Colorful subgraphs in Kneser-like graphs

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Abstract

Combining Ky Fan's theorem with ideas of Greene and Matoušek we prove a generalization of Dol'nikov's theorem. Using another variant of the Borsuk-Ulam theorem due to Tucker and Bacon, we also prove the presence of all possible completely multicolored t-vertex complete bipartite graphs in t-colored t-chromatic Kneser graphs and in several of their relatives. In particular, this implies a generalization of a recent result of G. Spencer and F. E. Su.

1 Introduction

The solution of Kneser's conjecture in 1978 by László Lovász [19] opened up a new area of combinatorics that is usually referred to as topological combinatorics [5, 9]. Many results of this area, including the first one by Lovász, belong to one of its by now most developed branches that applies the celebrated Borsuk-Ulam theorem to graph coloring problems. An account of such results and other applications of the Borsuk-Ulam theorem in combinatorics is given in the excellent book of Matoušek [21].

Recently it turned out that a generalization of the Borsuk-Ulam theorem found by Ky Fan [13] in 1952 can give useful generalizations and variants of the Lovász-Kneser theorem. Examples of such results can be found in [23, 27, 29, 30].

In this paper our aim is twofold. In Section 2 we show a further application of Ky Fan's theorem. More precisely, we give a generalization of Dol'nikov's theorem, which is itself a generalization of the Lovász-Kneser theorem. The proof will be a simple combination of Ky Fan's result with the simple proof of Dol'nikov's theorem given by Matoušek in [21] that was inspired by Greene's recent proof [15] of the Lovász-Kneser theorem.

In Section 3 we use another variant of the Borsuk-Ulam theorem due to Tucker [33] and Bacon [3] to show a property of optimal colorings of certain t-chromatic graphs, including Kneser graphs, Schrijver graphs, Mycielski graphs, Borsuk graphs, odd chromatic rational complete graphs, and two other types of graphs appearing in [27]. The claimed property (in a somewhat weakened form) is that all the complete bipartite graphs $K_{l,m}$ with $l+m = t$ will have totally multicolored copies in proper t-colorings of the above graphs.

When applied to rational complete graphs this implies a new proof of an earlier result about the circular chromatic number (see Subsection 3.3.4). It was originally obtained in [27] and independently in [23] for some special cases.

When applied to Kneser graphs the above property implies a generalization of a recent result due to G. Spencer and F. E. Su [29, 30] which we will present in the last subsection.

2 A generalization of Dol'nikov's theorem

We recall some concepts and notations from [21]. For any family $\mathcal{F}$ of subsets of a fixed finite set we define the general Kneser graph $\mathrm{KG}(\mathcal{F})$ by

$$\begin{aligned} V(\mathrm{KG}(\mathcal{F})) &= \mathcal{F}, \\ E(\mathrm{KG}(\mathcal{F})) &= \{\{F, F'\} : F, F' \in \mathcal{F}, F \cap F' = \emptyset\}. \end{aligned}$$

When we refer to a Kneser graph (without the adjective "general") we mean the "usual" Kneser graph $\mathrm{KG}(n, k)$ that is identical to the general Kneser graph of a set system consisting of all k-subsets of an n-set.

A hypergraph H is m-colorable, if its vertices can be colored by (at most) m colors so that no hyperedge becomes monochromatic. As isolated vertices (not contained in any hyperedge) play no role we may identify hypergraphs with the set of their edges as done in the following definition. The m-colorability defect of a set system $\emptyset \notin \mathcal{F}$ is

$$\mathrm{cd}_m(\mathcal{F}) := \min\{|Y| : \{F \in \mathcal{F} : F \cap Y = \emptyset\} \text{ is } m\text{-colorable}\}.$$

Dol'nikov's theorem ([10]) *For any finite set system $\emptyset \notin \mathcal{F}$, the inequality*

$$\mathrm{cd}_2(\mathcal{F}) \leq \chi(\mathrm{KG}(\mathcal{F}))$$

holds.

This theorem generalizes the Lovász-Kneser theorem, as it is easy to check that if $\mathcal{F}$ consists of all the k-subsets of an n-set with $k \leq n/2$, then $\mathrm{cd}_2(\mathcal{F}) = n - 2k + 2$ which is the true value of $\chi(\mathrm{KG}(\mathcal{F}))$ in this case. On the other hand, as also noted in [21], equality between $\mathrm{cd}_2(\mathcal{F})$ and $\chi(\mathrm{KG}(\mathcal{F}))$ does not hold in general.

Recently Greene [15] found a very simple new proof of the Lovász-Kneser theorem. In [21] Matoušek observed that one can generalize Greene's proof so that it also gives Dol'nikov's theorem. Here we combine this proof with Ky Fan's theorem (see below) to obtain the following generalization. We say that a (sub)graph is *completely multicolored* in a coloring if all its vertices receive different colors.

Theorem 1 *Let $\mathcal{F}$ be a finite family of sets, $\emptyset \notin \mathcal{F}$ and $\mathrm{KG}(\mathcal{F})$ its general Kneser graph. Let $r = \mathrm{cd}_2(\mathcal{F})$. Then any proper coloring of $\mathrm{KG}(\mathcal{F})$ with colors $1, \ldots, m$ (m arbitrary) must contain a completely multicolored complete bipartite graph $K_{\lceil r/2\rceil, \lfloor r/2\rfloor}$ such that the r different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.*

This theorem generalizes Dol'nikov's theorem, because it implies that any proper coloring must use at least $\mathrm{cd}_2(\mathcal{F})$ different colors.

Remark 1. Theorem 1 is clearly in the spirit of the Zig-zag theorem of [27] the special case of which for Kneser graphs was already established by Ky Fan in [14]. This theorem claims that if a specific topological parameter of a graph G, the value of which is a lower bound on its chromatic number, is at least t then any proper coloring of the graph must contain a completely multicolored $K_{\lceil t/2\rceil, \lfloor t/2\rfloor}$ where the colors also alternate on the two sides with respect to their natural order. As we will show in Remark 2 the proof below can be modified to show that the topological parameter mentioned is at least $\mathrm{cd}_2(\mathcal{F})$ for any $\mathrm{KG}(\mathcal{F})$. A similar inequality is also shown in [22]. We will say more about this topological parameter in Section 3. $\Diamond$

To prove the theorem we first have to state Ky Fan's theorem [13]. Its form fitting best for our purposes is the following. (To find exactly this form, see [3].) For a set A on the unit sphere S^h we denote by $-A$ its antipodal set, i.e., $-A = \{-\boldsymbol{x} : \boldsymbol{x} \in A\}$.

Ky Fan's theorem ([13]) *Let $A_1, \ldots, A_m$ be open subsets of the h-dimensional sphere S^h satisfying that none of them contains antipodal points (i.e., $\forall i\ (-A_i) \cap A_i = \emptyset$) and that at least one of $\boldsymbol{x}$ and $-\boldsymbol{x}$ is contained in $\cup_{i=1}^m A_i$ for all $\boldsymbol{x} \in S^h$.*

Then there exists an $\boldsymbol{x} \in S^h$ and $h+1$ distinct indices $i_1 < \ldots < i_{h+1}$ such that $\boldsymbol{x} \in A_{i_1} \cap -A_{i_2} \cap \ldots \cap (-1)^h A_{i_{h+1}}$.

Proof of Theorem 1. Let $h = \mathrm{cd}_2(\mathcal{F}) - 1$ and consider the sphere S^h. We assume without loss of generality that the base set $X := \cup\mathcal{F}$ is finite and identify its elements with points of S^h in general position, i.e., so that at most h of them can be on a common hyperplane through the origin. Consider an arbitrary fixed proper coloring of $\mathrm{KG}(\mathcal{F})$ with colors $1, \ldots, m$. For every $\boldsymbol{x} \in S^h$ let $H(\boldsymbol{x})$ denote the open hemisphere centered at $\boldsymbol{x}$. Define the sets $A_1, \ldots, A_m$ as follows. Set A_i will contain exactly those points $\boldsymbol{x} \in S^h$ that have the property that $H(\boldsymbol{x})$ contains the points of some $F \in \mathcal{F}$ which is colored by color i in the coloring of $\mathrm{KG}(\mathcal{F})$ considered. The sets A_i are all open. None of them contains an antipodal pair of points, otherwise there would be two disjoint open hemispheres both of which contain some element of $\mathcal{F}$ that is colored i. But these two elements of $\mathcal{F}$ would be disjoint contradicting the assumption that the coloring was proper. Now we show that there is no $\boldsymbol{x} \in S^h$ for which neither $\boldsymbol{x}$ nor $-\boldsymbol{x}$ is in $\cup_{i=1}^m A_i$. Color the points in $X \cap H(\boldsymbol{x})$ red, the points in $X \cap H(-\boldsymbol{x})$ blue and delete the points of X not colored, i.e., those on the "equator" between $H(\boldsymbol{x})$ and $H(-\boldsymbol{x})$. Since at most $h < \mathrm{cd}_2(\mathcal{F})$ points are deleted there exists some $F \in \mathcal{F}$ which became completely red or completely blue. All points of F are either in $H(\boldsymbol{x})$ or in $H(-\boldsymbol{x})$. This implies that $\boldsymbol{x}$ or $-\boldsymbol{x}$ should belong to A_i where i is the color of F in our fixed coloring of $\mathrm{KG}(\mathcal{F})$.

Thus our sets $A_1, \ldots, A_m$ satisfy the conditions and therefore also the conclusion of Ky Fan's theorem. Let $F_j \in \mathcal{F}$ be the set responsible for $\boldsymbol{x} \in A_{i_j}$ for j odd and for $-\boldsymbol{x} \in A_{i_j}$ for j even in the conclusion of Ky Fan's theorem. Then all the F_j's with odd j must be disjoint from all the F_j's with even j. Thus they form the complete bipartite graph claimed. □

2.1 An example

We would like to point out that Theorem 1 is really stronger than Dol'nikov's theorem, especially if looked at as an upper bound for $\mathrm{cd}_2(\mathcal{F})$. It is shown in [22] that every graph G is isomorphic to the general Kneser graph $\mathrm{KG}(\mathcal{F})$ of some set system $\mathcal{F}$.

Consider the following graphs defined in [11].

Definition 1 ([11]) *For positive integers $r \le m$ the graph $U(m,r)$ is defined as follows.*

$$\begin{aligned} V(U(m,r)) &= \{(i,A) : i \in [m], A \subseteq [m], |A| = r-1, i \notin A\} \\ E(U(m,r)) &= \{\{(i,A),(j,B)\} : i \in B, j \in A\} \end{aligned}$$

Let $\mathcal{F}(m,r)$ denote a set system for which $U(m,r) \cong \mathrm{KG}(\mathcal{F}(m,r))$. It follows from results in [11] that if $r \ge 3$ is fixed and m goes to infinity then $\chi(U(m,r))$ also grows above all limits. Thus Dol'nikov's theorem would not give any finite upper bound for $\mathrm{cd}_2(\mathcal{F}(m,r))$ if $r \geq 3$ is fixed and m goes to infinity. The same is true if we consider only the size (and not the coloring) of largest complete bipartite subgraphs in $U(m,r)$ which, due to combination of results in [8] and [22] can also provide an upper bound for $\mathrm{cd}_2(\mathcal{F}(m,r))$. (Indeed, it is easy to check that $U(m,r)$ contains $K_{\binom{m-2}{r-2},\binom{m-2}{r-2}}$ subgraphs.) In contrast, consider the proper coloring given as $(x,A) \mapsto x$. One easily checks that the largest balanced completely multicolored complete bipartite subgraph in this coloring is $K_{r-1,r-1}$, from which Theorem 1 gives the upper bound $2r-2$ on $\mathrm{cd}_2(\mathcal{F}(m,r))$, which is independent of m.

For more about the relevance of the graphs $U(m,r)$, see Subsection 3.3.2.

3 Applying a theorem of Tucker and Bacon

3.1 Preliminaries

First we give a very brief introduction of some topological concepts we need. All this can be found in detail, e.g., in [21]. A $\mathbb{Z}_2$-space is a pair (T,ν), where T is a topological space and $\nu : T \to T$ is an involution, that is, a continuous map satisfying $\nu(\nu(x)) = x$ for all $x \in T$. A $\mathbb{Z}_2$-space (T,ν) is free if $\nu(x) \neq x$ for every $x \in T$. If ν is clear from the context we write T in place of (T,ν). Accordingly, we write S^h for the most important $\mathbb{Z}_2$-space we deal with, the h-dimensional sphere with the usual antipodal map as involution.

A continuous map $f : (T,\nu) \to (W,\mu)$ is a $\mathbb{Z}_2$-map if it respects the respective involutions, that is, $f(\nu(x)) = \mu(f(x))$ for every $x \in T$. We write $(T,\nu) \to (W,\mu)$ if there exists a $\mathbb{Z}_2$-map from (T,ν) to (W,μ). Two important parameters of a $\mathbb{Z}_2$-space are its $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex that are defined as

$$\mathrm{ind}(T,\nu) := \min\{h \geq 0 : (T,\nu) \to S^h\},$$

and

$$\mathrm{coind}(T,\nu) := \max\{h \geq 0 : S^h \to (T,\nu)\},$$

respectively. The inequality

$$\mathrm{coind}(T,\nu) \leq \mathrm{ind}(T,\nu)$$

always holds and is equivalent to the celebrated Borsuk-Ulam theorem.

In applications of the topological method one often associates so-called box complexes to graphs. These give rise to topological spaces the index and coindex of which can be used to obtain lower bounds for the chromatic number of the graph. Following ideas in earlier works by Alon, Frankl, Lovász [2] and others, the paper [22] introduces several box complexes two of which we also define below.

Definition 2 *The box complex $B(G)$ is a simplicial complex on the vertices $V(G) \times \{1,2\}$. For subsets $S, T \subseteq V(G)$ the set $S \uplus T := S \times \{1\} \cup T \times \{2\}$ forms a simplex if and only* *if $S \cap T = \emptyset$, the vertices in S have at least one common neighbor, the same is true for T, and the complete bipartite graph with sides S and T is a subgraph of G. The $\mathbb{Z}_2$-map $S \uplus T \mapsto T \uplus S$ acts simplicially on $B(G)$ making the topological realization $||B(G)||$ of the complex a free $\mathbb{Z}_2$-space.*

It is explained in [22] and [21] that $B(G)$ is a functor, meaning for example, that whenever there exists a homomorphism from a graph F to another graph G then $B(F) \to B(G)$. For the $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex of $||B(G)||$ we simply write $\mathrm{ind}(B(G))$ and $\mathrm{coind}(B(G))$, respectively, and we will do similarly for the other box complex $B_0(G)$ defined below. It is not hard to see that $||B(K_n)||$ is $\mathbb{Z}_2$-homotopy equivalent to S^{n-2} (this means homotopy equivalence established by $\mathbb{Z}_2$-maps). In particular, $\mathrm{ind}(B(K_n)) = \mathrm{coind}(B(K_n)) = n-2$. Since a graph is t-colorable if and only if it admits a homomorphism to K_t, the foregoing implies

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2 \geq \mathrm{coind}(B(G)) + 2. \tag{1}$$

Another box complex $B_0(G)$ defined in [22] differs from $B(G)$ only by containing all those simplices $S \uplus T$, too, where one of S or T is empty independently of the existence of common neighbors required in the definition of $B(G)$.

Definition 3 *The box complex $B_0(G)$ is a simplicial complex on the vertices $V(G) \times \{1, 2\}$. For subsets $S, T \subseteq V(G)$ the set $S \uplus T := S \times \{1\} \cup T \times \{2\}$ forms a simplex if and only if $S \cap T = \emptyset$, and the complete bipartite graph with sides S and T is a subgraph of G. The $\mathbb{Z}_2$-map $S \uplus T \mapsto T \uplus S$ acts simplicially on $B_0(G)$ making the topological realization $||B_0(G)||$ of the complex a free $\mathbb{Z}_2$-space.*

(For a nice illustration of the two kinds of box complexes just defined we refer the reader to the figures in Matoušek and Ziegler's paper [22] showing $B(C_5)$ and $B_0(C_5)$.)

Csorba [6] proved a strong topological relationship between $B(G)$ and $B_0(G)$, namely, that $||B_0(G)||$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of $||B(G)||$. This extends (1) to the following longer chain of inequalities, cf. [22] and also [27].

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2 \geq \mathrm{ind}(B_0(G)) + 1 \geq \mathrm{coind}(B_0(G)) + 1 \geq \mathrm{coind}(B(G)) + 2 \tag{2}$$

Note that $B_0(G)$ is also a functor, and it is easy to see even without Csorba's result that $||B_0(K_n)||$ is $\mathbb{Z}_2$-homotopy equivalent (in fact, $\mathbb{Z}_2$-homeomorphic) to S^{n-1}.

There are several interesting graph families the members of which satisfy the inequalities in (2) with equality. These include, for example, Kneser graphs, and a longer list is given in Corollary 4 below. (We note that some of the graphs in Corollary 4 give equality only in the first three of the above inequalities, cf. Subsection 3.3.2.)

Remark 2. The topological parameter mentioned in Remark 1 is $\mathrm{coind}(B_0(G)) + 1$. Thus our claim in Remark 1 is that the proof of Theorem 1 implies $\mathrm{coind}(B_0(\mathrm{KG}(\mathcal{F}))) \geq$

$\mathrm{cd}_2(\mathcal{F}) - 1$ for any $\mathcal{F}$ not containing the empty set. The slightly weaker inequality $\mathrm{ind}(B_0(\mathrm{KG}(\mathcal{F}))) \geq \mathrm{cd}_2(\mathcal{F}) - 1$ is proved in [22]. Here we sketch the proof of our claim which is similar to the proof of Proposition 8 in [27]. Assume again without loss of generality that $X = \cup\mathcal{F}$ is finite and identify its elements with points of S^h in general position as in the proof of Theorem 1 with $h = \mathrm{cd}_2(\mathcal{F}) - 1$. For each vertex v of $\mathrm{KG}(\mathcal{F})$ and $\boldsymbol{x} \in S^h$ let $D_v(\boldsymbol{x})$ be the smallest distance of a point in v (this point is an element of X) from the set $S^h \setminus H(\boldsymbol{x})$. Notice that $D_v(\boldsymbol{x}) > 0$ iff $H(\boldsymbol{x})$ contains all points of v. Set $D(\boldsymbol{x}) := \sum_{v \in \mathcal{F}}(D_v(\boldsymbol{x}) + D_v(-\boldsymbol{x}))$. The argument in the proof of Theorem 1 implies $D(\boldsymbol{x}) > 0$. Therefore the map $f(\boldsymbol{x}) = (1/D(\boldsymbol{x}))(\sum_v D_v(\boldsymbol{x})||(v,1)|| + \sum_v D_v(-\boldsymbol{x})||(v,2)||)$ is a $\mathbb{Z}_2$-map from S^h to $||B_0(\mathrm{KG}(\mathcal{F}))||$, thus $\mathrm{coind}(B_0(\mathrm{KG}(\mathcal{F}))) \geq h$ as claimed. $\Diamond$

3.2 A colorful $K_{l.m}$-theorem

In their recent paper [8] Csorba, Lange, Schurr, and Waßmer proved that if $\mathrm{ind}(B(G)) = l+m-2$ then G must contain the complete bipartite graph $K_{l,m}$ as a subgraph. They called this "the $K_{l,m}$-theorem". In case of those graphs that satisfy $\mathrm{coind}(B_0(G))+1 = \chi(G)$ (see Corollary 4), the following statement generalizes their result. We use again the notation $[t] := \{1, \dots, t\}$.

Theorem 2 *Let G be a graph for which $\chi(G) = \mathrm{coind}(B_0(G))+1 = t$. Let $c : V(G) \to [t]$ be a proper coloring of G and let $A, B \subseteq [t]$ form a bipartition of the color set, i.e., $A \cup B = [t]$ and $A \cap B = \emptyset$.*

Then there exists a complete bipartite subgraph $K_{l,m}$ of G with sides L, M such that $|L| = l = |A|$, $|M| = m = |B|$, and $\{c(v) : v \in L\} = A$, and $\{c(v) : v \in M\} = B$. In particular, this $K_{l,m}$ is completely multicolored at c.

For the proof we will use a modified version of the following theorem.

Tucker-Bacon theorem ([33, 3]) *If $C_1, \dots, C_{h+2}$ are closed subsets of S^h,*

$$\cup_{i=1}^{h+2} C_i = S^h, \quad \forall i: \ C_i \cap (-C_i) = \emptyset,$$

and $j \in \{1, \dots, h+1\}$, then there is an $\boldsymbol{x} \in S^h$ such that

$$\boldsymbol{x} \in \cap_{i=1}^{j} C_i, \text{ and } -\boldsymbol{x} \in \cap_{i=j+1}^{h+2} C_i.$$

Tucker proved the above theorem in the 2-dimensional case and claimed the general statement, the proof of which is a straightforward generalization of his argument. Bacon [3] shows that this theorem is equivalent to 14 other statements that include standard forms of the Borsuk-Ulam theorem, and also Ky Fan's theorem.

It is a routine matter to see that the Tucker-Bacon theorem also holds for open sets C_i. (One can simply use the fact that for a collection of open sets C_i covering S^h one can define closed sets C_i' so that $C_i' \subseteq C_i$ for all i and $\cup_{i=1}^{h+2} C_i' = \cup_{i=1}^{h+2} C_i = S^h$. Cf. [1], Satz VII, p. 73, quoted also in [3].)

The modified version we need is the following.

Tucker-Bacon theorem, second form. *If $C_1, \ldots, C_{h+1}$ are open subsets of S^h,*

$$\cup_{i=1}^{h+1}(C_i \cup (-C_i)) = S^h, \quad \forall i: \ C_i \cap (-C_i) = \emptyset,$$

and $j \in \{0, \ldots, h+1\}$, then there is an $\boldsymbol{x} \in S^h$ such that

$$\boldsymbol{x} \in C_i \text{ for } i \leq j \text{ and } -\boldsymbol{x} \in C_i \text{ for } i > j.$$

Proof. Let $D_{h+2} = S^h \setminus \left(\cup_{i=1}^{h+1} C_i\right)$. Then $D_{h+2} \cap (-D_{h+2}) = \emptyset$ by the first condition on the sets C_i. Since D_{h+2} is closed there is some $\varepsilon > 0$ bounding the distance of any pair of points $\boldsymbol{x} \in D_{h+2}$ and $\boldsymbol{y} \in -D_{h+2}$ from below. Let C_{h+2} be the open $\frac{\varepsilon}{2}$-neighborhood of D_{h+2}. Then the open sets $C_1, \ldots, C_{h+2}$ satisfy the conditions (of the open set version) of the Tucker-Bacon theorem. Therefore its conclusion holds. Neglecting the set C_{h+2} in this conclusion the proof is completed for $j > 0$. To see the statement for $j = 0$ one can take the negative of the value $\boldsymbol{x}$ guaranteed for $j = h+1$. $\square$

Remark 3. Frédéric Meunier [24] noted that the second form of the Tucker-Bacon theorem can also be deduced directly from the given form of Ky Fan's theorem by exchanging some of the given open sets with their antipodal sets (separately for each possible value of j) and indexing appropriately. $\Diamond$

Proof of Theorem 2. Let G be a graph with $\chi(G) = \mathrm{coind}(B_0(G)) + 1 = t$ and fix an arbitrary proper t-coloring $c : V(G) \to [t]$. Let $g : S^{t-1} \to ||B_0(G)||$ be a $\mathbb{Z}_2$-map that exists by $\mathrm{coind}(B_0(G)) = t - 1$.

We define for each color $i \in [t]$ an open set C_i on S^{t-1}. For $\boldsymbol{x} \in S^{t-1}$ we let $\boldsymbol{x}$ be an element of C_i iff the minimal simplex $S_{\boldsymbol{x}} \uplus T_{\boldsymbol{x}} \in B_0(G)$ whose topological realization contains $g(\boldsymbol{x})$ has a vertex $v \in S_{\boldsymbol{x}}$ for which $c(v) = i$. These C_i's are open. If an $\boldsymbol{x} \in S^{t-1}$ is not covered by any C_i then $S_{\boldsymbol{x}}$ must be empty, which also implies $T_{\boldsymbol{x}} \neq \emptyset$. Since $S_{-\boldsymbol{x}} = T_{\boldsymbol{x}}$ this further implies $-\boldsymbol{x} \in \cup_{i=1}^{t} C_i$, thus $\cup_{i=1}^{t}(C_i \cup (-C_i)) = S^{t-1}$ follows.

If a set C_i contained an antipodal pair $\boldsymbol{x}$ and $-\boldsymbol{x}$ then $S_{-\boldsymbol{x}} = T_{\boldsymbol{x}}$ would contain an i-colored vertex u, while $S_{\boldsymbol{x}}$ would contain an i-colored vertex v. Since $S_{\boldsymbol{x}} \uplus T_{\boldsymbol{x}}$ is a simplex of $B_0(G)$ u and v must be adjacent, contradicting that c is a proper coloring. Thus the C_i's satisfy the conditions of (the second form of) the Tucker-Bacon theorem.

Let $j = |A| = l$ and relabel the colors so that colors $1, \ldots, j$ be in A, and the others be in B. The indices of the C_i's are relabeled accordingly. We apply the second form of the Tucker-Bacon theorem with $h = t - 1$. It guarantees the existence of an $\boldsymbol{x} \in S^{t-1}$ with the property $\boldsymbol{x} \in C_i$ for $i \in A$ and $-\boldsymbol{x} \in C_i$ for $i \in B$. Then $S_{\boldsymbol{x}}$ contains vertices $u_1, \ldots, u_l$ with $c(u_i) = i$ for all $i \in A$ and $S_{-\boldsymbol{x}} = T_{\boldsymbol{x}}$ contains vertices $v_1, \ldots, v_m$ with $c(v_i) = l + i$ for all $(l + i) \in B$. Since all vertices of $S_{\boldsymbol{x}}$ are connected to all vertices in $T_{\boldsymbol{x}}$

by the definition of $B_0(G)$, they give the required completely multicolored $K_{l,m}$ subgraph. $\square$

3.3 Graphs that are subject of the colorful $K_{l,m}$ theorem

Let us put the statement of Theorem 2 into the perspective of our earlier work in [27]. There we investigated the local chromatic number of graphs that is defined in [11] as

$$\psi(G) := \min_{c} \max_{v\in V(G)} |\{c(u) : uv \in E(G)\}| + 1,$$

where the minimum is taken over all proper colorings c of G. That is $\psi(G)$ is the minimum number of colors that must appear on some star subgraph in any proper coloring.

With similar techniques to those applied in this paper we have shown using Ky Fan's theorem that $\mathrm{coind}(B_0(G)) \geq t-1$ implies that G must contain a completely multicolored $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ subgraph in *any* proper coloring with the colors alternating with respect to their natural order on the two sides of this complete bipartite graph. This is the Zig-zag theorem in [27] we already referred to in Remark 1. The Zig-zag theorem implies that any graph satisfying its condition must have $\psi(G) \geq \lceil t/2\rceil + 1$. In [27] we have shown for several graphs G for which $\chi(G) = \mathrm{coind}(B_0(G)) + 1 = t$ that it can be colored with $t+1$ colors so that no vertex has more than $\lfloor t/2\rfloor + 1$ colors on its neighbors. When t is odd, this established the exact value $\psi(G) = \lceil t/2\rceil + 1$ for these graphs. For odd t this also means that the only type of $K_{l,m}$ subgraph with $l + m = t$ that must appear completely multicolored when using $t+1$ colors is the $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ subgraph guaranteed by the Zig-zag theorem (apart from the empty graph $K_{t,0}$). If we use only t colors, however, then the situation is quite different. It is true for any graph F that if it is properly colored with $\chi(F)$ colors then each color class must contain a vertex that sees all other colors on the vertices adjacent to it. If it were not so, we could completely eliminate a color class by recoloring each of its vertices to a color which is not present on any of its neighbors. In the context of local chromaticity this means that if $\psi(F) < \chi(F)$ then it can only be attained by a coloring that uses strictly more than $\chi(F)$ colors. Now Theorem 2 says that if G satisfies $\chi(G) = \mathrm{coind}(B_0(G)) + 1 = t$ then all t-colorings give rise not only to completely multicolored $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$'s that are guaranteed by the Zig-zag theorem, and $K_{1,t-1}$'s that must appear in any optimal coloring (with all possible choices of the color being on the single vertex side), but to all possible completely multicolored complete bipartite graphs on t vertices.

To conclude this subsection we list explicitly some classes of graphs G that satisfy the $\chi(G) = \mathrm{coind}(B_0(G)) + 1$ condition of Theorem 2. We recall that in [27] we used the term *topologically t-chromatic* for graphs G with $\mathrm{coind}(B_0(G)) \geq t - 1$. With this notation we are listing graphs G that are topologically $\chi(G)$-chromatic, i.e., for which this specific lower bound on their chromatic number is tight.

3.3.1 Standard examples

For the following graph families it is well known that they (or their appropriate members) satisfy the condition in Theorem 2. For a detailed discussion we refer to [21] and [27].

Kneser graphs. The argument presented in Remark 2 proves that the general Kneser graph $G = \mathrm{KG}(\mathcal{F})$ satisfies $\chi(G) = \mathrm{coind}(B_0(G)) + 1$ as long as $\chi(\mathrm{KG}(\mathcal{F})) = \mathrm{cd}_2(\mathcal{F})$. This family includes the graphs $\mathrm{KG}(n, k)$.

Schrijver graphs. The graph denoted by $\mathrm{SG}(n, k)$ is just the general Kneser graph $\mathrm{KG}(\mathcal{F})$ for the set system $\mathcal{F}$ consisting of exactly those k-subsets of $[n]$ that contain neither a pair $\{i, i+1\}$, nor $\{1, n\}$. These graphs were introduced by Schrijver who proved that they are vertex color-critical with $\chi(\mathrm{SG}(n, k)) = n-2k+2$. Though $\chi(\mathrm{KG}(\mathcal{F})) \neq \mathrm{cd}_2(\mathcal{F})$ for this $\mathcal{F}$ if $k > 1$, all graphs $\mathrm{SG}(n, k)$ satisfy the condition in Theorem 2.

Borsuk graphs. The graph $B(n, \alpha)$ has S^{n-1} as its vertex set and two vertices are adjacent if their distance is at least $\alpha < 2$, cf. [12, 20]. If α is close enough to 2, then the chromatic number of these graphs is $n + 1$. The paper [20] shows that some finite subgraphs of $B(n, \alpha)$ also have the required properties.

Mycielski and generalized Mycielski graphs. The generalized Mycielski construction first appeared probably in [31], cf. also [16]. It also reappears in [18, 32]. This construction creates from a graph G its generalized Mycielskian $M_r(G)$, where r denotes the number of "levels" in the construction. (The ordinary Mycielski construction [25] is the $r = 2$ special case.) For a graph G with vertices $v_1, \ldots, v_n$ the vertex set of $M_r(G)$ is $\{v_1^{(p)}, \ldots, v_n^{(p)} : 0 \leq p \leq r-1\} \cup \{z\}$ and the pair $v_i^{(p)} v_j^{(q)}$ forms an edge iff $v_i v_j \in E(G)$ and either $p = q = 0$ or $p = q \pm 1$. The additional vertex z is connected to $v_1^{(r)}, \ldots, v_n^{(r)}$.

When applying this construction to an arbitrary graph, the clique number does not increase (except in the trivial case when $r = 1$) while the chromatic number may or may not increase. If it does it increases by 1. Generalizing Stiebitz's result [31] (see also in [16, 21]) Csorba [7] proved that $B(M_r(G))$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of $B(G)$ for every graph G. (Csorba's result is in terms of the so-called homomorphism complex $\mathrm{Hom}(K_2, G)$ but this is known to be $\mathbb{Z}_2$-homotopy equivalent to $B(G)$ by results in [6, 22, 35].) Together with Csorba's already mentioned other result in [6] stating the $\mathbb{Z}_2$-homotopy equivalence of $B_0(G)$ and the suspension of $B(G)$, the foregoing implies that if a graph G satisfies $\chi(G) = \mathrm{coind}(B_0(G)) + 1$, then the analogous equality will also hold for the graph $M_r(G)$. In this case the chromatic number does increase by 1. Thus iterating the construction d times (perhaps with varying parameters r) we arrive to a graph which has chromatic number $\chi(G) + d$ and still satisfies that its chromatic number is equal to its third (in fact, for $d > 0$ also the fourth) lower bound given in (2). For further explanation of these relations we refer to [27].

In the following subsections we list three more families of graphs that satisfy the condition in Theorem 2. Members of the third family are well-known graphs but probably not in the present context. The other two families are less known. Their relevance here is implicit in [27].

3.3.2 Homomorphism universal graphs for local colorings

Recall the definition of graphs $U(m, r)$ from Subsection 2.1. It is shown in [11] that these graphs characterize local colorability in the following sense: a graph G has an m-coloring showing $\psi(G) \leq r$ if and only if G admits a homomorphism to $U(m, r)$. As mentioned above, in [27] for all odd $t \geq 3$ we showed for several t-chromatic graphs satisfying the conditions of Theorem 2 that their local chromatic number is $\lceil t/2 \rceil + 1$ and it is attained with a coloring using $t + 1$ colors. It follows that for odd t the t-chromatic graph $U(t + 1, \frac{t+3}{2})$ also satisfies the conditions of Theorem 2. (Indeed, $\chi(U(t + 1, r)) \leq t$ is straightforward for any $r \leq t$ and $\mathrm{coind}(B_0(U(t + 1, \frac{t+3}{2}))) \geq t - 1$ follows from the $F \to U(t + 1, \frac{t+3}{2})$ homomorphism from a graph with $\mathrm{coind}(B_0(F)) = t - 1$.)

The graphs $U(t + 1, \frac{t+2}{2})$ with t even also belong here. It is proven in [28] that $\mathrm{coind}(B_0(U(t+1, \frac{t+2}{2})) = t-1$, but the proof is rather different from the previous argument above. We also mention the result from [28] according to which the fourth lower bound on the chromatic number in (2) is not tight for these graphs (while it is for the graphs of the previous paragraphs). This shows that Theorem 2 in its present form is somewhat stronger than it would be with the stronger requirement $\chi(G) = \mathrm{coind}(B(G)) + 2$ in place of $\chi(G) = \mathrm{coind}(B_0(G)) + 1$. We needed the second form of the Tucker-Bacon theorem for obtaining this stronger form.

We mention that $\chi(U(t + 1, \lfloor \frac{t+3}{2} \rfloor)) = t$ is a special case of Theorem 2.6 in [11].

3.3.3 Homomorphism universal graphs for wide colorings

Definition 4 *Let H_s be the path on the vertices $0, 1, 2, \ldots, s$ (i and $i - 1$ connected for $1 \leq i \leq s$) with a loop at s. We define $W(s, t)$ to be the graph with*

$$
\begin{aligned}
V(W(s,t)) &= \{(x_1 \ldots x_t) : \forall i\ x_i \in \{0, 1, \ldots, s\}, \exists! i\ x_i = 0,\ \exists j\ x_j = 1\} \\
E(W(s,t)) &= \{\{x_1 \ldots x_t, y_1 \ldots y_t\} : \forall i\ \{x_i, y_i\} \in E(H_s)\}.
\end{aligned}
$$

The graphs $W(2, t)$ are defined in [16] in somewhat different terms. It is shown there that a graph can be colored properly with t colors so that the neighborhood of each color class is an independent set if and only if it admits a homomorphism into $W(2, t)$. The described property is equivalent to having a t coloring where no walk of length 3 can connect vertices of the same color. Similarly, a graph F admits a homomorphism into $W(s, t)$ if and only if it can be colored with t colors so that no walk of length $2s - 1$ can connect vertices of the same color. Such colorings are called s-wide in [27]. Other graphs having the mentioned property of $W(s, t)$ are also defined in [16]. The graphs $W(s, t)$ are defined and shown to be minimal with respect to the above property in [27] and independently also in [4]. The t-colorability of $W(s, t)$ is obvious: $c(x_1 \ldots x_t) = i$ if $x_i = 0$

gives a proper coloring. It is also shown in [27] that several of the above mentioned t-chromatic graphs (e.g., $B(t-1,\alpha)$ for α close enough to 2 and $\mathrm{SG}(n,k)$ with $n-2k+2=t$ and n,k large enough with respect to s and t) admit a homomorphism to $W(s,t)$. This implies $\mathrm{coind}(B_0(W(s,t))) \geq t-1$ (with equality, since $\chi(W(s,t))=t$).

3.3.4 Rational complete graphs

Our last example of a graph family satisfying the conditions of Theorem 2 consists of certain rational (or circular) complete graphs $K_{p/q}$, as they are called, for example, in [17]. The graph $K_{p/q}$ is defined for positive integers $p \geq 2q$ on the vertex set $\{0,\dots,p-1\}$ and $\{i,j\}$ is an edge if and only if $q \leq |i-j| \leq p-q$. The widely investigated chromatic parameter $\chi_c(G)$, called the circular chromatic number of graph G (cf. [34], or Section 6.1 in [17]) can be defined as the infimum of those values p/q for which G admits a homomorphism to $K_{p/q}$. It is well known that $\chi(G)-1 < \chi_c(G) \leq \chi(G)$ for every graph G. In [18] it is shown that certain odd-chromatic generalized Mycielski graphs can have their circular chromatic number arbitrarily close to the above lower bound. Building on this we showed similar results also for odd chromatic Schrijver graphs and Borsuk graphs in [27]. As it is also known that $K_{p/q}$ admits a homomorphism into $K_{r/s}$ whenever $r/s \geq p/q$ (see, e.g., as Theorem 6.3 in [17]), the above and the functoriality of $B_0(G)$ together imply that $\mathrm{coind}(B_0(K_{p/q}))+1 = \chi(K_{p/q}) = \lceil p/q \rceil$ whenever $\lceil p/q \rceil$ is odd.

We remark that the oddness condition is crucial here. It also follows from results in [27] (cf. also [23] for some special cases) that the graphs $K_{p/q}$ with $\lceil p/q \rceil$ even and p/q not integral do not satisfy the conditions of Theorem 2. Here we state more: the conclusion of Theorem 2 does not hold for these graphs. Indeed, let us color the vertex i with the color $\lfloor i/q \rfloor + 1$. This is a proper coloring with the minimal number $\lceil p/q \rceil$ of colors, but it does not contain a complete bipartite graph with all the even colors on one side and all the odd colors on the other. (The remaining case of p/q even and integral is not especially interesting as $K_{p/q}$ with p/q integral is homomorphic equivalent to the complete graph on p/q vertices and therefore trivially satisfies the $\chi(G) = \mathrm{coind}(B_0(G))+1$ condition.)

Taking the contrapositive in the above observation we obtain a new proof of Theorem 6 in [27] the special case of which for Kneser graphs and Schrijver graphs was independently obtained by Meunier [23].

Corollary 3 ([27], cf. also [23]) *If* $\mathrm{coind}(B_0(G))$ *is odd for a graph* G*, then* $\chi_c(G) \geq \mathrm{coind}(B_0(G))+1$*. In particular, if* G *satisfies* $\chi(G) = \mathrm{coind}(B_0(G))+1$ *and this number is even, then* $\chi_c(G) = \chi(G)$*.* □

We remark that the above result includes a partial solution of two conjectures about the circular chromatic number that are mentioned in [34]. For a detailed discussion of implications and references we refer to [27].

The proof of Corollary 3 in [27] and also the proof in [23] relies on Ky Fan's theorem. The above argument shows that Ky Fan's theorem can be substituted by (the second form of) the Tucker-Bacon theorem in obtaining this result. Nevertheless, it is worth noting, that the missing bipartite graph in the above optimal coloring of an even-chromatic $K_{p/q}$ is one the presence of which would also be required by the Zig-zag theorem.

3.3.5 The entire collection

Our examples are collected in the following corollary.

Corollary 4 *For any proper t-coloring of any member of the following t-chromatic families of graphs the property described as the conclusion of Theorem 2 holds.*

(i) *Kneser graphs* $\mathrm{KG}(n,k)$ *with* $t = n - 2k + 2$,

(ii) *Schrijver graphs* $\mathrm{SG}(n,k)$ *with* $t = n - 2k + 2$,

(iii) *Borsuk graphs* $B(t-1,\alpha)$ *with large enough* $\alpha < 2$ *and some of their finite subgraphs,*

(iv) $U(t+1, \lfloor \frac{t+3}{2} \rfloor)$, *for any* $t \geq 2$,

(v) $W(s,t)$ *for every* $s \geq 1, t \geq 2$,

(vi) *rational complete graphs* $K_{p/q}$ *for* $t = \lceil p/q \rceil$ *odd,*

(vii) *the t-chromatic graphs obtained by* $1 \leq d \leq t-2$ *iterations of the generalized Mycielski construction starting with a* $(t-d)$*-chromatic version of any graph appearing on the list above.*

$\square$

3.4 Generalization of Spencer and Su's result

Recently Gwen Spencer and Francis Edward Su [29, 30] found an interesting consequence of Ky Fan's theorem. They prove that if the Kneser graph $\mathrm{KG}(n,k)$ is colored optimally, that is, with $t = n - 2k + 2$ colors, but otherwise arbitrarily, then the following holds. Given any bipartition of the color set $[t]$ into partition classes B_1 and $B_2 = [t] \setminus B_1$ with $|B_1| = \lfloor t/2 \rfloor$, there exists a bipartition of the ground set $[n]$ into E_1 and E_2, such that, the k-subsets of E_i as vertices of $\mathrm{KG}(n,k)$ are all colored with colors from B_i and every color in B_i does occur (i=1,2).

Theorem 2 implies an analogous statement where no special requirement is needed about the size of B_1 and B_2. It can also be obtained by simply replacing Ky Fan's theorem by the Tucker-Bacon theorem in Spencer and Su's argument.

Corollary 5 *Let* $t = n - 2k + 2$ *and fix an arbitrary proper t-coloring c of the Kneser graph* $\mathrm{KG}(n,k)$ *with colors from the color set* $[t]$*. Let* B_1 *and* B_2 *form a bipartition of* $[t]$*, i.e.,* $B_1 \cup B_2 = [t]$ *and* $B_1 \cap B_2 = \emptyset$*. Then there exists a bipartition* (E_1, E_2) *of* $[n]$ *such that for* $i = 1,2$ *we have* $\{c(v) : v \in V(\mathrm{KG}(n,k)), v \subseteq E_i\} = B_i$.

Proof. Set $A = B_1$ and $B = B_2$ and consider the complete bipartite graph Theorem 2 returns for this bipartition of the color set. Let the vertices on the two sides of this bipartite graph be $u_1, \ldots, u_{|A|}$ and $v_1, \ldots, v_{|B|}$. All vertices u_i and v_j are subsets of $[n]$. Since u_i is adjacent to v_j for every i, j we have that $E_1' := \cup_{i=1}^{|A|} u_i$ and $E_2' := \cup_{j=1}^{|B|} v_j$ are disjoint. If there are elements of $[n]$ that are neither in E_1' nor in E_2' then put each such element into either one of the sets E_i' thus forming the sets E_1 and E_2. We show that these sets E_i satisfy our requirements. It follows from the construction that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = [n]$. It is also clear that all colors from B_1 appear as the color of some k-subset of E_1, namely, the k-subsets $u_1, \ldots, u_{|A|}$. Since E_2 is disjoint from E_1 no k-subset of E_2 can be colored by any of the colors from B_1. Thus each k-subset of E_2 is colored by a color from B_2, and all these colors appear on some k-subset of E_2 by the presence of $v_1, \ldots, v_{|B|}$. Exchanging the role of E_1 and E_2 we get that all k-subsets of E_1 are colored with some color of B_1 and the proof is complete. $\square$

Remark 4. The same argument proves a similar statement for the general Kneser graph $\mathrm{KG}(\mathcal{F})$ in place of $\mathrm{KG}(n, k)$ as long as we have $t = \chi(\mathrm{KG}(\mathcal{F})) = \mathrm{coind}(B_0(\mathrm{KG}(\mathcal{F}))) + 1$. Such graphs include the Schrijver graphs $\mathrm{SG}(n, k)$ with $t = n - 2k + 2$ and (by the argument presented in Remark 2) the graphs $\mathrm{KG}(\mathcal{F})$ with a family $\emptyset \notin \mathcal{F}$ satisfying $t = \chi(\mathrm{KG}(\mathcal{F})) = \mathrm{cd}_2(\mathcal{F})$. $\Diamond$

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