Connected matchings and Hadwiger's conjecture

Zoltán Füredi* András Gyárfás † Gábor Simonyi ‡

Hadwiger's well known conjecture (see the survey of Toft [9]) states that any graph G has a $K_{\chi(G)}$ minor, where $\chi(G)$ is the chromatic number . Let $\alpha(G)$ denote the independence (or stability) number of G, the maximum number of pairwise nonadjacent vertices in G. It was observed in [1], [4], [10] that through the inequality $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$, Hadwiger's conjecture implies

Conjecture 0.1 *Any graph G on n vertices contains a $K_{\lceil \frac{n}{\alpha(G)} \rceil}$ as a minor.*

During the last five years it was a popular question to consider Conjecture 0.1 for graphs G with $\alpha(G) = 2$:

Conjecture 0.2 *Suppose G is a graph with n vertices and with $\alpha(G) = 2$. Then G contains $K_{\lceil \frac{n}{2} \rceil}$ as a minor.*

Duchet and Meyniel proved [1] that every graph G with n vertices has a $K_{\lceil \frac{n}{2\alpha(G)-1} \rceil}$ minor, thus the statement of Conjecture 0.2 is true if $n/2$ is replaced by $n/3$ (for follow up and for some improvements see [4], [5], [2], [6]). The problem of improving $n/3$ is attributed to Seymour [7]:

Conjecture 0.3 *There exists $\epsilon > 0$ such that every graph G with n vertices and with $\alpha(G) = 2$ contains $K_{\lceil (\frac{1}{3}+\epsilon)n \rceil}$ as a minor.*

Conjecture 0.3 has a fairly interesting reformulation with some "Ramsey flavor". A set of pairwise disjoint edges $e_1, e_2, \ldots, e_t$ of G is called a *connected matching of size t* ([8]) if for every $1 \leq i < j \leq t$ there exists at least one edge of G connecting an endpoint of e_i to an endpoint of e_j.

*Alfréd Rényi Institute, Hungarian Academy of Sciences, Budapest 1364, P. O. Box 127 and Dept. of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL61801, USA. Email: furedi@renyi.hu Research partially supported by Hungarian National Science Foundation Grants OTKA T032452 and T037846 and by NSF grant DMS 0140692

†Computer and Automation Research Institute of the Hungarian Academy of Sciences, Budapest, P. O. Box 63, Hungary-1518. Email: gyarfas@sztaki.hu

‡Alfréd Rényi Institute, Hungarian Academy of Sciences, Budapest 1364, P. O. Box 127. Email: simonyi@renyi.hu
Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) Nos. T037846 and T046376.

Conjecture 0.4 *There exists some constant c such that every graph G with ct vertices and with $\alpha(G) = 2$ contains a connected matching of size t.*

Conjecture 0.4 is probably discovered independently by several people working on Conjecture 0.3. Thomassé [8] notes that Conjectures 0.4 and 0.3 are equivalent (a proof is in [2]).

This note risks the stronger conjecture that $f(t)$, the minimum n such that every graph G with n vertices and $\alpha(G) = 2$ must contain a connected matching of size t, is equal to $4t - 1$. The lower bound $f(t) \geq 4t - 1$ is obvious, shown by the union of two disjoint complete graphs K_{2t-1}.

Conjecture 0.5 *Every graph G with $4t-1$ vertices and with $\alpha(G) = 2$ contains a connected matching of size t.*

A modest support of Conjecture 0.5 is the following.

Theorem 0.6 *$f(t) = 4t - 1$ for $1 \leq t \leq 17$.*

Proof. Assume G is a graph with $4t-1$ vertices and with $\alpha(G) = 2$. Suppose, first, that the maximum degree of $\overline{G}$ is at least $t - 1$ and let v be a maximum degree vertex in $\overline{G}$. Let $A \subset V(G)$ consist of t (or all if there are only $t - 1$) non-neighbors of v (in G), thus $t - 1 \leq |A| \leq t$. Consider the bipartite subgraph $H = [A, B]$ of G, where $B = V(G) \setminus (A \cup \{v\})$. If H contains a matching of size t then it is a connected matching, since A induces a clique in G. Also, if $|A| = t - 1$ and H contains a matching of size $t-1$, it can be extended by an edge incident to v to a connected matching of size t. If the required matching does not exist, by König's theorem, there is a $T \subset V(G)$ with $|T| \leq t - 1$ (or $|T| \leq t - 2$ if $|A| = t - 1$) meeting all edges of H. As $|B| \geq 3t - 2$, this implies that there exists a vertex in $A \setminus T$ nonadjacent to at least $2t$ vertices of G. Thus $K_{2t} \subset G$ which clearly contains a connected matching of size t.

Therefore the maximum degree of $\overline{G}$ is at most $t - 2$. Now let A_v denote the set of non-neighbors and B_v the set of neighbors of v in G. Some vertex $w \in B_v$ is nonadjacent to at most

$$\frac{|A_v|(t-3)}{|B_v|} \leq \frac{(t-2)(t-3)}{3t} \tag{1}$$

vertices of A_v. The right hand side of (1) is less than 4 if $t \leq 16$. If $t = 17$ then, as all vertices cannot have odd degree, v can be selected as a vertex nonadjacent to at most 14 vertices and the estimate (1) still gives a $w \in B_v$ nonadjacent to at most $14^2/51 < 4$ vertices of A_v. Thus we have found an edge vw in G such that the set $C \subset V(G)$ nonadjacent to both v and w satisfies $|C| \leq 3$. This allows to carry out the inductive proof: removing v, w and two further vertices (as many from C as possible) the remaining graph has a connected matching of size $t - 1$ and the edge vw extends it to a connected matching of size t. (Of course, it is trivial to start the induction with $f(1) = 3$.) □

An obvious upper bound for $f(t)$ comes from the Ramsey function: $f(t) \leq R(3, 2t)$ (which has order of magnitude $\frac{t^2}{logt}$, see [3] and the references therein). Using the proof method of Theorem 0.6 we give a better bound for $g(t) \geq f(t)$ where $g(t)$ is the minimum n such that every graph G with n vertices and with $\alpha(G) = 2$ contains a "2-connected matching of size t": a set of pairwise disjoint edges $e_1, e_2, \ldots, e_t$ of G such that for every $1 \leq i < j \leq t$ there exists at least *two* edges of G connecting an endpoint of e_i to an endpoint of e_j.

Theorem 0.7 *Every graph G with $\alpha(G) = 2$ and with at least $2^{3/4}t^{3/2} + 2t + 1$ vertices contains a 2-connected matching of size t.*

Proof. Set $c = 2^{5/4}$ which is the positive root of $\frac{4}{c} = \frac{c\sqrt{2}}{2}$. We want to establish the recursive bound $g(t) \leq g(t-1) + ct^{1/2} + 2$, for the function $g(t)$ $(t \geq 2, g(1) = 3)$. Then (using the inequality between the arithmetic and quadratic means)

$$g(t) \leq c(\sum_{i=2}^{t} i^{1/2}) + 2(t-1) + g(1) \leq c\frac{(\sqrt{2})}{2}t^{3/2} + 2t + 1 = 2^{3/4}t^{3/2} + 2t + 1,$$

the theorem follows (for $t = 1$ it holds vacuously).

Using the argument of Theorem 0.6, let N be the smallest integer satisfying $N \geq 2^{3/4}t^{3/2} + 2t + 1$, let G be a graph with N vertices and with $\alpha(G) = 2$. Assuming G has no 2-connected matching of size t, any $v \in V(G)$ is nonadjacent to at most $2t-1$ vertices of G. Using the argument from the proof of Theorem 0.6, for any $v \in V(G)$ there is a $w \in B_v$ such that there are at most $M = \frac{(2t-1)(2t-2)}{N-2t}$ vertices of G nonadjacent to both v and w. Therefore, it is possible to remove at most $M + 2$ vertices of G so that the remaining graph does not contain 2-connected matchings of size $t-1$. Thus,

$$g(t) < g(t-1) + \frac{(2t-1)(2t-2)}{N-2t} + 2. \tag{2}$$

Notice that $\frac{(2t-1)(2t-2)}{N-2t} \leq ct^{1/2}$ because otherwise we get

$$N < (\frac{4}{c})t^{3/2} + 2t = 2^{3/4}t^{3/2} + 2t$$

implying

$$2^{3/4}t^{3/2} + 2t + 1 \leq N < 2^{3/4}t^{3/2} + 2t,$$

contradiction. Thus (2) gives the claimed recursive bound for $g(t)$. □

It is natural to conclude this note by introducing $h(t)$, the minimum n such that every graph G with n vertices and with $\alpha(G) = 2$ contains a 3-connected matching of size t: a set of pairwise disjoint edges $e_1, e_2, \ldots, e_t$ of G such that for every $1 \leq i < j \leq t$ there exists at least *three* edges of G connecting an endpoint of e_i to an endpoint of e_j.

Problem 0.8 *Separate the functions $f \leq g \leq h \leq R(3, 2t)$.*

References

[1] P. Duchet, H. Meyniel, On Hadwiger's number and the stability number, *Annals of Discrete Math.* **13** (1982), 71-74.

[2] K. Kawarabayashi, M.D. Plummer, and B. Toft, Improvements of the theorem of Duchet and Meyniel on Hadwiger's conjecture, manuscript, 2004.

[3] J.H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures Algorithms*, **7** (1995), 173–207.

[4] F. Maffray, H. Meyniel, On the relationship between Hadwiger and stability numbers, *Discrete Math.* **64** (1987), 39-42.

[5] M.D. Plummer, M. Stiebitz and B. Toft, On a special case of Hadwiger's Conjecture, *Discussiones Mathematicae Graph Theory* **23** (2003), 333–363.

[6] B. Reed, P. D. Seymour, Fractional colouring and Hadwiger's conjecture, *J. Combin. Theory Ser. B*, **74** (1998), 147–152.

[7] P. D. Seymour, unpublished

[8] Stephan Thomassé, in: Problems from Graph Theory 2002 meeting, Odense, University of Southern Denmark, August 19-23, 2002, page 4.

[9] B. Toft, A survey of Hadwiger's conjecture, *Congressus Numerantium* **115** (1996) 249-283.

[10] D. R. Woodall, Subcontraction-equivalence and Hadwiger's conjecture, *Journal of Graph Theory* **11** (1987), 197-204.