Different capacities of a digraph

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Abstract

Sperner product is the natural generalization of co–normal product to digraphs. For every class of digraphs closed under Sperner product, the cardinality of the largest subgraph from the given class, contained as an induced subgraph in the co–normal powers of a graph G, has an exponential growth. The corresponding asymptotic exponent is the capacity of G with respect to said class of digraphs. We derive upper and lower bounds for these capacities for various classes of digraphs, and analyze the conditions under which they are tight.

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1 Introduction

An induced complete subgraph is often called a clique and correspondingly, the cardinality of the largest clique of G is called its clique number. The analysis of its asymptotic growth in large product graphs leads to one of the most formidable problems in modern combinatorics. This problem was posed by Shannon [14] in 1956 in connection with his analysis of the capability of certain noisy communication channels to transmit information in an error–free manner. Shannon associated a graph with every channel. In our notation (which is different from his), the vertex set of the graph represents the symbols that can be transmitted through the channel and two vertices are connected by an edge if the corresponding symbols can never get confused by the receiver. Put in other words this means that our two vertices never lead to the reception of the same output symbol. Any graph can be obtained in this manner.

Shannon's model naturally leads to a product of graphs through the repeated use of the channel for the transmission of symbol sequences of some fixed length n, say. If the graph G has vertex set $V = V(G)$, then G^n denotes the graph with vertex set V^n whose edge set contains those pairs of sequences which can never get confused by the receiver. Formally,

$$(\mathbf{x}, \mathbf{y}) \in E(G^n) \quad \text{if and only if} \quad \exists i \quad (x_i, y_i) \in E(G),$$

where we suppose that x_i and y_i are the i'th coordinate of $\mathbf{x}$ and $\mathbf{y}$, respectively. As usual, $E(G)$ denotes the edge set of the graph G. Following Berge [1], G^n is called the n'th co–normal power of G. Shannon [14] observed that if K is a clique in G then K^n is a clique in G^n, whence the clique number of G^n is at least as big as the n'th power of the clique number of G. In fact, these two quantities coincide whenever the clique number of G equals its chromatic number. (This observation led Berge [2] to his celebrated concept of perfect graphs.) On the other hand, Shannon [14] noticed that for the smallest graph whose chromatic number exceeds its clique number, the now famous pentagon C_5, the clique number of C_5^2 is 5 while the square of the clique number of C_5 is just 4. Thus it was logical to ask as Shannon did for the determination of the (always existing)

$$C(G) = \lim_{n \to \infty} \frac{1}{n} \log \omega(G^n),$$

where $\omega(G)$ is the usual notation for the clique number of G, while $C(G)$ is called the Shannon capacity of G. A very good exposition of this problem and the new important developments around it in the late seventies is in the volume [15].

Notice further that this super-multiplicativity of the clique number is in contrast with the growth of the independence number $\alpha(G^n)$ of the powers of G. Recall that $\alpha(G)$ is the cardinality of the largest induced edge–free subgraph of G. In fact, it is perfectly trivial that every maximal independent set of G^n is the Cartesian product of n independent sets of G whence

$$\alpha(G^n) = [\alpha(G)]^n.$$

An analogous approach to directed graphs was initiated in [10] and further developed in [7], where the notion of Sperner capacity was introduced as the natural counterpart of Shannon capacity in case of graphs with directed edges. At this point, a few words about terminology. As usual, we call a graph directed if its edge set is an arbitrary subset of the Cartesian square of its vertex set. Graphs in which every edge is present with at most one of its possible orientations are called oriented. Finally, a directed graph is called symmetrically directed if each of its edges is present with both of its possible orientations.

Sperner capacity is a generalization of Shannon capacity to directed graphs and is therefore even harder to determine, cf. Calderbank, Frankl, Graham, Li and Shepp [4] and Blokhuis [3]. This does not mean, however, that this generalization is an academic exercise of no immediate use. To the contrary, Sperner capacity became the key to the solution of some important open problems in extremal set theory, [8], [9].

The study of the extension of a poset to sequences of its elements leads to the equally justified notion of antichain capacity that we will introduce in this paper. Although our concept will be defined for arbitrary oriented graphs, not just those corresponding to posets, it is useful to present it first in the special case of posets where it is probably more intuitive.

Let P be a partial order on the set V and write $(x,y) \in P$ to indicate that $x \in V$ precedes $y \in V$ in the partial order P. Consider for some fixed natural integer n the Cartesian power V^n. Then we can extend P to the sequences $\mathbf{x} \in V^n$ and $\mathbf{y} \in V^n$ by saying that $\mathbf{x}$ precedes $\mathbf{y}$ and write $(\mathbf{x},\mathbf{y}) \in P^n$ if for some coordinate i, $1 \leq i \leq n$ we have $(x_i, y_i) \in P$, while $(y_j, x_j) \in P$ never holds. Although this relation of precedence is not necessarily a partial order on V^n, it nevertheless has a clear intuitive meaning. Now there are two different reasons why for two sequences $\mathbf{x}$ and $\mathbf{y}$ none precedes the other. Either in no coordinate i is any of (x_i, y_i) and (y_i, x_i) in P, or there are two coordinates, i and j such that both (x_i, y_i) and (y_j, x_j) are contained in P. In both cases, there is no meaningful way to break symmetry and say that one sequence precedes the other. We will call the set $C \subseteq V^n$ an antichain for P^n if

$$\mathbf{x} \in C, \quad \mathbf{y} \in C \quad \text{implies} \quad (\mathbf{x},\mathbf{y}) \notin P^n.$$

An antichain is a set of sequences on which the partial order implies no relation of precedence at all. Let $M(P,n)$ denote the cardinality of the largest antichain for P^n. We call the always existing limit

$$\lim_{n\to\infty} \frac{1}{n} \log M(P,n)$$

the antichain capacity of the partial order P and introduce for it the notation $A(P)$. The above concept has an immediate extension to the case of an arbitrary oriented graph.

Let G be an oriented graph with vertex set V and edge set $E(G) \subset V^2$. We say that two sequences $\mathbf{x}$ and $\mathbf{y}$ in V^n are G–incomparable, if there are two different coordinates, i and j such that both (x_i, y_i) and (y_j, x_j) are a directed edge in G. Following [7], we denote by $N(G,n)$ the maximum cardinality of any set $C \subset V^n$ such that any two elements of

C are G–incomparable. For the sake of completeness, we should mention that the always existing limit

$$\Sigma(G) = \lim_{n\to\infty} \frac{1}{n} \log N(G,n)$$

is called the Sperner capacity of the oriented graph G. (Notice that the definition can be extended to any directed graph and thus Sperner capacity is a formal generalization of Shannon capacity, cf. [9].)

Now, two sequences, $\mathbf{x}$ and $\mathbf{y}$ in V^n are called G–independent if x_i and y_i are equal or non-adjacent vertices of G for any i, $1 \le i \le n$. Finally, the sequences $\mathbf{x}$ and $\mathbf{y}$ in V^n are called G–unrelated if they are either G–incomparable or G–independent. Let $M(G,n)$ denote the largest cardinality of a subset $C \subset V^n$ such that any pair of elements of C are G–unrelated. We call the always existing limit

$$A(G) = \lim_{n\to\infty} \frac{1}{n} \log M(G,n)$$

the antichain capacity of the oriented graph G. We shall sometimes call a set as above an antichain defined by G.

In case of an oriented graph G one immediately sees from the definitions that the Sperner capacity $\Sigma(G)$ is always upper bounded by $A(G)$ and that the two notions coincide for tournaments. Since the Sperner capacity of tournaments is unknown except for the trivial case of transitive tournaments and a few other special examples for small vertex sets [4], [3], we immediately understand that determining the antichain capacity of a graph is not easy and we will be happy if we can find meaningful and strong enough lower and upper bounds. This is the principal aim of the present paper. Incidentally, we will find some interesting problems along the way.

In order to stress the analogy of both Sperner capacity and antichain capacity to the familiar Shannon capacity of graphs we will introduce an analogon of the co–normal power of graphs in the case of oriented (and more generally, directed) graphs. To avoid the somewhat cumbersome term co–normal we will call this power the Sperner power. The search for antichains in product graphs leads to the search for induced subgraphs in which the vertices are partitioned into independent sets and vertices belonging to different classes of the partition are adjacent. Moreover, the classes are linearly ordered and every edge is directed in accordance with the order of its two endpoints so as to point in the direction of the class that comes later in the order. Induced subgraphs with this property will be called waterfalls. Their growth in product graphs is our central problem in this paper.

Initially, we shall be interested in oriented graphs, a case for which our problems are easier. To the other extreme, we will later treat symmetrically directed graphs, which will be more conveniently considered as simple graphs. Although our problems can be stated more generally, for arbitrary directed graphs, still, we will not discuss these in detail.

All exp's and log's in this paper are binary.

2 Waterfalls

The (logarithm of the) clique number is a simple and altogether not very bad lower bound on the (logarithmic) Shannon capacity of a simple graph. The core of the problem, however, lies in the fact that in product graphs larger cliques exist than just the Cartesian powers of the cliques in the graph itself. In the present case the situation will be similar in some sense.

Let G be an oriented graph with vertex set V and n be a natural number. An induced subgraph W of G will be called a waterfall if its vertices can be colored by natural numbers in such a way that vertices of the same color are non–adjacent, while if the vertex x gets a smaller color than the vertex y , then this implies that $(x, y) \in E(G)$.

A directed graph G with vertex set V defines a directed graph G^n on V^n in which $\mathbf{x} = x_1x_2 \dots x_n$ and $\mathbf{y} = y_1y_2 \dots y_n$ are connected by a directed edge pointing from $\mathbf{x}$ to $\mathbf{y}$ if for at least one i, $1 \le i \le n$, we have $(x_i, y_i) \in E(G)$. Notice that we have called $\mathbf{x}$ and $\mathbf{y}$ G–incomparable if both $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{y}, \mathbf{x})$ form a directed edge in G^n. We call G^n the n'th Sperner power of G. Notice that Sperner power is the natural generalization to directed graphs of the co–normal power with which it coincides in case the graph is symmetrically directed.

The previous definition of a waterfall carries through literally to the case of directed graphs.

Definition 1 *An induced subgraph W of the directed graph $G = (V(G), E(G))$ is called a waterfall if its vertices can be colored with natural integers in such a way that two vertices x and y getting the same color are non–adjacent, while if the color of x is a smaller integer than that of y then necessarily $(x, y) \in E(G)$. (Notice that the latter does not imply the absence of the reversed edge $(y, x) \in E(G)$.)*

Denote by $W(G)$ the maximum cardinality of a waterfall in G and call it the waterfall number of G.

Waterfalls give a simple construction and an easy lower bound for the antichain capacity of G which is the content of the following lemma.

Lemma 1 *For every natural n the antichain capacity $A(G)$ of the oriented graph G is lower bounded by the quantity*

$$\frac{1}{n} \log W(G^n).$$

Proof

Fix some n and a waterfall induced by G^n on some set $W \subseteq [V(G)]^n$ with the property $|W| = W(G^n)$. For every natural m consider those sequences in $W^{m|W|}$ in which every element of W occurs m times. One easily sees that, for every natural m, this set of sequences from $W^{m|W|}$ is an antichain for G with elements from $[V(G)]^{nm|W|}$. The cardinality of this antichain, as $m \to \infty$, gives the lower bound stated in the lemma. □

It is easy to see that any Sperner power of a waterfall is again a waterfall, whence

$$W(G^n) \geq [W(G)]^n$$

for any directed graph G.

As we shall see later equality holds above in several special cases, but not in general. This justifies the following

Definition 2 *For an arbitrary directed graph G let $\Theta(G)$ denote the always existing limit*

$$\Theta(G) = \lim_{n \to \infty} \frac{1}{n} \log W(G^n)$$

and call it the cascade capacity of G.

The cascade capacity is an obvious lower bound for the antichain capacity of a graph. The bound given by $W(G)$ is not always tight.

Proposition 1 *For an arbitrary oriented graph G we have*

$$\log W(G) \leq A(G)$$

where the strict inequality can also occur.

Proof

Except for an example for the strict inequality, the assertion follows from Lemma 1. A case of strict inequality is given by the cyclically oriented pentagon graph C_5. Clearly, we have $W(C_5) = 2$, while we will see in a moment that C_5^2 has an antichain of cardinality 5. To see this, denote by the numbers 0,1,2,3,4 the vertices of the pentagon in the order of the cyclic orientation of the graph. Then C_5^2 induces an antichain on the subset of $\{0, 1, 2, 3, 4\}^2$ the elements of which are (00), (14), (23), (32), (40). □

To obtain upper bounds on the antichain capacity of an oriented graph we introduce an auxiliary graph in the next section.

3 Independence graphs

To an arbitrary oriented graph G we associate a simple graph G^* as follows. The vertex set of G^* is the set of all pairs (x, A) such that x is a vertex of G, A is an independent set of G and A contains x. (An independent set of G is an induced subgraph not containing any edge of G.) The pairs (x, A) and (x', A') are adjacent in G^* if either A and A' are disjoint and G induces a waterfall with non-empty edge set on $A \cup A'$, or $A = A'$ but $x \neq x'$.

Our aim in this section is to prove that the logarithm of the fractional chromatic number $\chi^*(G^*)$ of the independence graph of G is an upper bound for its antichain capacity.

For simplicity, we will first prove the following weaker statement:

Theorem 1 *For an arbitrary oriented graph G we have*

$$A(G) \leq \log \chi(G^*),$$

where $\chi(G^*)$ denotes the chromatic number of the simple graph G^*. More precisely, we have

$$M(G,n) \leq [\chi(G^*)]^{n-1}\alpha(G).$$

Proof

Let $C \subseteq V^n$ be an arbitrary antichain of maximum size induced by G^n on some subset of V^n that we consider fixed in the rest of the proof. For an arbitrary $\mathbf{x} \in V^{n-1}$ we define

$$J(\mathbf{x}) = \{a : \quad a \in V, \quad a\mathbf{x} \in C\},$$

where $a\mathbf{x}$ is the sequence obtained in V^n by prefixing a to $\mathbf{x}$. One easily sees that the elements of $J(\mathbf{x})$ form an independent set in G, whatever the sequence $\mathbf{x}$. For any independent set A in G we define

$$C(A) = \{\mathbf{x} : \mathbf{x} \in V^{n-1}, \quad J(\mathbf{x}) = A\}.$$

Then, clearly, for every independent set A of G the set $C(A)$ is either empty or an antichain of G^{n-1}. Moreover, for different independent sets A the corresponding sets $C(A)$ are disjoint. Denoting the set of independent sets of G by $S(G)$ this immediately implies that

$$M(G,n) = \sum_{A\in S(G)} |A||C(A)| = \sum_{(x,A)\in V(G^*)} |C(A)|. \tag{1}$$

However, much more than the above is true. Consider namely an arbitrary vertex–coloring of G^* and fix some color class in it. For brevity's sake this will be referred to as the first color class of G^*. Let the independent sets A and B of G have the property that for some vertices x and y of G both (x, A) and (y, B) belong to the first color class of G^*. In order to prove the theorem it is clearly sufficient to show that $C(A) \cup C(B) \subseteq V^{n-1}$ is an antichain of G^{n-1}. This is what we will do next.

(The above is in fact sufficient for it implies that

$$M(G,n) \leq \chi(G^*)M(G^{n-1}),$$

whence the statement follows by iteratated application of the last inequality.)

Let A and B be as above. Then either there exists an $a \in A$ and a $b \in B$ that are non–adjacent (including the possibility of $a = b$) in G, or else one of the two sets, say B, has an element b such that for some two different vertices of G, $c \in A$, $d \in A$ we have

$$(b,c) \in E(G) \qquad (d,b) \in E(G).$$

In the first hypothesis, consider any $\mathbf{x} \in C(A)$ and $\mathbf{y} \in C(B)$. The sequences $a\mathbf{x}$ and $b\mathbf{y}$ belong to C and hence are G–unrelated. Since a and b are non–adjacent, this implies that also $\mathbf{x}$ and $\mathbf{y}$ are G–unrelated as we have claimed. In the second hypothesis, suppose indirectly, that there exists a sequence $\mathbf{x} \in C(A)$ and a sequence $\mathbf{y} \in C(B)$ such that they are not G–unrelated. How can this happen? One reason might be that $(x_i, y_i) \in E(G)$ for some i, while $(y_j, x_j) \in E(G)$ never occurs for any j. But then $d\mathbf{x}$ and $b\mathbf{y}$ would not be G–unrelated, a contradiction. Likewise, if $(y_i, x_i) \in E(G)$ for some i, whereas $(x_j, y_j) \in E(G)$ does not occur for any j, then $c\mathbf{x}$ and $b\mathbf{y}$ do not satisfy the condition of being G–unrelated.

Consider now an optimal coloring of the vertices of G^*. For any color $j \in \{1, 2, \ldots, \chi(G^*)\}$ denote by $\mathcal{R}(j)$ the family of all the independent sets A of G appearing in the color class j for some vertex $x \in A$. Rephrasing what we just proved we see that

$$\sum_{A \in \mathcal{R}(j)} |C(A)| \leq M(G, n-1),$$

for $j = 1, 2, \ldots, \chi(G^*)$. Substituting these inequalities into (1) we get

$$M(G, n) \leq \chi(G^*) M(G, n-1)$$

as claimed.

In order to prove our more precise second statement it suffices to notice that any antichain in G itself is an independent set. □

Theorem 2 *For an arbitrary oriented graph G we have*

$$A(G) \leq \log \chi^*(G^*).$$

Proof

For fixed k and a k–fold covering of the vertices of G^* by independent subsets of G^* the previous argument applies. □

Theorem 1 has the following obvious consequence:

Proposition 2 *For any oriented graph G we have*

$$W(G) \leq \chi(G^*).$$

Proof

The statement is a direct consequence of Proposition 1 and Theorem 1. □

(Obviously the stronger inequality $W(G) \leq \chi^*(G^*)$ is also true; we stated the weaker one only since it will suffice throughout the paper in later references.)

We do not always have equality in the last inequality. The simplest example for strict inequality is provided by C_3, the triangle graph with cyclically oriented edges. For this particular graph we have $W(C_3) = 2$ while $\chi(C_3^*) = 3$. Going back to our starting point we can show however that equality holds for every oriented graph that describes the precedence relations of a poset. In other words this means that for a poset P the antichain capacity is determined in terms of the waterfall number of the corresponding graph. We can also prove equality for bipartite graphs. Both of these cases will be discussed in the last section.

4 Waterfalls in simple graphs

Beyond their application to derive lower bounds for antichain capacity, waterfalls interest us on their own. We are interested in the study of the asymptotic growth of waterfalls in simple graphs and in particular in individuating the class of graphs for which the waterfall number is multiplicative for powers of the graph.

A symmetrically directed graph can be identified with a simple graph on the same vertex set in the obvious manner. Although a waterfall and the waterfall number have already been defined for this case, we find it useful to start from scratch and give an equivalent reformulation of our problems.

Let G be a simple graph with vertex set V. Any induced subgraph of G which is a complete k–partite graph for some k will be called a *waterfall* in G. As before, the largest cardinality of any waterfall of G is called its waterfall number, denoted by $W(G)$. We are interested in determining the exponential asymptotics of $W(G^n)$, where G^n is the n'th co-normal power of the simple graph G. Our problem has the following quite natural interpretation. If the absence of an edge is understood as a relation of "sameness" (indistinguishability) then the waterfall number is the cardinality of the largest set on which this relation is an equivalence relation. Thus our question amounts to examine the natural extension of the relation of "sameness" to n–length sequences of symbols from the point of view of the growth of the cardinality of the largest equivalence relation induced on some subset of V^n. In what follows we will derive upper and lower bounds for the cascade capacity

$$\Theta(G) = \lim_{n\to\infty} \frac{1}{n} \log W(G^n)$$

of arbitrary simple graphs. Our main interest will be in seeing for which class of graphs do we have

$$\Theta(G) = \log W(G). \qquad (2)$$

We will show that the above relation is true for bipartite graphs and odd cycles.

Once again, we will upper bound waterfall numbers for the powers of the simple graph G in terms of what we call its independence graph, yet our previous setup does not have an automatic reformulation for the undirected case. We have to be careful, as it can already be seen from the fact that the symmetrically directed graph corresponding to a simple graph would have an antichain capacity equal to the logarithm of the cardinality of its vertex set, and thus our Theorem 1 cannot be saved in the present context. We define the *independence graph* G^* of an arbitrary simple graph G. The vertex set of G^* is the set of all pairs (x, A) such that x is a vertex of G contained in the independent set A of G. The pairs (x, A), (x', A') are adjacent in G^* if either A and A' are disjoint and every vertex of A is joined by an edge with every vertex of A', or $A = A'$ but $x \neq x'$. We will say that the independent sets A and A' are *isochromatic* if not every vertex of A is adjacent to all the vertices of A'.

Our aim in this section is to prove that the chromatic number $\chi(G^*)$ of the independence graph of G is an upper bound of all the renormalized waterfall numbers $\sqrt[n]{W(G^n)}$

of the powers of G. The proof of this is not unlike but more complicated than that of Theorem 1. We will start by establishing a few lemmas.

Unless otherwise stated, in what follows we will fix a graph G with vertex set $V = V(G)$, an integer $n > 0$ and a set $C \subseteq V^n$ on which the graph G^n induces a waterfall.

Observe that a waterfall cannot contain as an induced subgraph 3 points of which one is isolated and the other two are adjacent. This property clearly characterizes waterfalls. In the subsequent lemmas we will make repeated use of this fact and in doing so we will refer to this 3–point graph as the *forbidden configuration.*

Lemma 2 *For an arbitrary* $\mathbf{x} \in V^{n-1}$ *write*

$$J(\mathbf{x}) = \{a : \quad a \in V, a\mathbf{x} \in C\},$$

where $a\mathbf{x}$ *denotes the juxtaposition of* a *and the sequence* $\mathbf{x}$*. Then* $J(\mathbf{x})$ *is a waterfall in* G*.*

Proof

Obvious. □

To any independent set $A \subseteq V$ of G we associate the set $C(A)$ consisting of those sequences $\mathbf{x} \in V^{n-1}$ for which A is a maximal independent set in the waterfall $J(\mathbf{x})$.

Lemma 3 *If the independent sets* A *and* B *of* G *are isochromatic then* $C(A)$ *and* $C(B)$ *are disjoint and* G^{n-1} *induces a waterfall on their union.*

Proof

Observe first of all that $C(A)$ and $C(B)$ are disjoint. In fact, if they had a common element $\mathbf{y}$, say, then certainly $A \cup B$ would contain at least one edge for otherwise it were a subset of $J(\mathbf{y})$. Then, however, $\mathbf{y}$ would be contained in $C(D)$ for some independent set D of G containing $A \cup B$, which contradicts our hypothesis that A and B are maximal independent sets in the waterfall $J(\mathbf{y})$. Thus there is an element in either A or B that is adjacent to some but not all the elements of the other set. Suppose without loss of generality that this element is $a \in A$. Hence there exist $b \in B$, $c \in B$ such that $(a, b) \in E(G)$ while $(a, c) \notin E(G)$. But then $a\mathbf{y}$, $b\mathbf{y}$ and $c\mathbf{y}$ would form a configuration that cannot be present in a waterfall. This proves the disjointness of $C(A)$ and $C(B)$.

Suppose now to the contrary that A and B are isochromatic independent sets of G and yet G^{n-1} does not induce a waterfall on $C(A) \cup C(B)$. This means that there is a forbidden configuration in G^{n-1} with vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in $C(A) \cup C(B)$. Suppose in particular that $\mathbf{x}$ and $\mathbf{y}$ form an edge. Further, since A and B are isochromatic, there must be an $a \in A$ and a $b \in B$ such that $(a, b) \notin E(G)$. If in the forbidden configuration the two endpoints of the edge are contained one in $C(A)$ and the other in $C(B)$, i. e., $\mathbf{x} \in C(A)$ and $\mathbf{y} \in C(B)$, say, then $a\mathbf{x}$, $b\mathbf{y}$ and $b\mathbf{z}$ form a forbidden configuration in C, yielding a contradiction. Suppose therefore that the endpoints $\mathbf{x}$ and $\mathbf{y}$ of the edge in the forbidden configuration induced by G^{n-1} on $C(A) \cup C(B)$ fall both into the same set;

$C(A)$ or $C(B)$. We can suppose without loss of generality that they both fall into $C(A)$. Then if $\mathbf{z} \in C(B)$, we find that $a\mathbf{x}$, $a\mathbf{y}$ and $b\mathbf{z}$ form a forbidden configuration in C, once again a contradiction. Finally, if $\mathbf{z} \in C(A)$, the desired contradiction is achieved by the forbidden configuration $a\mathbf{x}, a\mathbf{y}, a\mathbf{z}$. □

Lemma 4 *If for the independent sets A and B of G the graph G^{n-1} does not induce a complete bipartite graph on $C(A) \cup C(B)$, then G induces a waterfall on $A \cup B$.*

If further A and B are isochromatic, then $A \cup B$ is an independent set of G.

Proof

By our hypothesis, there exist $\mathbf{x} \in C(A)$, $\mathbf{y} \in C(B)$ such that $(\mathbf{x}, \mathbf{y}) \notin E(G^{n-1})$. To reason indirectly, suppose that G does not induce a waterfall on $A \cup B$. Then, without loss of generality we can suppose that $a \in A$, $b \in B$, $c \in B$ form a forbidden configuration in which $(a, b) \in E(G)$. Then, however, $a\mathbf{x}$, $b\mathbf{y}$, $c\mathbf{y}$ form a forbidden configuration in C, yielding a contradiction.

As for the second statement of the lemma, it is immediate from the foregoing that if A and B are isochromatic independent sets in G and the latter induces a waterfall on $A \cup B$, then A and B cannot form a complete bipartite graph and hence the union of A and B must be an independent set of G.

□

Now we are ready to prove our main lemma:

Lemma 5 *If any two of the independent sets A, B and D of G are iso–chromatic, then G^{n-1} induces a waterfall on $C(A) \cup C(B) \cup C(D)$.*

Proof

By Lemma 3 the sets $C(A)$, $C(B)$ and $C(D)$ are disjoint and G^{n-1} induces a waterfall on the union of any two of them. We reason indirectly, and suppose that despite the above, G^{n-1} does not induce a waterfall on $C(A) \cup C(B) \cup C(D)$. By the foregoing, this can only happen if there are 3 sequences falling into the 3 different sets, $\mathbf{x} \in C(A)$, $\mathbf{y} \in C(B)$ and $\mathbf{z} \in C(D)$ on which the graph G^{n-1} induces a forbidden configuration. Without loss of generality, we can suppose that in this configuration $(\mathbf{x}, \mathbf{y}) \in E(G^{n-1})$. By Lemma 4 this implies that both $D \cup A$ and $D \cup B$ are independent sets of G. Hence there exist 3 vertices, $d \in D$, $a \in A$ and $b \in B$ such that d is not adjacent to either a or b. But then $d\mathbf{z}$, $a\mathbf{x}$, $b\mathbf{y}$ form a forbidden configuration in C, a contradiction. □

The last three lemmas immediately yield

Theorem 3

$$W(G^n) \leq [\chi(G^*)]^n.$$

Proof

Clearly, it is enough to show that

$$W(G^n) \leq \chi(G^*)W(G^{n-1}).$$

To prove this inequality, consider a waterfall $C \subseteq G^n$ of maximum cardinality. Let further $\mathcal{R}$ denote a coloring of the vertices of the independence graph of G with the minimum number of colors $\chi(G^*)$. Let us fix an arbitrary color class of this coloring and consider all the independent sets A of G such that there is a vertex x of A for which the vertex (x, A) of G^* belongs to this color class. By the definition of the independence graph all these independent sets A from the fixed color class will occur just once among these pairs and any two of them will be isochromatic. Hence, by Lemma 3 the subgraphs induced by G^{n-1} on the corresponding sets $C(A)$ will all be vertex–disjoint. Let us denote, for every color $j \in \{1, 2, \ldots, \chi(G^*)\}$ of $\mathcal{R}$ by $\mathcal{R}(j)$ the family of all the independent sets associated with this color class in the above manner. Then, clearly,

$$|C| \leq \sum_{j=1}^{\chi(G^*)} \sum_{A \in \mathcal{R}(j)} |C(A)|. \tag{3}$$

Observe, however, that by Lemma 5 the graph G^{n-1} induces a waterfall on the union of all those sets $C(A)$ for which the corresponding independent set A belongs to the same fixed color class of $\mathcal{R}$. (In fact, G^{n-1} induces a waterfall on any three of them by said lemma. But any possible forbidden configuration would be contained in the union of at most three among the sets $C(A)$.) Hence our last inequality implies that

$$W(G^n) = |C| \leq \chi(G^*)W(G^{n-1}).$$

To conclude the proof, we notice that for $n = 1$

$$W(G) \leq \omega(G^*) \leq \chi(G^*).$$

□

Remark: Just as in the oriented case, we can prove the stronger statement

$$W(G^n) \leq [\chi^*(G^*)]^n$$

using the same ideas as above.

We conclude this section with a last remark about waterfalls. In general, it is interesting to study the growth of the largest subgraph of a certain type in product graphs. If we restrict ourselves to classes closed under the co–normal product, we get supermultiplicative (including multiplicative) behavior. If we further restrict ourselves to hereditary classes, i. e., those which contain every induced subgraph of their elements then only four non-empty classes remain as we will see in the next proposition. One of these classes is the trivial class containing all graphs. Another one is the class of empty graphs for which the growth in products is always multiplicative as mentioned in the Introduction. The third class is the class of complete graphs, the corresponding question of asymptotic growth in products is the Shannon capacity problem. The only remaining class is the class of waterfalls for which the question of asymptotic growth in products leads to the problem of determination of the cascade capacity.

Proposition 3 *The largest class of graphs that is different from the class of all graphs and has the property of being closed under the two operations of taking co–normal products and taking induced subgraphs is the class of waterfalls. Moreover, the only other (non-empty) such classes of graphs are the classes containing all empty graphs and all complete graphs, respectively.*

Proof

We prove the first statement first. It is easy to see that the class of waterfalls is closed under the above two operations. What we really have to prove is that a larger class of graphs with this property should contain every graph. To this end we first show that in the co–normal powers of our forbidden configuration every graph appears as an induced subgraph. Let F be an arbitrary graph and Z be the graph of our forbidden configuration, a graph on vertices 0, 1 and 2, with a single edge between 1 and 2. We build up $|V(F)|$ ternary sequences of length $|V(F)|-1$ over $V(Z)$ with the property of inducing a graph isomorphic to F in $Z^{|V(F)|-1}$. For simplicity of the presentation we assume that we already know which sequence will belong to which vertex of F and we will refer to the sequence belonging to the ith vertex of F (according to some arbitrary order) as the ith sequence. Now let the ith coordinate $(1 \leq i \leq |V(F)|-1)$ of the sequences be given as follows. It is 1 for the ith sequence, 2 for the sequences belonging to vertices that are adjacent with the ith vertex and 0 for the rest. It is easy to check that the jth and kth sequences are adjacent in $Z^{|V(F)|-1}$ if and only if the jth and kth vertices are adjacent in F. Now suppose we have a class of graphs that is closed under the two operations we have in the statement. If it contains a graph that is not a waterfall then it necessarily contains Z and so by the foregoing it should contain every graph.

For the second statement notice that once we have a non-empty but not-complete graph in our class then it also contains the graph with three vertices and two edges. It is easy to verify that the co–normal products of this graph contain every possible waterfall as an induced subgraph. This implies our statement.

□

5 When are the bounds tight?

We shall call a directed graph G conservative if every induced subgraph $F \subseteq G$ satisfies

$$\Theta(F) = \log W(F). \tag{4}$$

This definition is inspired by the concept of perfect graphs. Our bounds allow us to show that several classes of graphs are conservative. First we will show, as promised at the end of Section 3, that comparability digraphs (see definition below) and oriented bipartite graphs give equality in the inequality of Proposition 2. This immediately implies the conservativeness of these graphs. Then we will deal with simple graphs again, showing that bipartite graphs and odd cycles are conservative. An example found by Tomasz Łuczak shows the existence of non-conservative simple graphs. His example can easily be extended to an example of a non-conservative oriented graph once we have Proposition 4. The paper concludes with an application to extremal set theory.

Let us first recall the definition of a comparability graph and its extension to digraphs.

A graph G is a comparability graph if there exists a poset P on $V(G)$ for which $u, v \in V(G)$ are comparable if and only if they are adjacent in G. We call G a comparability digraph if there exists a poset P on $V(G)$ in which $u \in V(G)$ precedes $v \in V(G)$ in P if and only if there is an edge from u to v in G.

It is well known that every comparability graph is perfect, cf. e.g. [1].

It is straightforward from the definition of the independence graph that

$$\omega(G^*) = W(G) \tag{5}$$

for every directed graph G that does not contain a cyclically oriented triangle.

We will associate with an arbitrary digraph G its reduced independence graph $G^{(*)}$ through the following definition. The vertex set of $G^{(*)}$ consists of the independent sets of G and two vertices are adjacent in $G^{(*)}$ if and only if the corresponding independent sets form a waterfall which is not a single independent set in G. Notice that G^* can be obtained from $G^{(*)}$ if we substitute to each vertex a clique of the size of the independent set it represents in $G^{(*)}$.

Theorem 4 *For any comparability digraph G we have*
$W(G) = \chi(G^*)$.

Proof

We claim that if G is a comparability digraph then $G^{(*)}$ will be a comparability graph. Indeed, it is easy to check that the following is a partial order on the independent sets of G. Let the independent set A precede the independent set B if every element of A precedes every element of B in the original partial order (cf. [6]). The comparability graph defined by this partial order is just $G^{(*)}$. This implies that $G^{(*)}$ is perfect. Using the result of Lovász [11] saying that the substitution of a vertex of a perfect graph by another perfect graph preserves perfectness, we see that G^* is perfect. This implies $\chi(G^*) = \omega(G^*)$ and since a comparability digraph cannot contain a cyclically oriented triangle the statement follows by (5). □

Corollary 1 *The antichain capacity of a partial order equals* $\log W(G)$ *where G is the comparability digraph of the partial order.*

Proof The statement immediately follows from Proposition 1 and Theorem 1 by the previous theorem. □

Corollary 2 *Every comparability digraph is conservative.*

Proof

We already know by the foregoing that for any oriented graph F we have

$$\log W(F) \leq \Theta(F) \leq A(F) \leq \log \chi(F^*) \tag{6}$$

and so the previous theorem and the trivial observation that every induced subgraph of a comparability digraph is a comparability digraph implies the statement. □

Another class of oriented graphs for which we can prove the tightness of the upper bound in Theorem 1 is the class of oriented bipartite graphs (regardless of the actual orientation of the edges).

Theorem 5 *For any oriented bipartite graph G we have*
$A(G) = \log W(G)$.

Proof

What we will actually prove is again the statement $\chi(G^*) = W(G)$. By Theorem 1 this implies what we need. Consider the reduced independence graph $G^{(*)}$. We claim that if G is bipartite then so is $G^{(*)}$. To see this consider a covering of the nodes of G with two independent sets, A and B. (Such a covering exists since G is bipartite.) Each vertex of $G^{(*)}$ represents an independent set of G that should have a nonempty intersection with at least one of A and B. Now look at those independent sets whose intersection with A is nonempty. No pair of them can induce a waterfall with two independent sets, since they have non-adjacent (or possibly common) vertices, those that are also in A. So all these independent sets are non-adjacent as nodes of $G^{(*)}$. However, all other nodes of $G^{(*)}$ represent independent sets intersecting B and thus they are non-adjacent for the same reason. This proves that $G^{(*)}$ is bipartite. Since a bipartite graph is always perfect and cannot contain any triangle (cyclic or not), we conclude that $\chi(G^*) = W(G)$ as in the proof of Theorem 4. □

Corollary 3 *Every oriented bipartite graph is conservative.*

Proof

The proof is identical with that of Corollary 2 if replacing Theorem 4 by Theorem 5. □

Before turning to simple graphs we make a final observation about the oriented case.

It is also natural to ask whether the growth of the largest transitive tournament (which is a special waterfall) has a multiplicative nature. The answer is no and a counterexample is given by a specific orientation of C_5. The size of the largest transitive tournament in this graph is obviously two.

Proposition 4 *There exists an orientation of C_5 for which the second power of the resulting graph contains a transitive tournament on five points.*

Proof

Let the vertices be $0, 1, 2, 3, 4$ in a cyclic order. The orientation now is not cyclic, however, and the edges are oriented as follows: $(0,1), (0,4), (2,1), (2,3), (4,3)$. One easily checks that the vertices (00), (24), (12), (43) and (31) induce a transitive tournament in the second power of the graph. □

Remark: The above observation also implies, that the Sperner capacity of the pentagon oriented as in the proof equals its Shannon capacity, $1/2 \log 5$. Rob Calderbank has shown us [5] by their method [4] that all the other oriented versions of the five length cycle have Sperner capacity 1.

Now we come to the analysis of conservativeness in case of simple graphs. Although a simple graph can be identified with a symmetrically directed graph in the obvious way, let us be explicit again and repeat the definition for this case. We say that a simple graph G is conservative if every induced subgraph $F \subseteq G$ satisfies (4). ($\Theta(F)$ and $W(F)$ are meant as defined for simple graphs in the previous section.)

Theorem 6 *Bipartite graphs are conservative.*

Proof

The proof of Theorem 5 can be repeated literally to prove that $\chi(G^{(*)}) = W(G)$ for any bipartite graph G. (In fact, we did not use anything about the orientation when proving Theorem 5). Then the statement follows from Theorem 3. □

One might suspect that every perfect graph is conservative. However, this is not true. In fact, there are non–conservative perfect graphs just as there exist non–perfect graphs that are conservative. We limit ourselves to showing that

Proposition 5 *The cycle C_k is conservative for any $k \geq 3$.*

Proof

It is easy to see what C_k^* looks like. The points related to independent sets of size at least 3 form separate components of their own. Each of these components is a clique of the size of the independent set it belongs to. The rest is a 3–chromatic component. Now if $k > 7$ then the chromatic number of C_k^* is determined by the size of the largest previously mentioned clique component. Since its size is that of the largest independent set, i.e., of a special waterfall in C_k, we are done by Theorem 3. If $k \leq 7$ then both $\chi(C_k^*)$ and $W(C_k)$ are 3, so again by Theorem 3 we are done. □

The following example found by Tomasz Łuczak [13] proves that not every graph is conservative. (After this example was found, one of us proved that the complement of C_8, –a perfect graph– is not conservative.)

Proposition 6 *(Łuczak) Let D_{10} denote the graph we obtain by substituting a clique of size two to each vertex of C_5, the cycle of length five. This graph is not conservative.*

Proof

It is easy to check that $W(D_{10}) = 4$. On the other hand it is well known that $\omega(C_5^2) = 5$, cf. [14], [12]. If the vertices of C_5 are $0, 1, 2, 3, 4$ in a cyclic order, such a clique is induced by the sequences $00, 12, 24, 31, 43$. Let the vertices of our D_{10} be $0, 0', 1, 1', .., 4, 4'$ where aa' is the clique substituted in place of a of C_5. Then replacing each sequence ab in the above clique of C_5^2 by the four sequences $ab, a'b, ab', a'b'$ we obtain twenty sequences of length 2 inducing a clique in D_{10}^2. This means that $W(D_{10}^2) \geq \omega(D_{10}^2) \geq 20 > [W(D_{10})]^2$. □

Combining the previous example with that of Proposition 4 one easily obtains an example of a non-conservative oriented graph. Indeed, let us consider the following orientation of the graph D_{10} of the previous proposition. (The vertices are labelled as in the previous proof.) Any edge joining vertices a and a' is oriented from a to a'. Edges joining vertex a or a' to vertex b or b' $(a \neq b)$ are oriented as the edge between a and b in Proposition 4. Let the resulting oriented graph be denoted by D_{10}^+.

Proposition 7 *The oriented graph D_{10}^+ is not conservative.*

Proof It is easy to check that $W(D_{10}^+) = 4$. Consider the construction of Proposition 4 over the five vertices $0, 1, 2, 3, 4$. The five sequences of length 2 obtained this way can be extended to twenty sequences of length 2 in the same way as in Proposition 6. (To be explicit these twenty sequences are as follows: $00, 00', 0'0, 0'0', 24, 2'4, 24', 2'4', 12, 1'2, 12', 1'2', 43, 4'3, 43', 4'3', 31, 3'1, 31', 3'1'$.) These twenty sequences induce a transitive tournament in the second power of D_{10}^+ proving the strict supermultiplicativity of the waterfall number for this graph. □

Some of the results in the paper might have applications in extremal set theory as was the case with Sperner capacity. Let us conclude with a simple example.

Proposition 8 *Given an n–set X, let the family of pairs $\{(A_l, B_l)\}_{l=1}^m$ of subsets of X have the properties*

$$A_i \cap B_j = \emptyset \quad \textit{if and only if} \quad A_j \cap B_i = \emptyset,$$

$$A_i \cap B_i = \emptyset, \quad i = 1, 2, \dots n.$$

The maximum $M(n)$ of the number m of such pairs from an n–set is 2^n.

Proof

We start by showing that

$$M(n) \leq 2^n.$$

To this end consider the oriented graph G with vertex set $V(G) = \{0, 1, 2\}$ and edge set $E(G) = \{(1, 2)\}$. Now let us have any family of set pairs as in the statement of the proposition. We can identify any set pair (A_l, B_l) with a ternary sequence, i. e., an element of $\{0, 1, 2\}^n$ in the following manner. We identify X with the numbers from 1 to

n and define the ternary sequence $\mathbf{x} = x_1 x_2 \ldots x_n$ by setting $x_k = 1$ if $k \in A_i$, $x_k = 2$ if $k \in B_i$ and $x_k = 0$ else. Thus we can see that the family of pairs with our property defines an antichain in $[V(G)]^n$ for G and hence, by Theorem 1 we see that $M(n) \leq 2^n$.

The next observation giving the lower bound $M(n) \geq 2^n$ is due to Zsolt Tuza [16]. Divide X into two disjoint parts A and B any way you like and then consider the family of pairs

$$\{(Y \cap A, Y \cap B) : Y \subseteq X\}.$$

□

For an exhaustive bibliography of extremal problems for set pairs we refer the reader to Tuza's survey [17].

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