Colourful theorems and topological lower bounds on chromatic numbers

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Abstract

We introduce the cross-index, which refines the usual topological lower bounds on the chromatic number of a graph. We use the cross-index to prove colourful complete bipartite subgraph theorems in the spirit of [15, 16]. We investigate the complexity of computing the cross-index of $\mathbb{Z}_2$-posets.

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1. Introduction

For any proper colouring c of a graph G and any linear ordering of the colours, G contains a path with $k = \chi(G)$ vertices all of which have different colours, appearing in increasing order along the path. Indeed this is a consequence of the classical result of Gallai on colourings and orientations: If we orient each edge towards its endpoint with the larger colour, we get an acyclic orientation of a k-chromatic graph, which must contain a directed k-path, that is, a path with k colours appearing in increasing order. Furthermore if c uses only $k = \chi(G)$ colours, we can say more: According to an exercise in Douglas West's textbook [18], if T is any tree on k vertices labeled by the colours, then G contains a coloured copy of T.

These are elementary examples of "colourful subgraph" theorems. The existence of graphs with large girth and large chromatic number shows that trees cannot be replaced by other types of graphs in these results. Also, the existence of graphs with local chromatic number 3 and large chromatic number (see [7]) shows that if c uses more than $k = \chi(G)$ colours, then G is not even guaranteed to contain a claw using four different colours. So these two colourful subgraph results cannot be extended in the general case.

Nonetheless, the context of the local chromatic number prompted the authors of [15, 16] to investigate analoguous results involving colourful complete bipartite subgraphs. They had to restrict their attention to classes of graphs with suitable structural properties. As their results show, effective criteria exist among the topological obstructions to small chromaticity, measured by indices and coindices of complexes associated with graphs, in the spirit of [5, 11, 12].

Theorem 1.1 (Zig-zag theorem [15]). *Let G be a graph which is topologically t-chromatic in the sense that* $\mathrm{coind}(B_0(G)) + 1 \geq t$. *Let c be an arbitrary proper colouring of G by an arbitrary number of colours. We assume the colours are linearly ordered. Then G contains a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ such that c assigns distinct colours to all t vertices of this subgraph and these colours appear alternating on the two sides of the bipartite subgraph with respect to their order.*

The parameter $\mathrm{coind}(B_0(G))$ used in this result is the "coindex of the box complex $B_0(G)$"; for its definition see Section 3. The inequality $\chi(G) \geq \mathrm{coind}(B_0(G)) + 1$ belongs to a hierarchy of topological lower bounds for $\chi(G)$

that will also be discussed in Section 3. When the bound is tight, more can be said about the structure of colourings:

Theorem 1.2 (Colourful $K_{l,m}$ theorem [16])**.** *Let G be a graph for which $\chi(G) = \mathrm{coind}(B_0(G)) + 1 = t$. Let $c : V(G) \to \{1, \ldots, t\}$ be a proper colouring of G and let $A, B \subseteq \{1, \ldots, t\}$ form a bipartition of the colour set, i.e., $A \cup B = \{1, \ldots, t\}$ and $A \cap B = \emptyset$.*

Then there exists a complete bipartite subgraph $K_{l,m}$ of G with sides L, M such that $|L| = l = |A|$, $|M| = m = |B|$, and $\{c(v) : v \in L\} = A$, and $\{c(v) : v \in M\} = B$. In particular, this $K_{l,m}$ is completely multicoloured by c.

Theorems 1.1 and 1.2 illustrate the fact that the known topological lower bounds on the chromatic number can reach high values only if the complete bipartite subgraphs of G have a rich enough structure. As in the case of colourful trees in general graphs, we get stronger results when the colouring considered uses the minimum number of colours.

Theorems 1.1 and 1.2 were proved using results from algebraic topology, namely Ky Fan's theorem for Theorem 1.1 and the Tucker-Bacon theorem for Theorem 1.2 (both of which are equivalent versions of the celebrated Borsuk-Ulam theorem, see [1]). Using an idea that appears in Xuding Zhu's presentation [19] of some results of [13, 15] one can give a simpler proof (also based on the Borsuk-Ulam theorem) for Theorem 1.2, using the more general hypothesis $\chi(G) = \mathrm{ind}(B_0(G)) + 1 = t$. (This was also observed by Carsten Schultz [14].)

Here we present a simple proof of Theorem 1.1 using a different parameter, yielding a more general result. This new parameter, which we call the "cross-index", is derived from the natural order-theoretic definition of the cross polytope (see [11]). It is purely order-theoretic, though it fits within the hierarchy of topological parameters sandwiched between the chromatic number and the clique number, yielding a sharper lower bound for the chromatic number than any of the four topological bounds considered in this paper.

This raises the question as to whether a stronger version of Theorem 1.2 can also be proved with our methods. Note that a negative answer would provide a combinatorial distinction between different topological lower bounds on the chromatic number, in the spirit of [17]. In Section 2 after proving our more general version of the Zig-zag theorem, we also prove a colourful

$K_{n+1,n-1}$ theorem using the cross-index. In Section 3, we compare the relative strength of the hypotheses of the colourful $K_{l,m}$ theorems. Finally, in Section 4, we investigate complexity aspects of the cross-index.

2. The cross-index

Two sets A, B of vertices of a graph G are said to be *totally joined* if every vertex of A is adjacent to every vertex of B. We denote $\mathrm{Hom}(K_2, G)$ the set of ordered couples (A, B) of totally joined nonempty sets of vertices of G. There is a natural ordering $\leq$ on $\mathrm{Hom}(K_2, G)$, defined by $(A, B) \leq (A', B')$ if $A \subseteq A'$ and $B \subseteq B'$. We also consider the inversion $-$ on $\mathrm{Hom}(K_2, G)$ defined by $-(A, B) = (B, A)$. $\mathrm{Hom}(K_2, G)$ endowed with $\leq$ and $-$ is a $\mathbb{Z}_2$*-poset*, that is, an ordered set with a fixed-point free automorphism $-$ of order 2. A $\mathbb{Z}_2$*-map* between $\mathbb{Z}_2$-posets P and Q is an order-preserving map $\phi : P \to Q$ such that $\phi(-x) = -\phi(x)$.

The notation $\mathrm{Hom}(K_2, G)$ stands for the *homomorphism complex* of K_2 in G: For every $(A, B) \in \mathrm{Hom}(K_2, G)$, we get a homomorphism of the complete graph K_2 (with vertex-set $\{0, 1\}$) to G by selecting any pair of elements $a \in A$, $b \in B$ as respective images of 0 and 1. For graphs G and H, if there exists a homomorphism $\psi : G \to H$, then we can define a $\mathbb{Z}_2$-map $\hat{\psi} : \mathrm{Hom}(K_2, G) \to \mathrm{Hom}(K_2, H)$ by $\hat{\psi}(A, B) = (\psi(A), \psi(B))$.

For an integer $n \geq 0$, let Q_n be the $\mathbb{Z}_2$-poset on the $2(n + 1)$-element set $\pm 0, \pm 1, \ldots, \pm n$, with its natural inversion and the order defined by $x < y$ (in Q_n) if $|x| < |y|$ (in $\mathbb{N}$). Let K_{n+2} be the complete graph with vertex-set $\{0, \ldots, n + 1\}$. There exists a $\mathbb{Z}_2$-map $\phi : \mathrm{Hom}(K_2, K_{n+2}) \to Q_n$ defined by

$$\phi(A, B) = \begin{cases} |A \cup B| - 2 \text{ if } \min(A \cup B) \in A \\ -(|A \cup B| - 2) \text{ if } \min(A \cup B) \in B. \end{cases}$$

Therefore, for any graph G with chromatic number at most $n+2$, $\mathrm{Hom}(K_2, G)$ admits a $\mathbb{Z}_2$-map to Q_n. The *cross-index* of a $\mathbb{Z}_2$-poset P, denoted $\mathrm{Xind}(P)$, is defined as the smallest t such that P admits a $\mathbb{Z}_2$-map to Q_t. Thus we have $\chi(G) \geq \mathrm{Xind}(\mathrm{Hom}(K_2, G)) + 2$.

Our version of the Zig-zag theorem uses the cross-index.

Theorem 2.1. *Let G be a graph such that $\mathrm{Xind}(\mathrm{Hom}(K_2, G)) + 2 \geq t$, and let c be an arbitrary proper colouring of G by an arbitrary number of colours. We assume the colours are linearly ordered. Then G contains a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ such that c assigns distinct colours to all t vertices of this* *subgraph and these colours appear alternating on the two sides of the bipartite subgraph with respect to their order.*

Proof. For $(A,B) \in \mathrm{Hom}(K_2,G)$, there exists a longest sequence $x_1,\dots,x_\ell$ alternating between elements of A and elements of B, and such that $c(x_1) < \cdots < c(x_\ell)$. The sequence itself may not be unique; however we can assume without loss of generality that $c(x_1)$ is the minimal colour used on $A \cup B$. We put $\ell(A,B)=\ell$, $\sigma(A,B)=+$ if $c(x_1)$ is used in A, and $\sigma(A,B)=-$ if $c(x_1)$ is used in B. We define $\phi : \mathrm{Hom}(K_2,G) \to Q_{|V(G)|-2}$ by

$$\phi(A,B) = \begin{cases} \ell(A,B)-2 \text{ if } \sigma(A,B)=+, \\ -(\ell(A,B)-2) \text{ if } \sigma(A,B)=-. \end{cases}$$

Then ϕ is a $\mathbb{Z}_2$-map. If $\mathrm{Xind}(\mathrm{Hom}(K_2,G))+2 \geq t$, then $\phi(\mathrm{Hom}(K_2,G)) \nsubseteq Q_{t-3}$, hence there exists $(A,B) \in \mathrm{Hom}(K_2,G)$ such that $\phi(A,B) \notin Q_{t-3}$. By definition of ϕ, there exists a sequence $x_1,\dots,x_t$ alternating between elements of A and elements of B, and such that $c(x_1) < \cdots < c(x_t)$. ☐

When the bound is even and is tight, we have the following result:

Theorem 2.2. *Let G be a graph such that $\chi(G) = \mathrm{Xind}(\mathrm{Hom}(K_2,G))+2 = 2n \geq 4$. Let $c : V(G) \to \{1,\dots,2n\}$ be a proper colouring of G. Then G contains a copy C of $K_{n+1,n-1}$ such that c uses all $2n$ colours on C.*

Proof. Suppose that $\chi(G)=2n$ and c is a $2n$-colouring of G. If there are no copies C of $K_{n+1,n-1}$ such that c uses all $2n$ colours on C, we can define a $\mathbb{Z}_2$-map $\phi : \mathrm{Hom}(K_2,G) \to Q_{2n-3}$ as follows. We partition $\mathrm{Hom}(K_2,G)$ into three sets

$$\begin{aligned} S_1 &= \{(A,B) : \max\{|c(A)|,|c(B)|\} \leq n \text{ and } |c(A\cup B)| \leq 2n-2\}, \\ S_2 &= \{(A,B) : \max\{|c(A)|,|c(B)|\} \geq n+1\}, \\ S_3 &= \{(A,B) : (|c(A)|,|c(B)|) \in \{(n-1,n),(n,n-1),(n,n)\}\}. \end{aligned}$$

If $(A,B) \in S_1$, we put

$$\phi(A,B) = \begin{cases} |c(A\cup B)|-2 \text{ if } \min c(A\cup B) \in c(A), \\ -(|c(A\cup B)|-2) \text{ if } \min c(A\cup B) \in c(B). \end{cases}$$

Note that ϕ clearly preserves the order and the inversion on S_1, and $\phi(S_1) \subseteq \{\pm 0,\dots,\pm 2n-4\}$. The remaining elements on $\mathrm{Hom}(K_2,G)$ will be mapped to $\pm(2n-3)$. If $(A,B) \in S_2$, we put

$$\phi(A,B) = \begin{cases} 2n-3 \text{ if } |c(A)| \geq n+1, \\ -(2n-3) \text{ if } |c(B)| \geq n+1. \end{cases}$$

Clearly, ϕ preserves the order and inversion on $S_1 \cup S_2$. It remains to define ϕ on S_3. Here we use the hypothesis that there is no colourful copy of $K_{n+1,n-1}$, which implies that no element of S_3 is below an element of S_2. For $(A, B) \in S_3$, let $(X_{A,B}, Y_{A,B})$ be the partition of $\{1, \ldots, 2n\}$ such that $c(A) \subseteq X_{A,B}$, $c(B) \subseteq Y_{A,B}$ and $|X_{A,B}| = |Y_{A,B}| = n$. Since no copy of $K_{n+1,n-1}$ is multicoloured by c, for $(A', B') \geq (A, B) \in S_3$ we have $(A', B') \in S_3$, $X_{A',B'} = X_{A,B}$ and $Y_{A',B'} = Y_{A,B}$. We put

$$\phi(A, B) = \begin{cases} 2n-3 \text{ if } 1 \in X_{A,B}, \\ -(2n-3) \text{ if } 1 \in Y_{A,B}. \end{cases}$$

Thus $\phi : \mathrm{Hom}(K_2, G) \to Q_{2n-3}$ is a $\mathbb{Z}_2$-map, whence $\mathrm{Xind}(\mathrm{Hom}(K_2, G)) \leq 2n-3$. □

In the next section, we will see that $\mathrm{Xind}(\mathrm{Hom}(K_2, G)) + 2 \geq \mathrm{coind}(B_0(G)) + 1$. Therefore Theorem 2.1 generalises Theorem 1.1. Theorem 2.2 is independent of Theorem 1.2, with a weaker hypothesis and a weaker conclusion.

3. The hierarchy of topological bounds

To any poset P, one can associate a simplicial complex whose simplices are the chains of P. We denote $\hat{P}$ the geometric realization of this complex, that is, the set of functions $f : P \to [0, 1]$ such that $\{p \in P : f(p) > 0\}$ is a chain, and $\sum_{p \in P} f(p) = 1$. In particular, $\hat{Q}_n$ is the n-dimensional cross polytope, which is homeomorphic to the n-dimensional sphere S_n (see [11]). More generally, if P is a $\mathbb{Z}_2$-poset, then $\hat{P}$ is a $\mathbb{Z}_2$-*space*, that is, a topological space with a continuous fixed-point free involution $-$. A $\mathbb{Z}_2$-*map* between $\mathbb{Z}_2$-spaces X, Y is a continuous map $f : X \to Y$ such that $f(-x) = -f(x)$. Note that the natural homeomorphism between $\hat{Q}_n$ and S_n is in fact a $\mathbb{Z}_2$-homeomorphism.

The *index* $\mathrm{ind}(X)$ of a $\mathbb{Z}_2$-space X is the least n such that there exists a $\mathbb{Z}_2$-map from X to S_n, and its *coindex* $\mathrm{coind}(X)$ is the largest n such that there exists a $\mathbb{Z}_2$-map from S_n to X. By the Borsuk-Ulam theorem (see [11]), we always have $\mathrm{coind}(X) \leq \mathrm{ind}(X)$.

Any $\mathbb{Z}_2$-map between $\mathbb{Z}_2$-posets P and Q lifts naturally to a $\mathbb{Z}_2$-map between the $\mathbb{Z}_2$-spaces $\hat{P}$ and $\hat{Q}$. Therefore since $\hat{Q}_n$ admits a $\mathbb{Z}_2$-homeomorphism to S_n, we always have $\mathrm{Xind}(P) \geq \mathrm{ind}(\hat{P})$. (See Section 4 for examples where this inequality is strict.) We will write $\mathrm{ind}(P)$ for $\mathrm{ind}(\hat{P})$.

In the case of a graph G, the above considerations yield the following chain of inequalities.

$$\begin{aligned}\chi(G) &\geq \operatorname{Xind}(\operatorname{Hom}(K_2, G)) + 2 \geq \operatorname{ind}(\operatorname{Hom}(K_2, G)) + 2 \\ &\geq \operatorname{coind}(\operatorname{Hom}(K_2, G)) + 2 \geq \omega(G).\end{aligned}$$

By analogy with the coindex, it is possible to define an order-theoretic "cross coindex". However this parameter is uninteresting since there is never a $\mathbb{Z}_2$-map from Q_1 to $\operatorname{Hom}(K_2, G)$ for any graph G. The important parameters missing in this hierarchy correspond to the box complex $B_0(G)$. In a slightly unconventionnal manner, we will define $B_0(G)$ to be $\operatorname{Hom}(K_2, G^+)$, where G^+ is the graph obtained from G by adding a universal vertex adjacent to all the vertices of G. (See [2, 3, 4] for the "conventional" definition of $B_0(G)$, and the homotopy equivalence with $B_0(G)$ as defined here.) The hierarchy of "topological bounds" on the chromatic number of a graph G is the following.

$$\begin{aligned}\chi(G) &\geq \operatorname{ind}(\operatorname{Hom}(K_2, G)) + 2 \geq \operatorname{ind}(B_0(G)) + 1 \\ &\geq \operatorname{coind}(B_0(G)) + 1 \geq \operatorname{coind}(\operatorname{Hom}(K_2, G)) + 2 \geq \omega(G).\end{aligned}$$

This hierarchy was first discussed by Matousek and Ziegler [12] who noted that it is not a comparison of the usefulness of the bounds. Indeed, it is the bounds involving coindices that are easier to use, and that have been helpful in determining chromatic numbers of classes of graphs such as the Kneser and Schrijver graphs. From a combinatorial point of view, it would be interesting to isolate structural properties that distinguish these topological bounds. Thus our Theorem 2.1 has a negative aspect, in that extending Theorem 1.1 to the most general hypothesis, it rules out the zig-zag property as a feature distinguishing the strength of topological bounds. In [17] it is shown that the minimum possible value of the local chromatic number is different when the third or the fourth bound above achieves a certain value. Namely, $\operatorname{coind}(B_0(G)) + 1 \geq 2k$ implies that the local chromatic number is at least $k + 1$, while $\operatorname{coind}(\operatorname{Hom}(K_2, G)) + 2 \geq 2k$ implies that it is at least $k + 2$. The former means the existence of completely multicoloured $K_{1,k}$, the latter the existence of completely multicoloured $K_{1,k+1}$ subgraphs in any proper colouring. The question whether $\operatorname{coind}(\operatorname{Hom}(K_2, G)) + 2 \geq 2k$ implies also the existence of a completely multicoloured $K_{k-1,k+1}$ subgraph of G in every proper colouring is also posed in [17]. Note that although the conclusion in this statement is identical to the conclusion of our Theorem 2.2, the hypotheses are not comparable: in Theorem 2.2 the topological condition is weaker but we insist on using the minimum number of colours.

Perhaps the colourful $K_{l,m}$ property can also be used to distinguish the strength of topological bounds. It is now known that the hypothesis of Theorem 1.2 can be relaxed to $\chi(G) = \mathrm{ind}(B_0(G)) + 1$. All that we have been able to deduce from the weaker hypothesis $\chi(G) = \mathrm{Xind}(\mathrm{Hom}(K_2, G)) + 2$ is the much weaker conclusion of Theorem 2.2. Among the open variants with intermediate hypotheses and intermediate conclusions is the following.

Problem 3.1. *Let G be a graph and l, m nonnegative integers such that $\chi(G) = \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 = l + m$. Let $c : V(G) \to \{1, \ldots, l + m\}$ be a proper vertex colouring of G. Does there exist a copy C of $K_{l,m}$ such that c uses all $l + m$ colours on C?*

According to [5], the condition $\mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 \geq l + m$ at least guarantees the existence of some $K_{l,m}$ in G, without reference to colourings. Our result of this section will show that the hypotheses $\chi(G) = \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2$ and $\chi(G) = \mathrm{ind}(B_0(G)) + 1$ are different. Csorba [4] proved that the difference between $\mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2$ and $\mathrm{ind}(B_0(G)) + 1$ is at most 1, and there are graphs G such that $\mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 > \mathrm{ind}(B_0(G)) + 1$. (Note that this could still allow the equality $\chi(G) = \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2$ to never hold for graphs with this property.) No "combinatorial" construction of such graphs is known.

Proposition 3.2. *There exists a graph H such that $\chi(H) = \mathrm{ind}(\mathrm{Hom}(K_2, H)) + 2 > \mathrm{ind}(B_0(H)) + 1$.*

Proof. By results of Csorba [4], there exists a graph G such that $\mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 = 5$ and $\mathrm{ind}(B_0(G)) + 1 = 4$. However $\chi(G)$ could be larger than 5.

Let H be the categorical product $G \times K_5$; that is, the vertex set of H is $V(G) \times V(K_5)$ and the edges of H join the pairs (u, v), (u', v') such that $\{u, v\} \in E(G)$ and $\{u', v'\} \in E(K_5)$. Then we have $\chi(H) \leq 5$. In fact, if $\chi(G) = 5$, then there are homomorphisms both ways between G and H, hence $\mathrm{ind}(\mathrm{Hom}(K_2, H)) = \mathrm{ind}(\mathrm{Hom}(K_2, G))$ and $\mathrm{ind}(B_0(H)) = \mathrm{ind}(B_0(G))$.

If $\chi(G) > 5$, then there is no homomorphism from G to K_5, hence no homomorphism from G to H. However, since $\mathrm{ind}(\mathrm{Hom}(K_2, G)) = 3$, there exists a $\mathbb{Z}_2$-map f from its geometric realization to S_3. It is well known that $\mathrm{coind}(\mathrm{Hom}(K_2, K_n)) = n - 2$ hence there exists a $\mathbb{Z}_2$-map g from S_3 to the geometric realization of $\mathrm{Hom}(K_2, K_5)$. Hence $(\mathrm{id}, g \circ f)$ is a $\mathbb{Z}_2$-map from the geometric realization of $\mathrm{Hom}(K_2, G)$ to that of $\mathrm{Hom}(K_2, G) \times \mathrm{Hom}(K_2, K_5) \subseteq \mathrm{Hom}(K_2, G \times K_5) = \mathrm{Hom}(K_2, H)$. Since the first projection on $G \times K_5$ is a homomorphism from H to G, we conclude that there exist $\mathbb{Z}_2$-maps both ways between the geometric realizations of $\mathrm{Hom}(K_2, G)$ and $\mathrm{Hom}(K_2, H)$, therefore $\mathrm{ind}(\mathrm{Hom}(K_2, H)) + 2 = \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 = 5$. Since there is a homorphism from H to G, we also have $\mathrm{ind}(B_0(H)) + 1 \leq \mathrm{ind}(B_0(G)) + 1 = 4$. □

According to [6], if G is connected and $\chi(G) > n$, then all n-colourings of $G \times K_n$ are derived from n-colourings of K_n, hence $G \times K_n$ contains colourful complete bipartite graphs corresponding to every partition of the colour set. Therefore the preceding proposition cannot be used to find witnesses to a negative answer to Problem 3.1.

4. Complexity aspects

Some remarks on complexity aspects of topological lower bounds on the chromatic number can be found in Kozlov's survey paper [9] (at the end of Subsection 1.1.3) and also in his book [10] (on page 295). He mentions that while Lovász's original lower bound expressed in terms of connectivity of a simplicial complex is difficult to compute, another lower bound based on the so-called Stiefel-Whitney characteristic classes is polynomially computable. (The latter bound also depends on a $\mathbb{Z}_2$-space and when it is chosen to be $\mathrm{Hom}(K_2, G)$, then it can be expressed as $h(\mathrm{Hom}(K_2, G)) + 2$, where $h(\mathrm{Hom}(K_2, G))$ is the so-called Stiefel-Whitney height of $\mathrm{Hom}(K_2, G)$, cf. page 328 in [10]. It is shown on page 123 of [10] that if X is any $\mathbb{Z}_2$-space, then $\mathrm{coind}(X) \leq h(X) \leq \mathrm{ind}(X)$ holds.)

As the value of $\mathrm{Xind}(\mathrm{Hom}(K_2, G))$ can be found by a finite computation it is natural to ask the computational complexity of this parameter. Although we do not know the precise complexity of this question, in the context of $\mathbb{Z}_2$-posets, we have the following.

Theorem 4.1. *For an integer $d \geq 0$, the problem of determining whether an input $\mathbb{Z}_2$-poset P satisfies $\mathrm{Xind}(P) \leq d$ is polynomial if $d = 0$ and NP-complete otherwise.*

Proof. It is obvious that the problem is in NP, for you can verify that a mapping from the elements of P to the elements of Q_d is a $\mathbb{Z}_2$-map in time polynomial in the size of P.

First, let us examine the case when $d = 0$. In this case Q_d has only two elements, $+0$ and -0, and they are incomparable. Consider the comparability graph of the poset P. This is a graph whose vertices are the elements of P, and $\{x, y\}$ is an edge iff $x < y$ or $y < x$. We claim that the $\mathbb{Z}_2$-map to Q_0 exists if and only if no element x is connected to its mirror image $-x$ by a path in this graph. To prove this, suppose first that there is a $\mathbb{Z}_2$-map ϕ. If now $\{x, y\}$ is an edge then necessarily $\phi(x) = \phi(y)$ (because Q_0 does not have comparable but unequal elements), so the same must be true for any two path connected vertices x and y as well. However, if there was a path from some element x to $-x$, that would imply that $\phi(x) = \phi(-x)$, but that is a contradiction since we also know $\phi(-x) = -\phi(x)$. Now suppose for the other hand that there is no such path, and we want to construct the $\mathbb{Z}_2$-map. For this, notice that $\{x, y\}$ being an edge implies $\{-x, -y\}$ also being an edge, so the connected components of the graph can be grouped into pairs where the pair of a component consists of the mirror image of the vertices of the component. We can then take each such pair of components and let ϕ map the vertices of one of them to $+0$ and the vertices of the other one to -0. The two required identities are now obvious: $\phi(-x) = -\phi(x)$, and for any $x, y \in P$ if $x < y$ then $\phi(x) \leq \phi(y)$ (in fact they are equal). As the graph can be constructed from the poset and the condition of no paths from x to $-x$ can be checked in polynomial time, we have proved that the problem corresponding to $d = 0$ is polynomially decidable.

The following consequence of the first part of the proof is worth remembering. The only obstacle that can exclude a $\mathbb{Z}_2$-map to Q_0 is a sequence of elements $x_0, y_0, x_1, y_1, \ldots, x_{k-1}, y_{k-1}, x_k = -x_0$ such that $x_i < y_i$ and $x_{i+1} < y_i$ for each i. As a special case, for $k = 1$ this obstacle is simply two elements such that $x < y$ and $-x < y$, which is the reason why there is no $Q_1 \to Q_0$ map.

Now we shall prove that the problem is NP-hard if $d = 1$. For this, we give a Karp reduction from the satisfiability problem of boolean expressions in conjunctive normal form (CNF; for the definition and the NP-completeness of this problem, see, e.g., [8]). What this means is that our proof will have three parts: given a boolean formula in CNF, we first construct a $\mathbb{Z}_2$-poset P from it in polynomial time, then we show how to construct a $\mathbb{Z}_2$-map from P to Q_1 if we are given an evaluation of the variables that satisfies the formula, and finally we show the reverse construction of such an evaluation from a $\mathbb{Z}_2$-map.

To define P, we will give the list of its elements and the involution, and we will give some defining relations in the form $x < y$. The partial order $<$ is then understood to be the least defined transitive relation invariant to the involution and satisfying these defining relations (this is analogous to defining a poset with its Hasse diagram). One can compute the full table for this relation from the defining relations by first adding the relation $-x < -y$ for each axiom $x < y$ given, then taking the transitive closure. This computation can indeed be done in polynomial time. It will be obvious from the construction that it generates the list of defining relations from the formula in polynomial time, and that the partial order we get is indeed irreflexive. Now if P is given this way, and we have a mapping ϕ from P to Q_1, if we want to verify that this is indeed a $\mathbb{Z}_2$-map, it is enough to check two identities: namely that $\phi(-x) = -\phi(x)$ for all $x \in P$, and that $\phi(x) \leq \phi(y)$ for each defining relation $x < y$ (ie., we don't need to check all pairs x, y).

Let the boolean variables used be $x_1, \ldots, x_N$, and the formula $C^1 \wedge \ldots \wedge C^K$. Each clause C^k has the form $x_{n_1} \vee \overline{x}_{n_2} \vee \ldots \vee x_{n_m}$. Here each variable independently may or may not be negated; the list n_i and its length m actually depend on k but we omit that index for readability; and we assume for convenience that no variable occurs twice in any one clause. The poset P we define has four elements for each variable and four more elements for each term in each clause. Namely, for each variable x_n, we take four elements called $p_n, -p_n, q_n, -q_n$, and for each clause C^k, we take $4m$ new elements, namely $r_1^k, -r_1^k, r_2^k, -r_2^k, \ldots, r_m^k, -r_m^k$, and $s_1^k, -s_1^k, s_2^k, -s_2^k, \ldots, s_m^k, -s_m^k$. For defining the partial order, we first need an auxiliary definition. For every variable x_n, define T_n as the set of all elements r_i^k where the i-th term of C^k is x_n, and define F_n as the set of all elements r_i^k where the i-th term of C^k is $\overline{x}_n$. (We depend on the order of terms in the clauses we fixed.) Notice that each of the r_i^k elements is a member of exactly one of the $2K$ sets defined here. Now we list all the defining relations of the partial order. Firstly, each variable x_n will have two corresponding relations for each occurrence in a term: $t < p_n$ and $t < q_n$ for each $t \in T_n$, and $f < p_n$ and $-f < q_n$ for each $f \in F_n$, respectively. Secondly, each clause C^k has two relations corresponding to each of the m terms in it: $s_i^k < r_i^k$ for each $1 \leq i \leq m$; and $s_{i+1}^k < r_i^k$ for each $1 \leq i < m$, and additionally $-s_1^k < r_m^k$. (One may notice that the two groups contain the same number of relations, in fact each r_i^k occurs twice in the first group and twice in the second group.)

An example for this construction is shown on the figure, which lists some clauses of the CNF expression we consider, and shows part of the Hasse

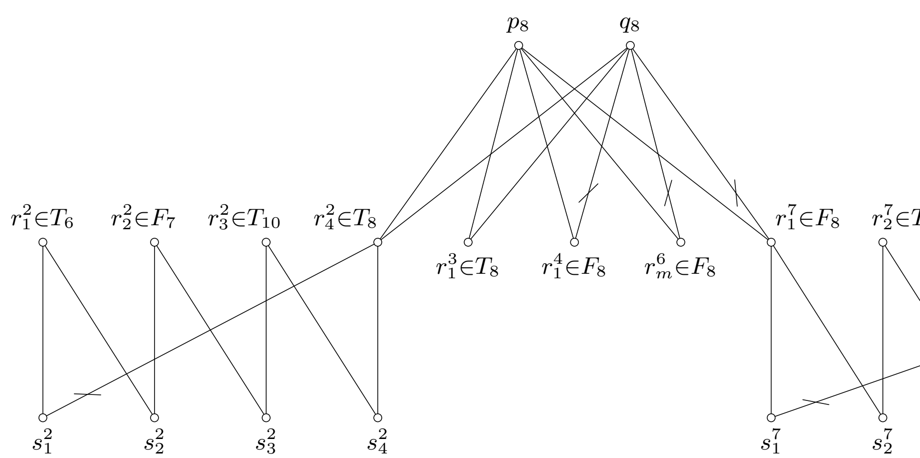

Figure 1: Part of the $\mathbb{Z}_2$-poset constructed from a formula some of whose clauses are: $C^2 = (x_6{\vee}\overline{x}_7{\vee}x_{10}{\vee}x_8)$; $C^3 = (x_8{\vee}\ldots)$; $C^4 = (\overline{x}_8{\vee}\ldots)$; $C^6 = (\ldots{\vee}\overline{x}_8)$; $C^7 = (\overline{x}_8{\vee}x_9{\vee}\ldots)$.

diagram of the poset, except that we use the convention that each element and its negation is drawn as only one point, and a crossed out edge means $-x < y$ where x is the endpoint of the edge that is lower on the diagram.

Let us examine some properties of this construction. Firstly, (though this does not really help us) notice that there always exists a $\mathbb{Z}_2$-map from P to Q_2: namely the one that maps $p_n \mapsto +2$; $q_n \mapsto +2$; $r^k_i \mapsto +1$; $s^k_i \mapsto +0$. Now suppose that there also is a $\mathbb{Z}_2$-map $\phi : P \to Q_1$. Observe that we may assume that this takes any p_n or q_n to ± 1 without loss of generality: indeed it is easy to amend ϕ to have this property by changing the image of such an element from ± 0 to $+1$, and changing the image of its mirror image to -1 accordingly. Similarly, we may assume that any s^k_i is always brought to ± 0. Now observe that for any fixed k, at least one of the elements $r^k_1, r^k_2, \ldots, r^k_m$ must be mapped to ± 1: indeed in the sequence $s^k_1, r^k_1, s^k_2, r^k_2, \ldots, s^k_m, r^k_m$ each element is comparable to the next one, and together with $-s^k_1 < r^k_m$ they form the exact kind of obstacle we mentioned that makes it impossible to map all these points to ± 0. Finally fix any n, and observe that if $\phi(p_n) = \phi(q_n)$ then all elements f of F_n must be mapped to ± 0, for we must keep both $\phi(f) \leq \phi(p_n)$ and $\phi(-f) \leq \phi(q_n)$. A similar statement is true in the other case when $\phi(p_n) \neq \phi(q_n)$ so necessarily $\phi(p_n) = -\phi(q_n)$: namely then ϕ maps all elements of T_n to ± 0 (the signs may vary).

Now we assume an evaluation σ of the variables $x_1, \ldots, x_N$ is given and satisfies the formula. We construct a $\mathbb{Z}_2$-map $\phi : P \to Q_1$ the following way.

Let

$$p_n \mapsto +1, \quad q_n \mapsto +1, \quad \text{if } x_n^\sigma \text{ is true; but}$$
$$p_n \mapsto +1, \quad q_n \mapsto -1, \quad \text{if } x_n^\sigma \text{ is false.}$$

Consider a clause C^k. The evaluation σ satisfies this clause, so at least one of its terms $x_{n_1}, \overline{x}_{n_2}, \ldots, x_{n_m}$ must be evaluated to true: so choose j to be an index of one such term x_{n_j} or $\overline{x}_{n_j}$. Let ϕ act on the elements corresponding to this clause the following way.

$$\begin{array}{lllllll} r_1^k \mapsto +0, & \ldots, & r_{j-1}^k \mapsto +0, & r_j^k \mapsto +1, & r_{j+1}^k \mapsto -0, & \ldots, & r_m^k \mapsto -0; \\ s_1^k \mapsto +0, & \ldots, & s_{j-1}^k \mapsto +0, & s_j^k \mapsto +0, & s_{j+1}^k \mapsto -0, & \ldots, & s_m^k \mapsto -0. \end{array}$$

It is easy to see that these latter assignments satisfy the requirements that $\phi(s_i^k) \leq \phi(r_i^k)$ and $\phi(s_{i+1}^k) \leq \phi(r_i^k)$ and $-\phi(s_1^k) \leq \phi(r_m^k)$. We must now check the restrictions given by the first group of defining relations of P. These, for elements of T_n, are that $\phi(r_i^k) \leq \phi(p_{n_i})$ and $\phi(r_i^k) \leq \phi(q_{n_i})$ if the i-th term of the clause C^k is x_{n_i}. If $i \neq j$ then these are satisfied automatically, because then $\phi(r_i^k) = \pm 0$. If, however, $i = j$, then use the fact that we chose j such that x_{n_j} is true, thus $\phi(p_{n_i}) = \phi(q_{n_i}) = \phi(r_i^k) = +1$. The defining relations involving the elements of F_n can be verified in a very similar way: if the i-th term of C^k is $\overline{x}_{n_i}$ then we need $\phi(r_i^k) \leq \phi(p_{n_i})$ and $-\phi(r_i^k) \leq \phi(q_{n_i})$, but $\phi(r_i^k) = \pm 0$ unless $i = j$, in which case x_{n_i} is false because of the choice of j, so $\phi(p_{n_i}) = \phi(r_i^k) = +1$ and $\phi(q_{n_i}) = -1$ satisfy the restrictions. This proves that ϕ is indeed a $\mathbb{Z}_2$-map.

As the last part of the proof for the $d = 1$ case, we have to prove that if there exists a $\phi : P \to Q_1$ mapping, then that induces an evaluation of the variables that satisfies the boolean expression. We construct the evaluation σ the following way: for any n, if $\phi(p_n) = \phi(q_n)$ then let x_n^σ be true, otherwise $\phi(p_n) = -\phi(q_n)$ and let x_n^σ be false. Recall our earlier observations stating that ϕ takes all members of F_n to ± 0 in the former case, but all members of T_n to ± 0 in the latter case. On the other hand, consider any clause C^k and the elements corresponding to it: we have observed that for at least one j, the element r_j^k is not mapped to ± 0. If the term corresponding to this index j in C^k is x_n then this element $r_j^k \in T_n$, so together with the above this means x_n is true; whereas if that term is $\overline{x}_n$ then similarly $r_j^k \in F_n$ which implies x_n is false. In either case, we have found a term in the clause C^k that is true in σ, and this can be repeated for each clause, thus σ indeed satisfies the boolean expression.

All that remains now is to prove that the cases of $1 < d$ are also NP-complete. This we do by modifying the above Karp reduction. The simple observation we need for this is the following: if we modify any $\mathbb{Z}_2$-poset P by adding two extra elements y and $-y$ that are greater than all other elements of the poset, then the cross-index of the resulting $\mathbb{Z}_2$-poset P^* is exactly one greater than the cross-index of the original. Indeed, we can extend a $P \to Q_d$ map to a $P^* \to Q_{d+1}$ map by setting the image of y to be $+(d+1)$; and conversely, by any $P^* \to Q_{d+1}$ map, no point other than y or $-y$ can be mapped to $\pm(d+1)$, thus restricting it to P gives a $P \to Q_d$ map. Thus, applying the reduction given in the $d = 1$ case then iterating this transformation $d-1$ times gives a $\mathbb{Z}_2$-poset that can be mapped to Q_d if and only if the original expression is satisfiable, and this construction still can be realized by a polynomial time computation. □

Remark. While $\text{Xind}(P)$ for the poset P constructed in the above proof depends on the satisfiability of the boolean formula from which it is constructed, one can prove that $\text{ind}(\hat{P})$ is always at most 1. The reason is that any two dimensional face appearing in $\hat{P}$ is a triangle which has at least one side that does not belong to any other two dimensional face. This makes it possible to retract $\hat{P}$ into a 1-dimensional complex.

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