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CODING FOR WRITE-UNIDIRECTIONAL MEMORIES AND CONFLICT RESOLUTION*

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Write-unidirectional memories (WUMs) have been introduced recently. They are binary storage devices having a close relationship to write-once memories (WOMs). When updating the information stored by a WUM the encoder can write 1's to some positions of the WUM or 0's to some positions of the WUM but it is not allowed to do both at the same time.

Wolf, Wyner, Ziv and Körner investigated WOMs in the four cases arising if the encoder and/or the decoder is informed/uninformed about the previous state of the memory. We investigate the WUMs under these circumstances.

We mostly deal with the two cases when the encoder is uninformed, and give bounds and conjectures on the best achievable rates.

In the last part, we discuss a related combinatorial problem: conflict resolution on a multiple access channel.

Introduction

A *write-unidirectional memory* (WUM)–as it is introduced in a recent paper of Borden [1]–is a binary storage medium which is constrained, during the updating of the information stored (we quote [1]) "to either writing 1's in selected bit positions or 0's in selected bit positions and is not permitted to write combinations of 0's and 1's. Such a constraint arises when the mechanism that chooses to write 0's or 1's operates much more slowly than the means of accessing and scanning a word".

Such a device has a close relationship with WOMs (*write-once memories*). The main difference is that in the WOM case it is allowed only to write 1's in selected bit positions and it is not allowed to write 0's. Therefore one can use a WOM only a finite number of times whereas a WUM can be used infinitely many times. In spite of this deep difference, many questions arise for WUMs in a very similar way as for WOMs. WOMs (and the related problems of defective memories) have been investigated in several papers during the past few years; [3, 6, 11, 16, 17] are only a few of them.

In the present paper we shall investigate some questions arising for WUMs in some special cases. The most important problem is: what is asymptotically the maximum achievable rate of a WUM, where the *rate* is defined as

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$$R = (\log M)/n.$$

Here M is the constant number of possible messages to be stored in each step (all logarithms are to the base 2 in this paper) and n is the number of binary positions in the WUM. We will consider, inspired by Wolf, Wyner, Ziv and Körner's work [17], the four cases when the encoder (writer) and/or decoder (reader) are informed or uninformed about the previous state of the memory. There are also some differences between the consideration of [17] and this paper. While the authors of [17] have investigated the WOMs in the case when errors can occur at decoding with arbitrarily small probability, we consider the case with WUMs when no error occurs. Another difference is that M must be the same number in each step in our work while that was not a condition in [17]. In spite of these differences we will use the same notation for the main cases that was used in [17], i.e.

- Case 1 means the case when encoder and decoder are informed about the previous state;
- Case 2 means encoder informed, decoder uninformed;
- Case 3 means encoder uninformed, decoder informed;
- Case 4 means encoder and decoder uninformed.

In [1], only Case 2 was investigated and Case 4 mentioned in an implicit way. Here we will focus on Cases 4 and 3, i.e. when the encoder is uninformed.

Now, before the real discussion, let us describe the four cases in a more precise way. A rigorous formulation of the model is given in [14]. Here we present a more intuitive description.

A WUM consists of n cells or binary positions that can be either in the 0 or 1 state. The encoding process is a mapping from the set of the possible messages to the union of the sets A and B where A is the set of n-tuples of the two symbols 1's and $\square$'s (we will call it a "hole") and B is the set of n-tuples of 0's and $\square$'s (the meaning of a hole is that the encoder writes nothing on that position, so the corresponding binary cell will store the same bit of information–0 or 1–as before). This mapping depends also on the previous state of the WUM in Cases 1 and 2.

The decoding process is a mapping from the set of the possible states of the memory, i.e. from a subset of all n-tuples of 0's and 1's, to the set of the messages. This mapping depends also on the previous state of the memory in Cases 1 and 3.

The elements of A and B we often call "*filters*", the elements of A being 1-*filters* and the elements of B 0-*filters*. We have to distinguish them from the possible states of the memory that we call "codewords".

A *good* coding system (or simply system) for a WUM is the set of the encoding and decoding functions in each generation. The system is *optimal* if M is as large as possible for fixed n.

1. Discussion of Borden's results

Denoting by $R(n)$ *the largest rate achievable* with n binary positions and setting

$\gamma := \log\frac{1}{2}(1+\sqrt{5}) \simeq 0.694$, it is proved in [1] that in Case 2 $R(n)<\gamma$ holds for $n \geq 5$. Furthermore

$$\lim_{n \to \infty} R(n) = \gamma. \tag{1.1}$$

The proof of (1.1) is by random coding. The best construction given in [1] for arbitrary large n has rate 0.5. A better one is presented in [15], with $R=0.517$ and generalized in [14], giving $R=0.5325$.

It is easy to see that Borden's proof is valid for Case 1 too, so we have the following theorem.

Theorem 1.1. *For $n \geq 5$ and in all four cases:*

$$R(n) \leq \gamma.$$

Definition 1.2. Two codewords, $\underline{x}=(x_1, x_2, \ldots, x_n)$ and $\underline{y}=(y_1, y_2, \ldots, y_n)$ are *comparable* if one of the next two implications is valid:

$$\forall i, \quad x_i = 1 \;\Rightarrow\; y_i = 1 \quad \text{or} \quad \forall i, \quad x_i = 0 \;\Rightarrow\; y_i = 0.$$

It is clear that two successive codewords stored in a WUM must be comparable.

Consider a good system for Cases 3 and 4 when the encoder does not know the previous state of the memory. This means that at a given generation the encoder uses a unique (i.e. independent of the memory's previous state) filter for each message.

If the encoder uses a certain filter it means that he writes 1's (or 0's) in the positions of the memory where the filter has 1's (or 0's) and writes nothing in the positions where the filter has holes. For example, if the encoder uses the filter 1□111□□1 and the previous state of the memory was 00010011, then the current state will be 10111011. It means that the encoder has M different filters (at each generation), one for each message, and any of them can be used in each step if just the corresponding message was chosen. If the system is good, it means that from every state of the memory which can arise, the decoder must know what the lastly used filter was.

2. Case 4 under special restrictions

In the case when neither the encoder nor the decoder knows the previous state of the memory, it is conjectured that the asymptotically best achievable rate is 0.5 (cf. [14]), although we do not have any better upper bound for this case than for the other cases.

In what follows, we prove that 0.5 is the best achievable rate in Case 4 under the restriction that each message can be represented only by one codeword at a given generation. It is easy to see that in this case it is enough to deal with two different generations.

So we have the following problem:

We have two families (of the same cardinality) of binary codewords and each codeword of the first family is covered by each codeword of the other family. We are interested in the maximum possible cardinality M of these families. It is clear that $M=2^{\lfloor n/2 \rfloor}$ is achievable by the simple construction given by Borden in [1]. We prove now that this is best possible.

In fact we prove a slightly stronger statement that asymptotically $R \lesssim \frac{1}{2}$ even if we require only that each element of one of our families is comparable with each element of the other. From the original practical point of view this strengthening is useless because the uninformed encoder, only being aware of the codeword to be written, could not decide whether to write the 0's or the 1's of this codeword. We prove our statement using the terminology of set-systems.

Theorem 2.1. *Let S be a set of size n, and $\mathscr{P}$ and $\mathscr{Q}$ two families of subsets of S, with $|\mathscr{P}|=|\mathscr{Q}|$. Suppose that for each pair $P\in\mathscr{P}$, $Q\in\mathscr{Q}$, one has $P\subseteq Q$ or $Q\subseteq P$. Then the maximum possible number $M(n):=|\mathscr{P}|=|\mathscr{Q}|$ is at most $2^{\lfloor n/2\rfloor}+n+1$, which asymptotically yields $R(n)\lesssim\frac{1}{2}$.*

If we require $P\subseteq Q$ for each pair $P\in\mathscr{P}$, $Q\in\mathscr{Q}$, then $M(n)=2^{\lfloor n/2\rfloor}$.

Proof. Consider all the subsets of S in the two families $\mathscr{P}$ and $\mathscr{Q}$ satisfying the conditions of Theorem 2.1. Order these subsets of S by their cardinality. Then we have $|\mathscr{P}|+|\mathscr{Q}|=2M$ subsets of S in a certain order

$$T_1, T_2, \ldots, T_{2M} \quad \text{where } |T_i| \geq |T_{i-1}|, \quad i=2,3,\ldots,2M.$$

We can assume that $T_1\in\mathscr{P}$. Define the integers k_i in the following way:

$$\begin{aligned} &k_0=0,\\ &k_{2j-1}=\min\{d:\ T_{k_0+\cdots+k_{2j-2}+d+1}\in\mathscr{Q}\},\\ &k_{2j}=\min\{d:\ T_{k_0+\cdots+k_{2j-1}+d+1}\in\mathscr{P}\},\quad j=1,2,\ldots. \end{aligned} \tag{2.1}$$

So we have the integers $k_1, k_2, \ldots, k_r$. Note that

$$|\mathscr{P}|=k_1+k_3+\cdots+k_{2f-1} \quad\text{and}\quad |\mathscr{Q}|=k_2+k_4+\cdots+k_{2g},$$

where $r=2g=2f$ or $r=2f-1=2g+1$.

Consider the following subfamilies of $\mathscr{P}$ and $\mathscr{Q}$:

$$H_i=\{T_j:\ k_0+\cdots+k_{i-1}<j<k_0+\cdots+k_{i-1}+k_i\},\quad i=1,\ldots,r. \tag{2.2}$$

Choose an arbitrary $s\in S$. The main idea of the proof is that there is at most one H_i for which s distinguishes some of its elements from some others. By this we mean:

If there are $T_{i_1}, T_{i_2} \in H_i$, $i_1 \neq i_2$, such that $s \in T_{i_1}$ and $s \notin T_{i_2}$, then there is no H_j, $j \neq i$, which has two elements $T_{j_1}, T_{j_2} \in H_j$, $j_1 \neq j_2$, such that $s \in T_{j_1}$ and $s \notin T_{j_2}$.

The above statement is a straightforward consequence of our conditions. If H_i has the above property, then for $j > i$, $s \in T_q$ for each $T_q \in H_j$, because $s \in T_{i_1} \subseteq T_q$. On the other hand, for $j < i$, $s \notin T_q$ for each $T_q \in H_j$, because $s \notin T_{i_2} \supseteq T_q$.

Let h_i be the number of elements of S that distinguish some subsets in H_i. Then $|H_i| \leq 2^{h_i}$. The above arguments result in

$$\sum_{i=1}^{m} h_i \leq n \quad \text{where } m = \max(2f-1, 2g). \tag{2.3}$$

Note that there are at most $n+1$ odd and at most $n+1$ even i's for which $h_i = 0$. This comes from the fact that if $h_i = 0$, then the unique element of H_i must have a unique cardinality in the corresponding family ($\mathscr{P}$ or $\mathscr{Q}$) which it belongs to (because $A \subseteq B$, $A \neq B \Rightarrow |A| < |B|$) and there are only $n+1$ possible different cardinalities for the subsets of S.

So we have

$$\begin{aligned} |\mathscr{P}| = |H_1| + |H_3| + \cdots + |H_{2f-1}| &\leq 2^{h_1} + 2^{h_3} + \cdots + 2^{h_{2f-1}} \\ &\leq 2^{h_1 + h_3 + \cdots + h_{2f-1}} + \sum_{\substack{i=1,\\ i \text{ is odd}}}^{2f-1} \delta(h_i) \\ &\leq 2^{h_1 + h_3 + \cdots + h_{2f-1}} + n + 1 \end{aligned}$$

and

$$\begin{aligned} |\mathscr{Q}| = |H_2| + |H_4| + \cdots + |H_{2g}| &\leq 2^{h_2} + 2^{h_4} + \cdots + 2^{h_{2g}} \\ &\leq 2^{h_2 + h_4 + \cdots + h_{2g}} + \sum_{\substack{i=2,\\ i \text{ is even}}}^{2g} \delta(h_i) \\ &\leq 2^{h_2 + h_4 + \cdots + h_{2g}} + n + 1, \end{aligned}$$

where $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ otherwise.

Now because of (2.3), one of the previous upper bounds is not greater than $2^{\lfloor n/2 \rfloor} + n + 1$. Being aware of $|\mathscr{P}| = |\mathscr{Q}|$ this proves $M(n) \leq 2^{\lfloor n/2 \rfloor} + n + 1$ which implies $R(n) \lesssim \frac{1}{2}$.

If we require $P \subseteq Q$ for every $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$, then we have an order $T_1, T_2, \ldots, T_{2M}$ with $|T_i| \geq |T_{i-1}|$ where $T_1, \ldots, T_M \in \mathscr{P}$ and $T_{M+1}, \ldots, T_{2M} \in \mathscr{Q}$. Then, from the above argument, we get

$$|\mathscr{P}| = |H_1| \leq 2^{h_1} \quad \text{and} \quad |\mathscr{Q}| = |H_2| \leq 2^{h_2},$$

where $h_1 + h_2 \leq n$. Finally

$$M(n) \leq 2^{\min(h_1, h_2)} \leq 2^{\lfloor n/2 \rfloor}.$$

$M(n) \geq 2^{\lfloor n/2 \rfloor}$ is proved by Borden's construction [1]:

Let A be a fixed subset of S with cardinality $\lfloor n/2 \rfloor$; take for $\mathscr{Q}$ the family of all sets containing A, and for $\mathscr{P}$ the family of all sets contained in A. Then $|\mathscr{Q}|=2^{\lceil n/2\rceil}$, $|\mathscr{P}|=2^{\lfloor n/2\rfloor}$ and for each $Q\in\mathscr{Q}$, $P\in\mathscr{P}$, $P\subseteq Q$ holds. □

Remarks. (1) It is easy to see that a similar argument extends to unequal size families, giving $(1/n)(\log M_1+\log M_2)\leq 1$ for $|\mathscr{P}|=M_1$, $|\mathscr{Q}|=M_2$. In other words, with $R_i:=(1/n)\log M_i$, $i=1,2$, we get $R_1+R_2\leq 1$.

(2) With the weaker constraint, one can get $M(n)>2^{\lfloor n/2\rfloor}$. For example $M(3)\geq 4$ (choose $\mathscr{P}=\mathscr{Q}=\{000,001,011,111\}$).

3. Bounds for Case 3

Definitions. (a) A step where the corresponding set of filters consists of 1-filters only is a *1-force-step* or simply a *1-force*.

(b) A step where the corresponding set of filters consists of 0-filters only is a *0-force-step* or simply a *0-force*.

(c) A step where the corresponding set of filters consists of both 0-, and 1-filters is a *mixed-step*.

Now we state a theorem about the structure of a good system. Its proof can be found in [13].

Theorem 3.1. *In Cases* 3 *and* 4, *if there exists a good system with* $M=M_0$, *then there exists a good system with* $M=M_0$ *where every (say) odd step is a* 1*-force-step and every even step is a* 0*-force-step. Such systems are called* alternating *in* [14].

Lemma 3.2. *If we have a construction for Cases* 3 *or* 4 *with* $n=n_0\geq 2$, *and* $R=R_0$, *then we can construct a good system for the same case with* $n>N$ *and* $R=R_0$ *where* N *is arbitrarily large.*

Note that this is not true for Cases 1 or 2. For example, it is easy to find a construction for Case 2 (naturally it is good for Case 1 too) with $n=3$ and $M=5$, i.e. with $R=\frac{1}{3}\log 5>\gamma$ although there is no construction with $R>\gamma$ for large n.

Proof of Lemma 3.2. If we have a good system for Case 3 or 4 (with $n\geq 2$), Theorem 2.1 tells us (since its proof is constructive) how to make a good system with alternating 0- and 1-forces from it. If we have two (not necessarily different) systems with alternating 0- and 1-forces and $n=n_1$ and n_2, $M=M_1$ and M_2 respectively for which $R_1=(1/n_1)(\log M_1)=(1/n_2)(\log M_2)=R_2$, then we can get a new system with $n=n_1+n_2$ and $M=M_1M_2$ (i.e. $R=(1/(n_1+n_2))(\log M_1+\log M_2)=R_1$) only by concatenating each filter of the first system to each filter of the corresponding set of the second system. It is obvious that the new system will work because of the alternating 0- and 1-forces, and it is clear by induction that in this way we can get a good system with the same rate and arbitrarily large n. □

As a consequence of Lemma 3.2, we have the following theorem.

Theorem 3.3. *The sequence $R(n)$ has a limit as n goes to infinity.*

Now we present a simple construction with $R>\frac{1}{2}$ in Case 3.

Construction (see [14]). We describe this construction with $n=3$. By Lemma 3.2 we know that we can make a construction to any $n=3k$ and the same rate, by concatenation.

Let the encoder have two sets of filters with cardinality three:

$$E_1=\{11\square, 1\square 1, \square 11\} \quad \text{for odd steps,}$$

and

$$E_2=\{00\square, 0\square 0, \square 00\} \quad \text{for even steps.}$$

This gives a rate $R=\frac{1}{3}\log 3 \simeq 0.528$. Let us prove that this system works by showing the decoding algorithm.

Decoding. After having written an odd (respectively even) number g of times, the memory contains a triple m of weight 2 or 3 (respectively 0 or 1). Assume first that g is odd. There are two cases to consider:

(1) $w(m)=2$; say $m=110$; then $M_1=11\square$ has been written;

(2) $m=111$; then, at time $g-1$ the memory contained a triple of weight exactly 1, known to the decoder, say $m'=100$. Hence $M_3=\square 11$ has been written at time g.

The same proof applies for even g.

Let $M=(x_1,x_2,\dots,x_n)$ and $M'=(x_1',x_2',\dots,x_n')$ be two filters in the same step. Then we define

$$M \triangle M' := \{i\colon x_i \neq x_i'\}.$$

With a slight abuse of notation M_i means the ith message and also its filter.

We call a configuration *ambiguous* if there exist two different messages M_i and M_j which change the content of the memory from m to m', which we denote by

$$m \xrightarrow[M_j]{M_i} m'.$$

W.l.o.g. assuming that we are at a 0-force-step, it is clear that such a configuration occurs iff

$$M_i \triangle M_j \subset \{\text{“zero” positions in } m\} := z(m).$$

For example consider a writing at time $2t$ (an even writing)

$$m_{2t-1} = 011100 \xrightarrow[M_j = \square 0 \square \square 00]{M_i = 00\square\square\square\square} 001100 = m_{2t}.$$

Here $M_i \triangle M_j = z(m_{2t-1}) = \{1,5,6\}$. But m_{2t-1} results from writing message M_k (say) at time $2t-1$. Hence $z(m_{2t-1}) \subset \{\square\text{-positions of } M_k\} := \square(M_k)$, and $M_i \triangle M_j \subset \square(M_k)$.

We have proved the following theorem.

Theorem 3.4. *A system is good if for any triple (M_i, M_j, M_k) of filters, with M_i, M_j in one step and M_k in the previous step the following holds:*

$$M_i \triangle M_j \text{ is not contained in } \square(M_k). \tag{3.1}$$

If we restrict our attention to systems with only two different steps with filters sets E_1 and E_2, then Condition (3.1) is necessary, i.e.:

Theorem 3.5. *If there exists a triple (M_i, M_j, M_k) of filters with $M_i, M_j \in E_2$ (respectively E_1), $M_k \in E_1$ (respectively E_2) satisfying $M_i \triangle M_j \subseteq \square(M_k)$, then the system is ambiguous.*

The formal proof of Theorem 3.5 is essentially the same as the one given in [14] for the symmetric case (see the definition below), so we do not repeat it here. To convince himself, the reader can assume that the all-zero state $\underline{0}$ occurs, and check the following:

$$\underline{0} \xrightarrow{M_k} m_1 \xrightarrow[M_j]{M_i} m_2.$$

Considering $\square$ as its own complement, we can define the *complement* of a filter (e.g. $\overline{11\square} = 00\square$) and of a step E: $\bar{E} = \{\bar{M}, M \in E\}$. Let us call a system with two different steps and $\bar{E}_2 = E_1$ *symmetric* (e.g. the previous construction). The determination of the rate $R^*(n)$ of an optimal symmetric system is equivalent to the following (as proved in [14]):

Problem. What is the maximum number $M(n)$ for which there exists a set E, $|E| = M(n)$ of binary n-tuples with the following property: There is no $\underline{e}_i, \underline{e}_j \in E$, $\underline{e}_i \neq \underline{e}_j$ for which $\underline{e}_i \oplus \underline{e}_j$ is covered by any $\underline{e}_k \in E$ (even if $k = i$ or j). ($\underline{e}_i \oplus \underline{e}_j$ = the componentwise modulo 2 sum of $\underline{e}_i$ and $\underline{e}_j$.)

This problem is already considered in [7] with a slightly weaker constraint:

Condition (E.K.). Equation (3.1) must hold for *distinct* M_i, M_j, M_k.

Conjecture [13]. Under Condition (E.K.), $M(n) \le 2^{n(\log 3)/3}$ holds.

We now derive an upperbound on $R(n)$ in Case 3, using a condition equivalent to (E.K.), called *cancellation* in [5].

Definition. A family E of subsets of an n-set is *cancellative* if it contains no triple (A, B, C) such that $A \cup B = A \cup C$.

Theorem 3.6. *A family satisfies* (E.K.) *iff it is cancellative.*

Proof. It is easy to check that both conditions are equivalent to the following: For any three *distinct* sets of E, at least two of them contain an element not contained in the union of the two others. □

An upperbound is derived in [5] for the maximal size $G(n)$ of a cancellative family:

$$G(n) < n \cdot (\tfrac{3}{2})^n.$$

In terms of the rate, this gives asymptotically

$$(1/n)\log G(n) \le \log \tfrac{3}{2} \simeq 0.585.$$

This yields:

Theorem 3.7. *The maximal rate of a symmetric system satisfies*

$$\tfrac{1}{3}\log 3 \le R^*(n) < \log \tfrac{3}{2}.$$

Conjecture.

$$\lim_{n\to\infty} R(n) = \lim_{n\to\infty} R^*(n) = \tfrac{1}{3}\log 3.$$

4. Conflict resolution [2, 9]

Suppose we have n users sharing a unique ressource (a multiple access channel). A *conflict* arises when 2 or more users send a message in the same slot of time. All users receive a feedback $0, 1, 2^+$ telling them that 0, 1 or at least 2 of them have tried to speak. We shall suppose that this feedback is delayed (because e.g. of transmission delay) and look for a conflict resolution strategy with the following rules:

(1) At time $t = 1, 2, \ldots$ some users are allowed to speak, until everybody has been satisfied (i.e. has been given the possibility of speaking alone). At time i, the characteristic vector of the set of users allowed to speak is a n-tuple called *query* or test.

(2) *Only users in conflict may speak* during the conflict resolution procedure. The other ones, even if asked to, and even if in the meantime they have something to say, remain silent.

We will assume known an upperbound on the number of users in conflict, say s.

Our problem is to find $m(n,s)$, the minimum number of queries needed to solve a conflict involving at most s users among n. We now make explicit the links with extremal set theory and WUMs.

Let $E=\{e_1, e_2, \dots, e_{|E|}\}$ be a family of non-void subsets of an n-set. The elements of E index a set of $|E|$ users (see example). If the characteristic vectors of the e_i's are written as columns, we get an $n \times |E|$ binary matrix M which can be used for conflict resolution:

The ith row r_i of M is the ith query, with the convention: User j is allowed to speak at time i iff $r_{ij}=1$.

Example (see Table 1). Here, any conflict involving $s=2$ users is solved (because the e_i's are a Sperner family). A conflict involving e_1, e_2, e_3 is solved, these users being satisfied at the 2nd, 3rd and 4th query respectively. If the conflict occurs between users 4, 5, 6, no one is satisfied after the 4 queries.

A stronger condition, called *s-surjectivity*, has been studied in [8,12]. It has applications to testing VLSI. Namely the family of columns of T must have the following property: If we pick any s columns in T, then the $n \times s$ matrix thus formed must contain as rows at least once every possible s-tuple (instead of at least k different rows of weight 1 in the case of a k among s family - see below).

The case $s=3$. The cancellative property is equivalent to: if we pick any 3 columns in T, then the $n \times 3$ matrix thus formed contains as rows at least 2 of the following 3 triples: 100, 010, 001.

This implies the following: in a conflict involving 3 users (at most), at least 2 will be satisfied, i.e. get the opportunity of being the only one asked to speak on the common channel. Let us call such a family of columns a 2 *among* 3 *family*.

With an obvious extension of the definition, we shall speak of a *k among s family*.

Table 1

	Users					
Queries	e_1	e_2	e_3	e_4	e_5	e_6
1	1	1	1	0	0	0
2	1	0	0	1	1	0
3	0	1	0	1	0	1
4	0	0	1	0	1	1

Remark. If the columns of a set of queries form an $(s-1)$ among s family, then, after feedback has been received, one ultimate question (or row of T), namely the "all-one" row is enough to solve the conflict: the last unsatisfied user, if he exists, will be asked to speak.

A stronger property is studied in [4]: Find the maximal size, $F(n)$, of a family E with the property that no set in E is covered by the union of two others. The authors obtain

$$1.134^n \leq F(n) \leq 1.25^n.$$

This property is easily seen to imply that any $n \times 3$ submatrix extracted as before from M will contain as rows the 3 triples 100, 010, 001. Thus the set of queries formed by the rows of M enables the resolution of any conflict involving 3 users.

Let us summarize.

Theorem 4.1. *The largest* 3 *among* 3 *family has rate R, with*

$$0.181 \simeq \log 1.134 \leq R \leq \log 1.25 \simeq 0.322.$$

The largest 2 *among* 3 *family has rate R′, with*

$$0.528 \leq R' \leq 0.585.$$

The case $s=4$. The case of 4 among 4 families is considered in [2].

We shall deal here with 1 among 4 families. Note that after feedback has been received, if we use a 1 among 4 family to solve a conflict involving 4 people, we are left with a conflict involving 3 people, which is the previous case.

A slightly weaker condition; *weakly union free* (WUF), has been considered in [5].

Definition. A family E of binary n-tuples is weakly union-free if it contains no 4-tuple A, B, C, D of elements such that the corresponding sets satisfy

$$A \cup B = C \cup D.$$

Theorem 4.2 [5]. *The largest* WUF *family has asymptotically rate R satisfying*

$$\tfrac{1}{3} \leq R \leq \tfrac{3}{4}.$$

Theorem 4.3. *If E is a* 1 *among* 4 *family, then it is* WUF.

Proof. If one writes as columns the characteristic vectors of E, getting a tableau T, then the WUF property is equivalent to saying that any $n \times 4$ subtableau of T contains in its rows a 4-tuple of weight one or 3 noncomplementary vectors of weight two (e.g. 1100, 1010 and 1001). The 1 among 4 property is stronger: any $n \times 4$ subtableau T must contain a 4-tuple of weight one. □

Theorem 4.4. *The best rate of a* 1 *among* 4 *family satisfies*

$$R \geq 0.25.$$

Proof. The proof is constructive, based on error-correcting codes (see [10] for definitions and properties). Consider a linear 2-error-correcting code C of length N and dimension K.

Let H be a parity-check matrix for C, i.e. an $(N-K)\times N$ matrix whose rows generate the dual of C. Then C has minimum distance d at least 5 $\Leftrightarrow$ any 4 columns *of* H are independent over $\mathbb{F}_2$ $\Leftrightarrow$ any four columns A, B, C, D contain a row of odd weight (1 or 3).

Concatenating H and its complement, we get a $2(N-K)\times N$ tableau L which is 1 among 4.

Now there exist such codes with $N-K=2\log(N+1)$, for example the BCH codes $[N=2^r-1, K=2^r-2r-1, d=5]$.

Setting $4\log(N+1)=n$, the columns of L give us a 1 among 4 family of n-tuples of size $2^{n/4}-1$. □

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