*Entropy splitting hypergraphs

Gábor Simonyi
Mathematical Institute
of the Hungarian Academy of Sciences
H-1364 Budapest, POB 127, Hungary
simonyi@math-inst.hu, simonyi@konig.elte.hu

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Abstract

Hypergraph entropy is an information theoretic functional on a hypergraph with a probability distribution on its vertex set. It is sub-additive with respect to the union of hypergraphs. In case of simple graphs, exact additivity for the entropy of a graph and its complement with respect to every probability distribution on the vertex set gives a characterization of perfect graphs. Here we investigate uniform hypergraphs with an analoguous behaviour of their entropy. The main result is the characterization of 3-uniform hypergraphs having this entropy splitting property. It is also shown that for $k \geq 4$ no non-trivial k-uniform hypergraph has this property.

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1 Introduction

Graph entropy $H(G, P)$, introduced by J. Körner [13], is an information theoretic functional on a graph G with a probability distribution P on its vertex set. A basic property of graph entropy is its sub-additivity under graph union, proved also by Körner in [14]. This means the following. Let F and G be two graphs on the same vertex set V with edge sets $E(F)$ and $E(G)$, respectively, and $F \cup G$ is the graph on V with edge set $E(F) \cup E(G)$. Then for any fixed probability distribution P on V we have

$$H(F \cup G, P) \leq H(F, P) + H(G, P). \tag{1}$$

The above inequality has provided a useful tool to obtain lower bounds in graph covering and complexity problems, for various applications, see Körner [14], Körner and Marton [18], Boppana [1], Newman, Ragde, and Wigderson [25], Radhakrishnan [26], Kahn and Kim [11]. In [17] Körner and Marton introduced hypergraph entropy. They used this new concept to improve upon the Fredman-Komlós bound of [6] generalizing its proof that relied on the sub-additivity of graph entropy, cf. [14]. This generalization was based on a similar inequality for hypergraphs. (For another application of hypergraph entropy, see Körner and Marton [19].)

In [16], [4], and [21] the conditions of equality in (1) were investigated. (This problem dates back to Körner and Longo [15] where a special case of (1) already appears.) The results of these investigations showed that there are close connections between graph entropy and some classical concepts of combinatorics, e.g., perfect graphs. One of the main questions in [15] was to characterize those graphs G that satisfy equality in (1) with $F = \bar{G}$ (where $\bar{G}$ stands for the complementary graph of G) and every P. It was conjectured in [16] and proved in [4] that these graphs are exactly the perfect graphs. (Perfect graphs were introduced by Berge as those graphs for which the chromatic number is equal to the clique number in every induced subgraph. They form an important class of graphs appearing in many different contexts; for more about perfect graphs, cf. Lovász [23], [24].) In this paper we investigate conditions for the similar equality in case of complementary uniform hypergraphs.

We need some definitions. The usual notation, $V(G), E(G)$, for the vertex and edge set of a (hyper)graph will be used throughout the paper.

Definition 1 *The vertex packing polytope $VP(F)$ of a hypergraph F is the convex hull of the characteristic vectors of the independent sets of F.*

We remark that an independent set of a hypergraph F is a subset of its vertex set $V(F)$ that contains no edge.

Definition 2 *Let F be a hypergraph on the vertex set $V(F) = \{1, ..., n\}$ and let $P = (p_1, ..., p_n)$ be a probability distribution on $V(F)$ (i.e., $p_1 + ... + p_n = 1$ and $p_i \geq 0$ for all* *i). The entropy of F with respect to P is then defined as*

$$H(F,P) = min_{\underline{a}\in VP(F)} - \sum_{i=1}^{n} p_i \log a_i. \tag{2}$$

Remark The results in [13] provide two equivalent definitions for graph entropy. A third equivalent definition was given in [4]. This is the one we have adopted. (Körner and Marton [17] generalized one of the earlier definitions when they introduced hypergraph entropy. The proof of equivalence in [4], however, literally applies to the hypergraph case, too.)

The union of two hypergraphs on the same vertex set V is a third hypergraph on V having as its edge set the union of the edge sets of the two original hypergraphs.

A hypergraph is k-uniform if all of its edges have size k. We denote the complete k-uniform hypergraph on n vertices by $K_n^{(k)}$. (Instead of $K_n^{(2)}$, however, we usually write simply K_n.) The complement of a k-uniform hypergraph F on n vertices is the k-uniform hypergraph $\bar{F}$ on the same vertex set that has a disjoint edge set from that of F and satisfies $F \cup \bar{F} = K_n^{(k)}$.

Considering graphs as 2-uniform hypergraphs, Definition 2 gives graph entropy as a special case. We remark that it is not difficult to see (cf. Lemma 3.1 in [3]) from this definition that the entropy of the complete graph, K_n, equals the Shannon-entropy of the probability distribution involved:

$$H(K_n,P) = H(P) = -\sum_{i=1}^{n} p_i \log p_i.$$

For a somewhat more complicated formula to compute $H(K_n^{(k)},P)$ for $k > 2$ see [5]. (The same formula was found independently by Gerards and Hochstättler [7], the statement of this result is also quoted in [29].)

In [17] Körner and Marton proved that hypergraph entropy is sub-additive in general, i.e., (1) holds not only for graphs but also for hypergraphs F and G.

The following definition is from [15] generalized to hypergraphs.

Definition 3 *A k-uniform hypergraph F is strongly splitting if for every probability distribution P on $V(F) = V$, we have*

$$H(F,P) + H(\bar{F},P) = H(K_{|V|}^{(k)},P). \tag{3}$$

As we have already mentioned, it was conjectured in [16] and proved in [4] that a graph is strongly splitting if and only if it is perfect.

Our aim here is to characterize strongly splitting k-uniform hypergraphs for $k \geq 3$. The main results are Theorems 1 and 2 of the next section that give this characterization for $k = 3$ and its generalization involving more than two 3-uniform hypergraphs. It turns out that for $k > 3$ no strongly splitting hypergraph exists except the trivial ones, $K_n^{(k)}$ and its complement. This is shown in Section 3. Section 4 deals with connections between the class of 3-uniform hypergraphs characterized in Theorem 1 and the well investigated class of cographs.

2 Splitting 3-uniform hypergraphs

All hypergraphs in this section will be 3-uniform, so we will often omit the full description and write simply hypergraph. (Graphs, however, still mean 2-uniform hypergraphs.) To state our result on 3-uniform hypergraphs we need the following definition.

Definition 4 *Let T be a tree and let us be given a two-coloring of its internal vertices with two colors that we call 0 and 1. The leaf-pattern of the two-colored tree T is the following* 3*-uniform hypergraph F. The vertices of F are the leaves of T and three leaves x, y, z form an edge if and only if the unique common point of the paths joining pairs of x, y and z is colored by* 1*.*

A 3*-uniform hypergraph F is said to be a leaf-pattern if there exists a two-colored tree T such that F is the leaf-pattern of T.*

It is obvious that the degree two vertices of a tree will have no effect on its leaf-pattern, so when concerned about the leaf-pattern, we can always think about trees with no degree two vertices. In fact, if a 3-uniform hypergraph F is the leaf-pattern of some tree then there is a unique two-colored tree not containing degree two vertices and having a proper coloring (i.e., a coloring in which neighbouring nodes have different colors), for which F is its leaf-pattern. For example, $K_n^{(3)}$ is the leaf-pattern of a star on $n+1$ points having 1 as the color of the middle point.

Strongly splitting 3-uniform hypergraphs are characterized by the following theorem.

Theorem 1 *A* 3*-uniform hypergraph is strongly splitting if and only if it is a leaf-pattern.*

For the proof of this theorem we will use two further descriptions of leaf-patterns. To this end we need some more definitions.

Duplicating a vertex x of a hypergraph F means that a new vertex x' is added to $V(F)$ thereby creating a new hypergraph F' as follows. For any set of vertices $S \subseteq V(F)$ not containing x, the set $S \cup \{x'\}$ is an edge of F' if and only if $S \cup \{x\}$ is an edge of F. A set $S \subseteq V(F)$ itself forms an edge of F' if and only if it is an edge in F. Notice that no edge of the new hypergraph contains both x and x'.

Definition 5 *A uniform hypergraph is called reducible if it can be obtained from a single edge by successive use of the following two operations in an arbitrary order:*

(i) duplication of a vertex,

(ii) taking the complementary uniform hypergraph.

For the next definition we have to describe a particular hypergraph on five vertices. Consider the five points $0, 1, 2, 3, 4$ and the five hyperedges of the form $\{i, i+1, i+2\}$ where $i = 0, 1, 2, 3, 4$ and the numbers are intended modulo 5. Notice that the hypergraph defined this way is isomorphic to its complement and let us call it *flower*.

Definition 6 *A* 3*-uniform hypergraph is light if the number of its edges induced by any four vertices is always even and it does not contain an induced flower.*

It turns out that the class of light hypergraphs is equivalent to the class of leaf-patterns. This was already proven by Gurvich in [8]. We state his result for reference.

Theorem G *A* 3*-uniform hypergraph is a leaf-pattern if and only if it is light.*

This theorem of Gurvich proves one of the equivalences in the following lemma. The lemma obviously implies Theorem 1.

Lemma 1 *The following four statements about a* 3*-uniform hypergraph F are equivalent:*
(i) F is strongly splitting
(ii) F is light.
(iii) F is a leaf-pattern
(iv) F is reducible

As a preparation for the proof we recall some consequences of already known results. As an immediate consequence of the definition of hypergraph entropy, notice that the minimizing vector $\underline{a}$ in (2) is always a maximal vector of $VP(F)$. (We call a vector $\underline{b}$ maximal in some set of vectors, if this set does not contain a $\underline{b}'$ with $b'_i \geq b_i$ for every i.) This also implies that all the independent sets of F that appear with positive coefficients in some convex combination representation of this minimizing $\underline{a}$ must be maximal.

For a hypergraph F let us denote the set of all maximal vectors in $VP(F)$ by $VP'(F)$. The following lemma is an immediate consequence of Corollary 7 in [4].

Lemma A *For every $\underline{a} \in VP'(F)$ there exists a probability distribution P such that $H(F,P) = -\sum_{i=1}^{n} p_i \log a_i$. Furthermore, if no $p_i = 0$ for this P then $\underline{a}$ is the unique minimizing vector in the definition of $H(F,P)$.*

The following lemma is also not new (cf. [15], [16], [4] Corollary 10, [21] Lemma 3), but since it is easy, we give a short proof for the sake of clarity.

Lemma B *Let F and G be two hypergraphs, P an everywhere positive probability distribution and let $\underline{a} \in VP(F)$, $\underline{b} \in VP(G)$, $\underline{c} \in VP(F \cup G)$ be the vectors achieving $H(F,P)$, $H(G,P)$, and $H(F \cup G, P)$, respectively. Now, if $H(F,P) + H(G,P) = H(F \cup G, P)$ then necessarily $a_i b_i = c_i$ for every i. Furthermore, then any two independent sets appearing with positive coefficients in some convex combination representations of $\underline{a}$ and $\underline{b}$, respectively, must intersect in a maximal independent set of $F \cup G$.*

Proof

Observe that the intersection of an independent set of F and an independent set of G is always an independent set of $F \cup G$. (In fact, sub-additivity is a consequence of this observation, cf. [17].) This implies that the vector $(a_1 b_1, ..., a_n b_n) \in VP(F \cup G)$. So if $H(F,P) + H(G,P) = -\sum_{i=1}^{n} p_i \log(a_i b_i) = H(F \cup G, P)$ then this vector should be the minimizing vector defining $H(F \cup G, P)$. The statement about the intersection is then obvious by the remark that only maximal independent sets can appear with positive coefficients in the representation of a vector achieving entropy. $\square$

For vectors $\underline{a}, \underline{b} \in \mathbf{R}^n$ we will use the notation $\underline{a} \circ \underline{b} = (a_1 b_1, a_2 b_2, ..., a_n b_n)$. Similarly, for two sets of vectors in $\mathcal{A}, \mathcal{B} \subseteq \mathbf{R}^n$, we write $\mathcal{A} \circ \mathcal{B} = \{\underline{a} \circ \underline{b} : \underline{a} \in \mathcal{A}, \underline{b} \in \mathcal{B}\}$. With this notation, the meaning of the previous lemma is that equality in (1) for every P is equivalent to $VP(F) \circ VP(G) = VP(F \cup G)$.

We will need three more lemmas.

Let us call two vertices of a uniform hypergraph F *twins* if they are duplicates of each other (in the sense of Definition 5) and *siblings* if they are twins either in F or in $\bar{F}$. Let $F - x$ denote the hypergraph induced by F on $V(F) - x$. We say that a vertex u distinguishes between two other vertices v and w in the hypergraph F, if v and w are siblings in $F - u$ but not in F. The following lemma states that reducibility is a hereditary property. For simplicity, we consider a hypergraph with no edge as a reducible hypergraph even if it has fewer than 3 nodes.

Lemma 2 *If F is a reducible 3-uniform hypergraph then so is every induced subhypergraph of F.*

Proof

We use induction on $|V(F)|$. For $|V(F)| \leq 3$ the statement is obvious. Assume we know it for $|V(F)| = n$. Now let $|V(F)| = n + 1$ and take an arbitrary (induced) sub-hypergraph of F that we denote by D. We know that F is reducible. Consider its "evolution" according to Definition 5. Let the vertex appearing last in this evolution be a and the one duplicated when a appears be b. (So a and b are siblings.) Now clearly $F - a$ is also a reducible hypergraph (it is just one step "behind" in the evolution of F). $F - b$ is a reducible hypergraph, too, since it is isomorphic to $F - a$. If D does not contain both, a and b, then D is an induced sub-hypergraph of at least one of the reducible hypergraphs $F - a$ and $F - b$, so we are done by the induction hypothesis. If D contains both a and b then identify a and b in D. What we get this way is isomorphic to an induced sub-hypergraph of $F - a$ again, so it is reducible. But now we can obtain D by duplicating a vertex of this reducible hypergraph (and maybe taking its complement). That means that D is reducible. $\square$

We remark that the above lemma is also an easy consequence of the fact that reducible 3-uniform hypergraphs are leaf-patterns of trees that will be proven later as part of Lemma 1

Lemma 3 *Let x be a vertex that distinguishes between vertices u and v in a 3-uniform hypergraph F that contains an even number of edges on every four vertices. If u and v are twins in $F - x$, then $\{x, u, v\} \in E(F)$, otherwise u and v are twins in $\bar{F} - x$ and $\{x, u, v\} \in E(\bar{F})$.*

Proof

Let F, x, u and v be as in the statement. We may assume that u and v are twins in $F - x$. This means that for no $y \in V(F - x)$ will $\{u, v, y\} \in E(F)$ be true. Assume indirectly that also $\{u, v, x\} \notin E(F)$. Since x distinguishes between u and v, there must be a vertex w with the property that either $\{u, x, w\} \in E(F)$ and $\{v, x, w\} \notin E(F)$ or vice versa. However, this would mean that the induced sub-hypergraph of F on the set $\{u, v, x, w\}$ contains exactly one edge, a contradiction, therefore the statement is true. $\square$

We remark that the assumption of the above lemma trivially holds for both, reducible and light 3-uniform hypergraphs, therefore so does its conclusion.

Finally, we need the following

Lemma 4 *If F is a reducible 3-uniform hypergraph on at least four vertices, then it has at least two disjoint pairs of siblings.*

Proof

Let $n = |V(F)|$. We use induction on n. For $n = 4$ the statement is obvious. We assume the lemma is true for $n = m$ and prove it for $m + 1$. Consider a reducible hypergraph F on $m + 1$ vertices. By definition, there must be at least one pair of vertices in F that form siblings, let us denote them by x and y. By Lemma 2 and the induction hypothesis there are at least two disjoint pairs of siblings in $F - x$, let them be called z, t and u, v, respectively. If z, t and u, v are siblings also in F, we are done. Assume that z and t are not siblings in F. This means that x distinguishes between them in F. We may assume that z and t are twins in $F - x$ otherwise we would consider $\bar{F}$. Then by Lemma 3 we have $\{x, z, t\} \in E(F)$. This, however, implies that $y \in \{z, t\}$, otherwise, since y is a sibling of x, we should have $\{y, z, t\} \in E(F)$ contradicting the assumption that z and t are twins in $F - x$, and therefore they cannot both occur in the same edge. Since y cannot be also one of u and v at the same time, x cannot distinguish between them, too, so u and v will be still siblings in F. Then F contains two disjoint pairs of siblings: u, v and x, y, hence the lemma is proven. $\square$

Proof of Lemma 1:

We prove by showing (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i). Among these implications (iii)$\Rightarrow$(iv) is more or less trivial, while for (ii)$\Rightarrow$(iii) we can refer to Theorem G.

(i)$\Rightarrow$(ii):

The class of light hypergraphs is defined by some forbidden configurations. We will show that none of these configurations is a strongly splitting hypergraph. This immediately implies that no strongly splitting hypergraph can contain these configurations as induced sub-hypergraphs. Indeed, otherwise we could concentrate a probability distribution violating (3) on this particular sub-hypergraph, all the entropy values would be the same as if the zero-probability vertices did not exist, and so (3) would be violated, too.

But if no strongly splitting hypergraph contains these sub-hypergraphs, then all strongly splitting hypergraphs are light.

There are three forbidden configurations in the definition of light hypergraphs: one or three edges on four vertices and the flower. Since the first two are complements of each other, these are essentially only two cases.

Consider the first pair of forbidden configurations, the 3-uniform hypergraph on four vertices with one edge and its complement that has three edges. Let us denote them by F and $\bar{F}$, respectively, and their four vertices by x, y, z, t, in such a way that the only edge of F is $\{x, y, z\}$. We will show that for no $\underline{a} \in VP(F)$ and $\underline{b} \in VP(\bar{F})$ can $a_i b_i = \frac{1}{2}, (i = 1, 2, 3, 4)$ be satisfied. Since $\underline{c} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in VP'(K_4^{(3)})$, this implies the statement by Lemmas A and B. (In fact, instead of the above $\underline{c}$ we could consider any $\underline{c} \in VP'(K_4^{(3)})$ satisfying $0 < c_i < 1$ for every i.)

First observe that every maximal independent set of $\bar{F}$ containing t has only two elements, and for having $b_t > 0$ it is necessary for at least one of these sets to get a positive coefficient in the convex combination representation of $\underline{b}$. We may assume that the set $\{x, t\}$ gets a positive coefficient. However, this set is a maximal independent set of $K_4^{(3)}$, too, therefore by Lemma B, all maximal independent sets of F that will get a positive coefficient in the representation of $\underline{a}$ must contain $\{x, t\}$ completely. There are only two such maximal independent sets in F: $\{x, y, t\}$ and $\{x, z, t\}$. Both of these two sets should get a positive coefficient in the representation of $\underline{a}$ in order to have $a_y > 0$ and $a_z > 0$. Now going back to $\bar{F}$, apart from $\{x, t\}$, it has only one maximal independent set that intersects both of the previous two independent sets of F in a maximal independent set of $K_4^{(3)}$, this is $\{x, y, z\}$. So, again by Lemma B, apart from $\{x, t\}$ only this set can get a positive coefficient in the representation of $\underline{a}$. Now observe that all the above mentioned sets contain x, so whatever convex combination of them is taken, we will have $a_x = b_x = 1$, therefore $a_x b_x = \frac{1}{2}$ will not be satisfied. By Lemma B, this proves that the hypergraphs in our first pair of forbidden configurations are not strongly splitting.

For the flower a similar proof can be carried out. The following argument is shorter, however, it was suggested by one of the referees. Let M denote a flower and let the vector $\underline{c} \in VP'(K_5^{(3)})$ we want to have in the form $\underline{c} = \underline{a} \circ \underline{b}$ with $\underline{a} \in VP(M)$, $\underline{b} \in VP(\bar{M})$, be $c_i = \frac{2}{5}, i = 1, .., 5$. Assume we have such an $\underline{a}$ and $\underline{b}$. Since the independence number of M is 3, we have $a_1 + ... + a_5 \leq 3$ and similarly for the b_i's. By the convexity of the function $\frac{1}{x}$ we can write

$$3 \geq \sum_{i=1}^{5} b_i = \frac{2}{5} \sum_{i=1}^{5} \frac{1}{a_i} \geq (\frac{2}{5}) 5 (\frac{5}{3}) = \frac{10}{3},$$

a contradiction. This concludes the proof of (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii):
This follows from Theorem G.

(iii)$\Rightarrow$(iv):

Let F be a leaf-pattern and T be the two-colored tree that F belongs to. We may assume that T has no vertex of degree two. T certainly has two leaves x and y, say, that are joint to the same inner node. If this inner node is colored by 0 then x and y are twins in F. By identifying these two nodes we arrive to $F-x$ being the leaf-pattern of the tree we obtain by deleting x. If the common neighbour of x and y is colored 1 in T then the same can be done after a complementation of both F and the coloring of T. This argument implies that from any leaf-pattern we can "go down" to one single edge by iterative use of two transformations: identifying twins and complementation. But since identifying twins is the inverse of vertex duplication, this means that any leaf-pattern is a reducible hypergraph.

(iv)$\Rightarrow$(i):

We use induction on $n=|V(F)|$. For $n=3$ the statement is trivial. We assume it is true for $n=m$ and prove it for $n=m+1$. Consider a reducible hypergraph F on $m+1$ vertices. Let us be given an arbitrary probability distribution P on the vertices of F and let $\underline{c}\in VP(K_{m+1}^{(3)})$ be the vector achieving $H(K_{m+1}^{(3)},P)$. Observe that $VP(K_n^{(3)})=\{\underline{h}: 0\le h_i\le 1,\sum_{i=1}^n h_i\le 2\}$ and since c must be a maximal vector in $VP(K_{m+1}^{(3)})$ we surely have $\sum_{i=1}^{m+1}c_i=2$.

By Lemma 4 there exist two disjoint pairs of siblings in F, let them be x,y and z,t. By $\sum_{i=1}^{m+1}c_i=2$ we have that at least one of the two inequalities, $c_x+c_y\le 1$ and $c_z+c_t\le 1$, holds. We may assume that the first one is valid and label the vertices so that $x=1$ and $y=2$. Then we have

$$\underline{c}'=(c_1+c_2,c_3,c_4,...,c_m,c_{m+1})\in VP'(K_m^{(3)}),$$

and by Lemma A there exists a probability distribution P' for which $H(K_m^{(3)},P')$ is achieved by $\underline{c}'$. Now consider the hypergraph on m vertices that we obtain by identifying the vertices x and y (i.e., 1 and 2) of F in the obvious manner. The new vertex will be denoted by x', and the hypergraph obtained this way we denote by F'. By the induction hypothesis, F' is strongly splitting, in particular, we have

$$H(F',P')+H(\bar{F}',P')=H(K_m^{(3)},P').$$

This means that the vectors $\underline{a}'$ and $\underline{b}'$ achieving $H(F',P')$ and $H(\bar{F}',P')$, respectively, satisfy $\underline{a}'\circ\underline{b}'=\underline{c}'$. Now we obtain an $\underline{a}\in VP(F)$ and a $\underline{b}\in VP(\bar{F})$ from $\underline{a}'$, $\underline{b}'$, respectively, that will satisfy $\underline{a}\circ\underline{b}=\underline{c}$. To this end we assume that 1 and 2 (the former x and y) are twins in F, otherwise we could change notation and consider $\bar{F}$. Look at the maximal independent sets of F' and $\bar{F}'$ that appear with positive coefficients in some representations of $\underline{a}'$ and $\underline{b}'$, respectively. Let the coefficient of the independent set I of F' be $\alpha'(I)$ in the representation of $\underline{a}'$. For $x'\notin I$ let $\alpha(I)=\alpha'(I)$ and for $x'\in I$ let $\alpha((I\backslash\{x'\})\cup\{x,y\})=\alpha'(I)$. The coefficient of an independent set J of $\bar{F}'$ we denote by $\beta'(J)$. For $x'\notin J$ we let $\beta(J)=\beta'(J)$ while for $x'\in J$ we let $\beta((J\backslash\{x'\})\cup\{x\})=\beta'(J)\frac{c_1}{c_1+c_2}$ and $\beta((J\backslash\{x'\})\cup\{y\})=\beta'(J)\frac{c_2}{c_1+c_2}$. It is easy to check that this way we gave coefficients to independent sets of F and $\bar{F}$, and that the $\underline{a} \in VP(F)$ and $\underline{b} \in VP(\bar{F})$ they represent are:

$$\underline{a} = (a'_1, a'_1, a'_3, a'_4, ..., a'_m, a'_{m+1})$$

and

$$\underline{b} = (b'_1 \frac{c_1}{c_1 + c_2}, b'_1 \frac{c_2}{c_1 + c_2}, b'_3, b'_4, ..., b'_m, b'_{m+1}).$$

Using $a'_i b'_i = c'_i$ this immediately gives $a_i b_i = c_i$ for every i and so

$$H(K^{(3)}_{m+1}, P) = -\sum_{i=1}^{m+1} p_i \log c_i = -\sum_{i=1}^{m+1} p_i \log a_i - \sum_{i=1}^{m+1} p_i \log b_i \geq H(F, P) + H(\bar{F}, P).$$

Together with the sub-additivity of hypergraph entropy this implies equality above and so F is strongly splitting.

This completes the proof of Lemma 1. $\square$

Proof of Theorem 1:

The statement of Theorem 1 is the (i)$\Leftrightarrow$(iii) part of Lemma 1 so it is proven by the foregoing. $\square$

Remark We note that the last part of the above proof does make use of the fact that we are in the case $k = 3$, that is, though it may sound plausible, it is not proven, moreover, it is not true in general that vertex duplication keeps the splitting property of a uniform hypergraph. If this were true then all reducible uniform hypergraphs were strongly splitting contradicting Theorem 3 of the next section. (In case of $k = 2$ the analoguous statement is true and follows from Lovász' result in [23] stating that vertex duplication keeps the perfectness of a graph.)

In [8] Gurvich has proved a generalization of his Theorem G, that we can use to obtain a generalization of Theorem 1. First a generalization of the concept of leaf-pattern is needed.

Definition 7 *Let T be a tree with its inner nodes colored by colors 1, 2,..., r. The leaf-factorization of the r-colored tree T is a collection $F_1, F_2, ..., F_r$ of 3-uniform hypergraphs with the following properties. The vertex set of F_i ($i = 1, .., r$) is the set of leaves of T and three leaves x, y, z form an edge in F_i if and only if the unique common point of the paths xy, yz, and zx is colored with color i in T.*

The collection of hypergraphs $F_1, ..., F_r$ is called a leaf-factorization if it is the leaf-factorization of some r-colored tree T.

The general result of Gurvich is the following.

Theorem GG *A collection $F_1, ..., F_r$ of 3-uniform hypergraphs is a leaf factorization if and only if all F_i's are light.*

Using this result we have

Theorem 2 *Let $F_1, ..., F_r$ be 3-uniform hypergraphs on a common vertex set V and their union be the complete 3-uniform hypergraph on V. Then having*

$$\sum_{i=1}^{r} H(F_i, P) = H(K_{|V|}^{(3)}, P)$$

for every distribution P on V is equivalent to $F_1, ..., F_r$ forming a leaf-factorization.

Proof

By Theorem 1 the equality in the statement implies that every F_i is a leaf-pattern, i.e., all of them are light (by Theorem G). Then by Theorem GG they form a leaf-factorization. All we have to show is that leaf-factorizations satisfy the above equality. This goes by a similar induction as that in the proof of the (iv)⇒(i) implication part of Lemma 1.

Let $F_1, ..., F_r$ be the leaf-factorization of the r-colored tree T. Since F_1 is light it has two disjoint pairs of siblings by Lemma 4. Let one such pair be x and y with the additional property that $c_x + c_y \leq 1$ where $(c_1, c_2, ..., c_{|V|})$ denotes the vector in $VP(K_{|V|}^{(3)})$ that gives $H(K_{|V|}^{(3)}, P)$ for some arbitrarily fixed P. Now observe that x and y are siblings in all F_i's, moreover, they are twins in each F_i except one, F_j, say. (This is because, if we exclude degenerate colorings, then x and y must be two leaves of T with a common neighbour that is colored by j.) After this observation we can more or less literally repeat the corresponding part ((iv)⇒(i)) of the proof of Lemma 1 with F_j playing the role of $\bar{F}$ there. □

Remark Theorem 2 is the analogon of Corollary 1 in [21] which states that if $(G_1, ..., G_r)$ is a collection of edge disjoint graphs with their union being the complete graph on their common vertex set, then

$$\sum_{i=1}^{r} H(G_i, P) = H(P)$$

for every P is equivalent to all G_i's being perfect and no triangle having its three edges in three different G_i's. It is interesting to note that while all G_i's being strongly splitting (i.e., perfect) is not enough for the above equality, all F_i's being strongly splitting is sufficient for the analoguous equality in the 3-uniform case.

3 The case $k \geq 4$

In this section we show that for $k > 3$ the only strongly splitting k-uniform hypergraphs are the two trivial ones.

Theorem 3 *If $k \geq 4$ and F is a strongly splitting k-uniform hypergraph on n vertices then $F = K_n^{(k)}$ or $F = \bar{K}_n^{(k)}$.*

Proof

It is enough to prove the above statement for $n = k+1$. This is because being strongly splitting is a hereditary property and a k-uniform hypergraph which is complete or empty on every $k+1$ vertices must be complete or empty itself. (The fact that being strongly splitting is hereditary follows from the argument that a probability distribution can be concentrated on any subset of the vertex set and then the entropy values are just the same as if the zero-probability vertices did not exist.) The proof for $n = k+1$ will use similar arguments as the proof of (i)$\Rightarrow$(ii) in Lemma 1.

Consider a k-uniform hypergraph F with $k+1$ vertices and m edges. Up to isomorphism, there is only one such hypergraph. Its complement $\bar{F}$ has $k+1-m$ edges. The maximal independent sets of F ($\bar{F}$) are the edges of $\bar{F}$ (F) and those $(k-1)$-element sets that are not contained in the former independent sets.

Like in the proof of (i)$\Rightarrow$(ii) of Lemma 1 our setting is this. We consider an arbitrarily given probability distribution P. This singles out a vector $\underline{c} \in VP(K_{k+1}^{(k)})$ that achieves the entropy of $K_{k+1}^{(k)}$ with respect to P. Now we look for an $\underline{a} \in VP(F)$ and a $\underline{b} \in VP(\bar{F})$ giving $\underline{a} \circ \underline{b} = \underline{c}$, and thereby additivity of hypergraph entropy for the given P. We will investigate which independent sets of F and $\bar{F}$ may have positive coefficients in the convex combination representations of $\underline{a}$ and $\underline{b}$, respectively. It will follow that not every $\underline{c} \in VP'(K_{k+1}^{(k)})$ can be represented this way if neither F nor $\bar{F}$ is complete, and then by Lemmas A and B the theorem follows.

So our next task is to choose a $\underline{c} \in VP'(K_{k+1}^{(k)})$ that we will not be able to obtain in the required form. By Lemma A this is enough, since then a corresponding P exists for which $\underline{c}$ achieves $H(K_{k+1}^{(k)}, P)$. Let this $\underline{c}$ be such that $0 < c_i < 1$ for every i, and furthermore, none of $\sum_{i=1}^{m}(1-c_i) = 1$ and $\sum_{i=m+1}^{k+1}(1-c_i) = 1$ holds. (In fact, the latter two are equivalent, since $\sum_{i=1}^{k+1} c_i = k-1$ for every $\underline{c} = VP'(K_{k+1}^{(k)})$.) It is easy to check that such a $\underline{c} \in VP'(K_{k+1}^{(k)})$ always exists. We show it cannot be represented as $\underline{a} \circ \underline{b}$ with $\underline{a} \in VP'(F)$, $\underline{b} \in VP'(\bar{F})$.

Assume the contrary. First observe that it cannot happen that in the representations of both, $\underline{a}$ and $\underline{b}$, some $(k-1)$-element independent set occurs with positive coefficient, because (since these sets could not be identical) the intersection of such two sets, would not be a maximal independent set of $K_{k+1}^{(k)}$, thereby violating Lemma B. We distinguish between two cases: either there is at least one $(k-1)$-element set with positive coefficient in the representation of, say, $\underline{a}$, or no $(k-1)$-element set appears with positive coefficient at all.

In the second case, for every vertex i there is at most one independent set with positive coefficient not containing i. This implies that for every i this unique independent set must get coefficient $(1-c_i)$. We get convex combinations this way only if $\sum_{i=1}^{m}(1-c_i) = 1$ and $\sum_{i=m+1}^{k+1}(1-c_i) = 1$. But this is not satisfied by the $\underline{c}$ we have chosen.

In the first case, only those two maximal independent sets may have positive coefficients in the representation of $\underline{b}$ that contain the $(k-1)$-element set appearing in the representation of $\underline{a}$. (This is again by Lemma B.) Since we must have $b_i > 0$ for every i, these two independent sets must really get positive coefficients there. This implies that only one $(k-1)$-element set can get positive coefficient in the representation of $\underline{a}$ (again, by Lemma B). Now observe that this way there are $m-2$ points that will be contained in all the independent sets that may appear in the representations of $\underline{a}$ or $\underline{b}$ with positive coefficient. For all such points i we will have $a_i b_i = 1$, a contradiction, unless we have $m \leq 2$.

If $m = 2$, then again, the coefficients of the k-element sets appearing in the representation of $\underline{a}$ are determined. Since the set missing element i is the only set that does not contain i, its coefficient must be $1 - c_i$. Labelling the vertices in such a way that 1 and 2 are the two vertices missed by our unique $(k-1)$-element set in the representation of $\underline{a}$, the previous observation implies $\sum_{i=3}^{k+1}(1-c_i) \leq 1$. We may assume, however, that c_1 and c_2 are just the two largest coordinates of $\underline{c}$, implying $c_1 + c_2 \geq \frac{2(k-1)}{k+1}$, i.e., $\sum_{i=3}^{k+1}(1-c_i) = ((k-1) - (k-1-(c_1+c_2)) \geq \frac{2(k-1)}{k+1}$. But $\frac{2(k-1)}{k+1} \leq 1$ implies $k \leq 3$.

It is already implicit in the above argument that $m \neq 1$. Indeed, if $m = 1$, then there is a vertex which is not contained in any independent set of $\bar{F}$ that is larger than $k-1$. Since some independent set of $\bar{F}$ containing this vertex must get positive coefficient, there must be a $k-1$-element independent set with positive coefficient in the representation of $\underline{b}$. But we assumed we have a $k-1$-element independent set with positive coefficient in the representation of $\underline{a}$. Since the latter two have too small an intersection, we have arrived to a contradiction.

The proof is complete now. □

Theorem 2 of [4] together with our Theorems 1 and 3 implies the following

Corollary 1 *If a k-uniform hypergraph F is strongly splitting then (at least) one of the following three statements should hold:*

(i) $k = 2$ and F is a perfect graph

(ii) $k = 3$ and F is a leaf-pattern

(iii) F is $K_n^{(k)}$ or $\bar{K}_n^{(k)}$.

□

4 Connections with cographs

Cographs are defined as those graphs one can obtain starting from a single vertex and successively and iteratively using two operations: taking the complement and taking vertex disjoint union. (For their algorithmic importance, history, and other details, cf. [2].) By a theorem of Corneil, Lerchs, and Stewart Burlingham [2] cographs are identical to reducible graphs (i.e., reducible 2-uniform hypergraphs) in the sense of Definition 5. In fact, Corneil, Lerchs and Stewart Burlingham [2] show the equivalence of eight different characterizations of cographs, relying also on earlier results by Jung [10], Lerchs [22], Seinsche [28], and Sumner [30]. (Related results can also be found in [8], cf. also

[12]). Among others, this theorem shows that cographs also admit a characterization by excluded configurations. In fact, they are equivalent to P_4-free graphs, i.e., graphs that have no induced subgraph isomorphic to a chordless path on 4 vertices.

The definition of reducible hypergraphs gives a natural (although not necessarily unique) way to describe the evolution of such a hypergraph. We obtain this description by simply ordering the vertices, telling for each vertex which preceding vertex it was originally a twin of and saying at which steps we should complement the hypergraph we have at hand. Since this means that after having fixed the first three vertices, the same description can describe a cograph and also a 3-uniform reducible hypergraph, it is natural that some correspondence can be found between them more directly. This is really easy to find.

Proposition 1 *A 3-uniform hypergraph F is reducible if and only if there exists a cograph G on $V(F)$ such that in each edge of F the number of edges of G has the same parity.*

□

The proof is straightforward and left to the reader.

Quoting results of Seidel [27], Hayward [9] defines the IP_3-structure of a graph G. This is the 3-uniform hypergraph on $V(G)$ the edges of which are exactly those triples of vertices that induce an even number of edges in G. (It is shown (cf. [27], [9]) that the IP_3-structures of graphs are exactly those 3-uniform hypergraphs that on every four vertices have an even number of edges.) Using this terminology and the fact that the complement of a cograph is also a cograph, the previous proposition says that leaf-patterns (reducible 3-uniform hypergraphs) are equivalent to the IP_3 structures that arise from cographs. For further details on the related topic of "Seidel's switching" cf. also [20].

Finally, it is interesting to note, that since all cographs are perfect (cf. Lovász [23], Seinsche [28]), Corollary 1, together with the above proposition, shows a kind of "continuity" as we consider strongly splitting graphs, strongly splitting 3-uniform hypergraphs and then strongly splitting k-uniform hypergraphs with $k > 3$.

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