = (V, A)$  is defined by*

$$\begin{aligned} V(L(D)) &= A \\ E(L(D)) &= \{(a, b), (c, d)\} : b = c \text{ or } a = d \} \end{aligned}$$

Let  $\mathcal{L}$  denote the family of all finite simple graphs that are isomorphic to the line graph of some directed graph with possibly multiple edges. It is a standard combinatorial problem to ask how many graphs belonging to a certain family of graphs are needed to cover all edges of a given complete graph, see, e.g., [3]. We show that the problem of determining  $K(\ell)$  is equivalent to this problem for the family  $\mathcal{L}$ .

Let the minimum number of graphs in  $\mathcal{L}$  the edge sets of which together can cover the edges of the complete graph  $K_n$  be denoted by  $h(n)$ .

**Proposition 7** *For any  $M \in \mathbb{N}$ , the minimum number  $\ell$  for which  $K(\ell) \geq M$  is equal to  $h(M)$ .*

**Proof.** Consider a construction attaining  $K(\ell)$ , that is a graph  $G$  with  $\ell$  edges and  $K(\ell)$  infinite permutations that are  $G$ -different. Let this set of permutations be denoted by  $W$  and let  $\{a, b\}$  be one of the edges of  $G$ . Define a graph  $T_{a-b}$  on  $W$  as its vertex set where an edge is put between two permutations if and only if there is a position where one of them has  $a$  while the other has  $b$ . In other words, the two permutations are  $G$ -different by the edge  $\{a, b\}$ . Consider the graphs  $T_{a-b}$  for all edges of  $G$ . These all have the same vertex set, while the union of their edge sets clearly covers the complete graph  $K_M$ , where  $M = K(\ell)$ .

Next we show that all the graphs  $T_{a-b}$  belong to  $\mathcal{L}$ . To this end fix an edge  $\{a, b\} \in V(G)$  and consider a graph  $D_{a-b}$  with its vertex set  $V(D_{a-b})$  consisting of those positions where any of the permutations in  $W$  has a non-isolated vertex of  $G$ . Since  $G$  and  $W$  are finite, so is  $V(D_{a-b})$ . For each element of  $W$  we define an edge of  $D_{a-b}$ . For  $\sigma \in W$ , let  $i$  and  $j$  be the two positions where  $\sigma$  contains  $a$  and  $b$ , respectively. Then let  $\sigma$  be represented by the directed edge  $(i, j)$  in  $D_{a-b}$ . (If there is another permutation in  $W$  with  $a$  and  $b$  being in the same positions as in  $\sigma$  then we have another arc  $(i, j)$  in  $D_{a-b}$  for this other permutation. Thus  $D_{a-b}$  is a directed multigraph.) Now it follows directly from the definitions that  $T_{a-b} = L(D_{a-b})$ , thus  $T_{a-b}$  is indeed the line graph of a digraph. Together with the previous paragraph this proves  $h(K(\ell)) \leq \ell$ .

For the reverse inequality consider a covering of  $K_M$  with  $h(M)$  graphs belonging to  $\mathcal{L}$ . Let the line graphs in this covering be  $L_1, \dots, L_{h(M)}$ . We may assume that  $V(L_i) = [M]$  for all  $i$  by extending the smaller vertex sets through the addition of isolated points. Let  $D_1, \dots, D_{h(M)}$  be directed graphs satisfying  $L_i = L(D_i)$  for all  $i$ . (Such  $D_i$ 's exist since  $L_i \in \mathcal{L}$ .) By  $E(D_i) = V(L_i) = [M]$  we can consider the edges of all  $D_i$ 's labelled by  $|E(D_i)|$  elements of  $1, \dots, M$ . (If  $L_i$  had some isolated vertices then the corresponding labels are not used.) Using these digraphs we define  $M$  permutations  $\sigma_1, \dots, \sigma_M$  that are  $G$ -different for the graph  $G = \ell K_2$  with  $\ell = h(M)$ . For all  $i$  define  $t_i = |V(D_i)|$  and identify  $V(D_i)$  with  $[t_i]$ . Consider  $D_1$ . If  $D_1$  has an edge labelled  $r$  and this edge is  $(i, j)$ , then put a 1 in position  $i$  of  $\sigma_r$  and put a 2 in position  $j$  of  $\sigma_r$ . Do similarly for all edges of  $D_1$ . Then consider  $D_2$ . If it has an edge labelled  $r$  which is  $(i', j')$  then put a 1 in position  $t_1 + i'$  of  $\sigma_r$  and put a 2 in position  $t_1 + j'$  of  $\sigma_r$ . In general, if  $D_s$  has an edge labelled  $r$  which is  $(a, b)$  then put a  $2s - 1$  in position  $(\sum_{k=1}^{s-1} t_k) + a$  and a  $2s$  in position

$(\sum_{k=1}^{s-1} t_k) + b$  of  $\sigma_r$ . When this is done for all edges of all  $D_i$ 's then extend the obtained partial sequences to infinite permutations of  $\mathbb{N}$  in an arbitrary manner. This way one obtains  $M$  permutations that are pairwise  $G$ -different. To see this consider two of these permutations, say,  $\sigma_q$  and  $\sigma_r$ . Look at the edge  $\{q, r\}$  of our graph  $K_M$  that was covered by line graphs. Let  $L_i$  be the line graph that covered the edge  $\{q, r\}$ . Then  $D_i$  has an edge labelled  $q$  and another one labelled  $r$  in such a way that the head of the one is the tail of the other. This common point of these two edges defines a position of  $\sigma_q$  and  $\sigma_r$  where one of them has  $2i - 1$  while the other has  $2i$  making them  $G$ -different.  $\square$

**Remark 3.** We note that the first part of the above proof makes no reference to the graphs  $\ell K_2$ , yet it leads to another proof of the inequality  $K(\ell) \leq 4^\ell$ . Our earlier proof of this fact in Proposition 3 relied on Corollary 1. Here we sketch a different proof. By Proposition 7 it is enough to prove  $h(M) \geq \log_4 M$ . Consider a line graph  $L$  of a digraph  $D$  with  $|V(D)| = t, |E(D)| = M$ . Let  $\hat{K}_t$  be the directed graph on  $t$  vertices having an edge between any two different vertices in both directions.  $D$  can certainly be obtained by deleting some (perhaps zero) edges of  $\hat{K}_t$  and multiplying some (perhaps zero) of its edges. Thus  $L(D)$  can be obtained by multiplying some vertices of a subgraph of  $L(\hat{K}_t)$ . (Multiplying a vertex means substituting it by an independent set of size larger than 1 in such a way that the out-neighbourhoods and in-neighbourhoods of all vertices in this independent set are the same as the corresponding neighbourhood of the original vertex.) This implies that the fractional chromatic number  $\chi_f(L(D))$  of  $L(D)$  is bounded from above by  $\chi_f(L(\hat{K}_t))$ . (For the notion and basic properties of the fractional chromatic number we refer to [6].) The graph  $L(\hat{K}_t)$  is vertex transitive so its fractional chromatic number is equal to  $|V(L(\hat{K}_t))|/\alpha(L(\hat{K}_t))$ , where  $\alpha(F)$  stands for the independence number of graph  $F$ . The latter ratio is bounded from above by 4 as  $\hat{K}_t$  contains  $\lfloor t/2 \rfloor \cdot \lceil t/2 \rceil$  edges that form a complete bipartite subgraph and give rise to pairwise independent vertices in the line graph. So  $\chi_f(L(D)) \leq 4$ . Now let  $L_1, \dots, L_h$  be a minimal collection of line graphs (of directed graphs) covering  $K_M$ . It is easy to show that we must have  $\prod_{i=1}^h \chi_f(L_i) \geq \chi_f(\bigcup_{i=1}^h L_i) \geq \chi_f(K_M) = M$ . Having  $\chi_f(L_i) \leq 4$  for all  $i$  this implies  $h \geq \log_4 M$ .  $\diamond$

## 3.2 Fixed suborders

It seems worthwhile to revisit the problem of the determination of  $\kappa(G)$  for the restricted class of infinite permutations in which the vertices of  $G$  appear in a predetermined order. We will study this problem in the case of complete graphs. Without restricting generality, we can suppose that the fixed order is the natural one.

Let  $\kappa_{id}(K_n)$  denote the maximum number of infinite permutations of  $\mathbb{N}$  that are  $K_n$ -different and contain the first  $n$  positive integers in their natural order.

**Proposition 8** *For every  $n \in \mathbb{N}$*

$$\kappa_{id}(K_n) \geq C_n$$

holds, where  $C_n = \frac{1}{n+1} \binom{2n}{n}$  is the  $n^{\text{th}}$  Catalan number.

**Proof.** For  $n = 1, 2$  we have equality:  $\kappa_{id}(K_1) = 1, \kappa_{id}(K_2) = 2$ . Set  $a_0 = 1$  and  $a_n := \kappa_{id}(K_n)$ . It is enough to prove that the numbers  $a_n$  satisfy the inequality

$$a_{n+1} \geq \sum_{i=0}^n a_i a_{n-i}$$

that has the well-known recursion of Catalan numbers on its right hand side. We will look at our infinite permutations as infinite sequences consisting of infinitely many  $\star$ 's and one of each of the symbols  $1, 2, \dots, n$ , where the  $\star$ 's refer to all other symbols. Clearly, only the positions of the elements of  $[n]$  are relevant with respect to the  $K_n$ -difference relation. Thus we will define the positions of the elements of  $[n]$  and then let the  $\star$ 's be substituted by the other numbers in any way that will result in infinite permutations of  $\mathbb{N}$ .

Our construction is inductive. Assume that we already know that  $a_k \geq C_k$  holds for  $k \leq n$  and thus it suffices to prove it for  $n + 1$ . Fix a position of our permutations which is "far away", meaning that it is far enough for having enough earlier positions for the following construction. Call this position  $j$ . For each  $i = 0, \dots, n$  we construct  $a_i a_{n-i}$  sequences having  $i + 1$  at their position  $j$ . Any two of these sequences that have a different symbol at position  $j$  are  $K_n$ -different. For those sequences that have  $i + 1$  at their position  $j$  do the following. Consider a construction of  $a_i$  pairwise  $K_i$ -different sequences consisting of symbols  $1, \dots, i, \star$ , where the symbols in  $[i]$  are all used somewhere in the first  $j - 1$  positions (this is possible if  $j$  is chosen large enough). Take the first  $j - 1$  coordinates of all these sequences,  $a_{n-i}$  times each, and continue each of them with an  $i + 1$  at the  $j^{\text{th}}$  position. So we have  $a_i a_{n-i}$  sequences of length  $j$  with  $i + 1$  at the  $j^{\text{th}}$  position, each of these sequences are one of  $a_i$  possible types and we have  $a_{n-i}$  copies from each type.

Now consider  $a_{n-i}$  sequences with the symbols  $1, \dots, n - i, \star$  that are pairwise  $K_{n-i}$ -different and shift each value in these sequences by  $i + 1$ . (The latter means that we change each value  $k$  to  $k + i + 1$  in these sequences while  $\star$ 's remain  $\star$ 's.) For each type of the previous sequences take its  $a_{n-i}$  copies and suffix to each of them one of the current  $a_i$  different sequences. This way one gets  $a_i a_{n-i}$   $K_{n+1}$ -different sequences with symbol  $i + 1$  at position  $j$ . Doing this for all  $i = 1, \dots, n$  one obtains  $\sum_{i=0}^n a_i a_{n-i}$   $K_{n+1}$ -different sequences proving the desired inequality.  $\square$

Observe that if we have a construction of  $M$  infinite permutations that are pairwise  $K_{n+1}$ -different and furthermore each of them contains the symbols  $1, \dots, n$  in their natural order then the number of those among them that have the symbol  $i + 1$  in a fixed position is at most  $\kappa_{id}(K_i) \kappa_{id}(K_{n-i})$ . This is simply because all such sequences must be made  $K_{n+1}$ -different entirely either by their smallest  $i$  or by their largest  $(n - i)$  "non-dummy" symbols. So in case we have a construction where at some position every permutation has a useful value, that is a natural number at most  $n + 1$ , then the number of these permutations is at most  $C_{n+1}$  and the construction in the proof of Proposition 8 is optimal. It seems plausible that the condition on this special coordinate can be dropped, implying that

Catalan numbers give the true optimum. If it is so then one feels it should be possible to find a bijection between our permutations and the objects of one of the many enumeration problems leading to Catalan numbers, cf. [8]. It seems to be a significant difficulty, however, that our permutations are not objects having some structural property on their own, as it happens in most of the enumeration problems leading to Catalan numbers. Rather, in our case the criterion is in terms of a relation between pairs of objects, and this seems to make an important difference.

## 4 Acknowledgment

The first author would like to thank Alexandr Kostochka for a stimulating discussion. Thanks also to Riccardo Silvestri for computer help.

## References

- [1] C. Berge, Sur une conjecture relative au problème des codes optimaux, In: Comm. 13ème assemblée générale de l'URSI, Tokyo, (1962),
- [2] C. Berge, J.L. Ramírez-Alfonsín, Origins and genesis, in: *Perfect Graphs*, (Reed, Ramírez-Alfonsín eds.), John Wiley and Sons, 2001, Chapter 1, 1–12.
- [3] J. Körner, Fredman-Komlós bounds and information theory *SIAM J. Algebraic and Discrete Methods*, **7** (1986), 560–570.
- [4] J. Körner, C. Malvenuto, Pairwise colliding permutations and the capacity of infinite graphs, *SIAM J. Discrete Mathematics*, **20** (2006), 203–212.
- [5] J. Körner, A. Orlitsky, Zero-error information theory, *IEEE Trans. Inform. Theory*, **44** (1998), 2207–2229.
- [6] E. R. Scheinerman, D. H. Ullman, *Fractional Graph Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons, Chichester, 1997.
- [7] C. E. Shannon, The zero-error capacity of a noisy channel, *IRE Trans. Inform. Theory* **2** (1956), 8–19.
- [8] R. P. Stanley, *Enumerative combinatorics. Vol. 2.*, Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.