Asymptotic values of the Hall-ratio for graph powers

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Abstract

The Hall-ratio of a graph G is the ratio of the number of vertices and the independence number maximized over all subgraphs of G. We investigate asymptotic values of the Hall-ratio with respect to different graph powers.

Dedicated to the memory of Claude Berge

1 Introduction

Several graph parameters show an interesting behaviour when investigated for different products or powers of graphs. One of the most famous examples for such behaviour is that of the Shannon capacity of graphs which is defined as a normalized limit of the independence number under the so-called normal power, cf. [23].

If $\Theta(G)$ denotes the Shannon capacity of graph G and $\bar{G}$ is the complementary graph, then $C(G) := \Theta(\bar{G})$ is known to lie between the clique number $\omega(G)$ and the chromatic number $\chi(G)$ of G (cf. [23], [17]). This property of Shannon capacity is known to have had a strong influence on Claude Berge when he introduced the celebrated concept of perfect graphs. (For a detailed account on this story, see [5].)

By using the complementary (to normal) concept of co-normal powers (definitions are given in the next section) $C(G)$ can be defined as the asymptotic value of the (appropriately normalized) clique number. The same kind of limit for the chromatic number is known to be equal to the fractional chromatic number of G, cf. [6], [20], [22], which is in general strictly larger than $C(G)$, cf. [17]. (For other descriptions and basic properties of the fractional chromatic number we refer to [22].)

The above discussion may already suggest that similar asymptotic values for parameters falling into the interval $[\omega(G), \chi(G)]$ are usually interesting. The parameter called Hall-ratio, we are concerned with in this paper, has the property of falling into the above interval. In Section 2 we will define and investigate its asymptotic value analoguous to Shannon capacity. (We remark that several other analogues of the Shannon capacity of graphs were already defined and investigated, cf., e. g., [14] and [11].)

It will be clear from its definition below that the Hall-ratio is also closely related to the independence ratio $i(G) := \frac{\alpha(G)}{|V(G)|}$ for which asymptotic values under other graph exponentiations, namely the Cartesian and the direct (or categorical) powers are investigated in [13], [15], [25] and [7]. In these cases the relevant limits are called *ultimate* values. We will define and investigate the corresponding ultimate values of the Hall-ratio in Section 3.

Motivated by problems of list coloring, the Hall-ratio of a graph G is investigated in [8] where it is defined as

$$\rho(G) = \max\{\frac{|V(H)|}{\alpha(H)} : H \subseteq G\},$$

that is, as the ratio of the number of vertices and the independence number maximized over all subgraphs of G. (In [8] *induced subgraph* is said in the definition of $\rho(G)$, but considering other subgraphs, as well, will not matter, since deleting edges from a subgraph H may only increase $\alpha(H)$.) It is clear that $\omega(G) \leq \rho(G)$ by considering any maximal clique as a subgraph. Also, $\rho(G) \leq \chi(G)$ is immediate just like the stronger inequality

$\rho(G) \le \chi_f(G)$, where $\chi_f(G)$ is the fractional chromatic number of G. In [8] the authors investigated the length of the intervals $[\omega(G), \rho(G)]$ and $[\rho(G), \chi(G)]$ and showed that both can be made arbitrarily large, even simultanously.

Thus the Hall-ratio is indeed an invariant that lies between the clique number and the chromatic number, so it coincides with them whenever these two are equal, e. g., for all perfect graphs. It will turn out that the relevant limit for co-normal powers will behave as it does for the upper bound and is always equal to the fractional chromatic number. This will be shown in Section 2 where we also discuss the behaviour of the Hall-ratio with respect to the normal power.

The later sections pursue a systematic study of asymptotic values of the Hall-ratio under the Cartesian, direct and lexicographic exponentiations, i.e., the other three powers coming from graph products treated as the most important ones in the book [16]. In case of the direct and the lexicographic powers we cannot solve the problem of determining the asymptotic values in general, only conjecture that the corresponding limits are expressed again by the fractional chromatic number. In case of the Cartesian power we can use results from [13] and [25] to show that the problem is equivalent to that of determining the ultimate independence ratio and it may actually differ from the value of the fractional chromatic number.

2 Normal and co-normal powers

We first define the co-normal and then the normal power of graphs.

Definition 1 *The co-normal product $G \cdot H$ of two graphs G and H is defined on the vertex set $V(G \cdot H) = V(G) \times V(H)$ with edge set $E(G \cdot H) = \{\{uv, xy\} : \{u, x\} \in E(G)$ or $\{v, y\} \in E(H)\}$. The n^{th} co-normal power G^n of G is the n-fold co-normal product $G \cdot G \cdot \ldots \cdot G$.*

That is, G^n is defined on the n-length sequences over $V = V(G)$ as vertices and two such sequences are adjacent in G^n iff there is some coordinate where the corresponding elements of the two sequences form an edge of G.

Definition 2 *The normal product $G \odot H$ of two graphs G and H is defined on the vertex set $V(G \odot H) = V(G) \times V(H)$ with edge set $E(G \odot H) = \{\{uv, xy\} : \{u, x\} \in E(G)$ and $\{v, y\} \in E(H)$, or $\{u, x\} \in E(G)$ and $v = y$, or $u = x$ and $\{v, y\} \in E(H)\}$. The n^{th} normal power $G^{(n)}$ of G is the n-fold normal product of $G \odot G \odot \ldots \odot G$.*

That is $G^{(n)}$ is defined on the n-length sequences over $V = V(G)$ as vertices and two such sequences are adjacent in $G^{(n)}$ iff their elements at every coordinate are either equal or form an edge in G.

The term *normal* product is used by Berge, for example in [4] (page 111), and is often substituted by several other names, like *AND* product ([2]) or strong product ([16]).

Similarly, the co-normal product is called OR product in [2] and disjunctive product in [16], [22].

It is easy to check that the above two graph powers are complementary in the sense that $\bar{G}^n = \overline{G^{(n)}}$.

2.1 Co-normal powers

The (normalized) asymptotic value of the chromatic number referred to in the Introduction is given by the following theorem of McEliece and Posner [20], cf. also Berge and Simonovits [6].

Theorem 1 *([20])*

$$\lim_{n\to\infty} \sqrt[n]{\chi(G^n)} = \chi_f(G).$$

Exchanging $\chi(G^n)$ to $\omega(G^n)$ above, we obtain the definition of the Shannon capacity of the complementary graph of G (cf. [23]). What we are interested in here is the similar asymptotic value of the Hall-ratio.

Definition 3

$$h(G) := \lim_{n\to\infty} \sqrt[n]{\rho(G^n)}.$$

The existence of the limit easily follows from the fact that $\rho(G^{k+m}) \geq \rho(G^k)\cdot\rho(G^m)$ which is a consequence of $|V(F_k\cdot F_m)| = |V(F_k)|\cdot|V(F_m)|$ and $\alpha(F_k\cdot F_m) = \alpha(F_k)\cdot\alpha(F_m)$ applied to the subgraphs F_i of G^i achieving $\rho(G^i)$ $(i = k, m)$. The existence of the limit is then implied by what is usually called Fekete's theorem, cf., e. g., [21].

We know that $\omega(G^n) \leq \rho(G^n) \leq \chi(G^n)$ and thus $h(G) \leq \chi_f(G)$ follows immediately from either the McEliece-Posner theorem or even more directly from the well known identity $\chi_f(G^n) = [\chi_f(G)]^n$ (cf., e. g., Corollary 3.4.2 of [22]) and the obvious inequality $\rho(G^n) \leq \chi_f(G^n)$.

In general, there is a gap between the Shannon capacity of $\bar{G}$ which is $C(G) := \lim_{n\to\infty}\sqrt[n]{\omega(G^n)}$ and $\chi_f(G)$. A famous example is $G = C_5$, for which $C(G) = \sqrt{5}$ and $\chi_f(G) = \frac{5}{2}$, see [17]. As $\rho(G)$ can be arbitrarily far from both $\omega(G)$ and $\chi(G)$ (cf. [8]), it is not immediately obvious where $h(G)$ should lie between the asymptotic values of $\omega(G)$ and $\chi(G)$, i.e., in the interval $[C(G), \chi_f(G)]$. The next theorem shows that it is always at the upper end.

Theorem 2 *For every graph G we have $h(G) = \chi_f(G)$.*

Remark: Observe that a possible interpretation of Theorems 1 and 2 is that the unique common point of the nested intervals $[\sqrt[i]{\rho(G^i)}, \sqrt[i]{\chi(G^i)}], (i = 1, 2, \ldots)$ is $\chi_f(G)$.

For the proof of Theorem 2 we need some preparation. (For a detailed account of the notions and techniques we will use, cf. [10].)

Definition 4 *For a sequence* $\mathbf{x} \in V^n$ *let the probability distribution defined by* $\forall a \in V : P_{\mathbf{x}}(a) = \frac{|\{i:x_i=a\}|}{n}$ *be called the type of* $\mathbf{x}$. *Let* $\mathcal{T}_P^n$ *denote the set of all sequences in* V^n *that have type* P.

We will use the notation $G[U]$ for the induced subgraph of G on $U \subseteq V$. Thus $G^n[\mathcal{T}_P^n]$ denotes the induced subgraph of G^n on all those sequences that have type P. Let $V(G) = V$.

Lemma 1

$$\chi_f(G^n) \leq (n+1)^{|V|} \max_P \chi_f(G^n[\mathcal{T}_P^n]).$$

Proof. This follows from the fact that the number of different types is at most $(n+1)^{|V|}$, since the number of appearences of each element of V in a sequence of length n can take only $n+1$ different values (cf. [10].) It is obvious that $\chi_f(G^n) \leq \Sigma_{i=1}^M \chi_f(G^n[\mathcal{T}_{P_i}^n])$ where $P_1, \ldots, P_M$ are all the possible types of sequences of length n. (This is easiest to see by representing all χ_f values in the inequality by maximal fractional cliques, that can be done by the duality theorem of linear programming.) Using our previous estimation on the number of different types we obtain

$$\chi_f(G^n) \leq (n+1)^{|V|} \max_i \chi_f(G^n[\mathcal{T}_{P_i}^n]).$$

$\square$

Lemma 2 *For every* P *which is a possible type of sequences of length* n, *we have*

$$\chi_f(G^n[\mathcal{T}_P^n]) = \frac{|\mathcal{T}_P^n|}{\alpha(G^n[\mathcal{T}_P^n])}.$$

Proof. Since sequences of the same type are permutations of each other and the order of the elements of a sequence does not matter in the definition of the co-normal power, the graph $G^n[\mathcal{T}_P^n]$ is vertex-transitive. It is well known that the fractional chromatic number of a vertex-transitive graph is just the ratio of its number of vertices and its independence number (see, e. g., Proposition 3.1.1 in [22].) $\square$

Proof of Theorem 2. We know $h(G) \leq \chi_f(G)$ thus it is enough to prove the reverse inequality.

Taking n^{th} root and limit in the inequality stated by Lemma 1 we get

$$\chi_f(G) \leq \lim_{n\to\infty} \sqrt[n]{\max_P \chi_f(G^n[\mathcal{T}_P^n])}.$$

But by Lemma 2 the right hand side here is just

$$\lim_{n\to\infty} \sqrt[n]{\max_P \frac{|\mathcal{T}_P^n|}{\alpha(G^n[\mathcal{T}_P^n])}} \leq h(G),$$

where the last inequality follows from considering the induced subgraphs $G^n[\mathcal{T}_P^n]$ as possible candidates to achieve the Hall-ratio. The last two inequalities give $\chi_f(G) \le h(G)$ what we needed. $\square$

We remark that the above proof used the standard information theoretic technique of partitioning an exponential size set of sequences according to their types. This is a very powerful method used throughout the book [10], see also [9].

2.2 Normal powers

Notice that Lemmas 1 and 2 remain true even if we use the normal exponentiation in place of the co-normal one. Thus by a similar argument we obtain an analoguous statement for the variant of $h(G)$ corresponding to the normal power.

Definition 5

$$h_n(G) := \lim_{n\to\infty} \sqrt[n]{\rho(G^{(n)})}.$$

The existence of the limit is somewhat less obvious here than in case of $h(G)$ because $\alpha(G^{(n)})$ may get larger than $[\alpha(G)]^n$. However, following the steps of Lemmas 1 and 2 we obtain

$$\sqrt[n]{\chi_f(G^{(n)})} \le \sqrt[n]{(n+1)^{|V|} \max_P \chi_f(G^{(n)}[\mathcal{T}_P^n])} =$$

$$= \sqrt[n]{(n+1)^{|V|} \max_P \frac{|\mathcal{T}_P^n|}{\alpha(G^{(n)}[\mathcal{T}_P^n])}} \le \sqrt[n]{(n+1)^{|V|}\rho(G^{(n)})} \le \sqrt[n]{(n+1)^{|V|}\chi_f(G^{(n)})}.$$

Taking the limit everywhere here, we readily obtain the existence of the limit $h_n(G)$ by its equality to $\lim_{n\to\infty} \sqrt[n]{\chi_f(G^{(n)})}$, whose existence will follow from its further equality to $\lim_{n\to\infty} \sqrt[n]{\chi(G^{(n)})}$. The existence of the latter limit follows again from Fekete's theorem by observing $\chi(G^{(k+m)}) \le \chi(G^{(k)}) \cdot \chi(G^{(m)})$, that can be seen by coloring the vertices of $G^{(k+m)}$ by the appropriate pairs of colors used in optimal colorings of $G^{(k)}$ and $G^{(m)}$. The equality of $\lim_{n\to\infty} \sqrt[n]{\chi_f(G^{(n)})}$ and $\lim_{n\to\infty} \sqrt[n]{\chi(G^{(n)})}$ can be shown similarly to the proof of Theorem 1 as follows. Key to the proof is a lemma of Lovász from [18] stating that $\chi(G) \le (1+\log\alpha(G))\chi_f(G)$ holds for every graph G. This implies

$$\lim_{n\to\infty} \sqrt[n]{\chi(G^{(n)})} \le \lim_{n\to\infty} \sqrt[n]{(1+\log\alpha(G^{(n)}))\chi_f(G^{(n)})} \le$$

$$\le \lim_{n\to\infty} \sqrt[n]{(1+n\log|V(G)|)\chi_f(G^{(n)})} \le \lim_{n\to\infty} \sqrt[n]{\chi_f(G^{(n)})}.$$

Since the reverse inequality is obvious this proves the equality we stated.

The corresponding asymptotic value for the chromatic number is also a graph invariant considered for its own right, cf. [24], [2], [3].

Definition 6 *([24])*

$$R(G) := \lim_{n\to\infty} \sqrt[n]{\chi(G^{(n)})}$$

is the Witsenhausen rate of the graph G.

Just like Shannon capacity, $R(G)$ was defined for its information theoretic meaning, see [24], [3]. It is similarly difficult to determine as to determine $C(G)$ and it can lie strictly between the clique number and the chromatic number. It follows from the results in [17] and [24], for example, that $R(C_5) = \frac{5}{C(C_5)} = \sqrt{5}$ which is also smaller than the fractional chromatic number of C_5.

The result thus obtained for $h_n(G)$ by the previous discussion is the following.

Corollary 1

$$h_n(G) = R(G).$$

$\Box$

Remark: A somewhat surprising feature of the above equality is that the parameter $R(G)$ may be both smaller and larger than $\rho(G)$, thus the asymptotic value $h_n(G)$ is sometimes larger, sometimes smaller than the original value $\rho(G)$. For the latter relation the already mentioned C_5 is an example as $\rho(C_5) = \frac{5}{2} > \sqrt{5} = R(G)$. An example when $R(G) > \rho(G)$ is provided by $G = W_5$, the 5-wheel, which is the graph we obtain from a C_5 by connecting each of its vertices to a sixth vertex. It takes an easy checking that $\rho(W_5) = 3$. On the other hand, $R(W_5)$ can be shown to be at least $1+\sqrt{5} > 3$. This follows from $R(G) > C(G)$ which is a consequence of results of Marton in [19] and from the fact that $C(W_5) = 1+\sqrt{5}$. What we need from the latter is only $C(W_5) \geq 1+\sqrt{5}$ which follows from a general construction given by Shannon [23] proving $\Theta(F \cup G) \geq \Theta(F) + \Theta(G)$, where $F \cup G$ means the disjoint union of graphs F and G. (To see that equality holds for $F = C_5$ and $G = K_1$ one can use the properties of Lovász's theta-function defined in [17]. Shannon actually conjectured that the last inequality is always an equality. This long-standing conjecture was recently disproved by Alon [1].)

3 Cartesian and direct powers

The book [16] treats four associative graph products as basic, three of which are commutative, these are the normal (under the name "strong"), the Cartesian and the direct (often called also "categorical") products. In this section we investigate the behaviour of the Hall-ratio under the powers we obtain from the Cartesian and the direct product.

3.1 Cartesian powers

Definition 7 *The Cartesian product $G \square H$ of two graphs G and H is defined on the vertex set $V(G \square H) = V(G) \times V(H)$ with edge set $E(G \square H) = \{\{uv, xy\} : \{u, x\} \in E(G)$ and $v = y$ or $u = x$ and $\{v, y\} \in E(H)\}$. The n^{th} Cartesian power $G^{\square n}$ of G is the n-fold Cartesian product $G \square G \square \dots \square G$.*

Thus the Cartesian power is also given on the n-length sequences of the original vertices and two such sequences form an edge if and only if they differ at exactly one place and at that place the corresponding coordinates form an edge of the original graph.

Definition 8 *The ultimate Hall-ratio with respect to the Cartesian power is defined as*

$$h_{\square}(G) = \lim_{n \to \infty} \rho(G^{\square n}).$$

The existence of the limit easily follows from $\alpha(F \square G) \leq \alpha(F)|V(G)|$ that implies $\rho(G^{\square(i+1)}) \geq \rho(G^{\square i})$ and from the obvious boundedness of $\rho(G^{\square n})$ by $|V(G)|$, say.

Notice that here we do not have the multiplicative-like behaviour of the Hall-ratio we faced in the powers of the previous section and so we do not have to take roots here.

A related notion to $h_{\square}(G)$ is the ultimate independence ratio. Formally introduced in [15] as $I(G) := \lim_{n \to \infty} i(G^{\square n})$ where $i(F) = \frac{\alpha(F)}{|V(F)|}$, it was extensively studied by Hahn, Hell, and Poljak in [13]. A (surprisingly non-trivial) result of the latter paper (see as Lemma 2.2 in [13]) immediately implies that $I(G)$ and $h_{\square}(G)$ are essentially the same notions.

Lemma 3 *([13]) If F is a subgraph of G then $I(G) \leq I(F)$.*

Corollary 2 $h_{\square}(G) = \frac{1}{I(G)}$.

Proof. Since $h_{\square}(G) \geq \frac{1}{I(G)}$ is obvious, it is enough to prove the reverse inequality. For every i let F_i denote the subgraph of $G^{\square i}$ achieving $\rho(G^{\square i})$. Now we can write

$$\rho(G^{\square i}) = \frac{1}{i(F_i)} \leq \frac{1}{I(F_i)} \leq \frac{1}{I(G^{\square i})} = \frac{1}{I(G)}$$

where the first inequality follows from the easy fact that $i(F^{\square r})$ is a decreasing sequence of r (see as Corollary 2.2 in [15]), the second is a consequence of Lemma 3, while the last equality is easy to see. Letting i go to infinity the required inequality follows. $\square$

The results proven in [13], [15] and [25] for $I(G)$ thus can also be stated in terms of $h_{\square}(G)$. We quote some of these to illustrate the relationship between $h_{\square}(G)$ and $\chi_f(G)$.

Theorem 3 *([13],[15])*

$$\chi_f(G) \leq h_{\square}(G) \leq \chi(G).$$

Zhu in [25] showed that $h_\square(G)$ (i. e., $\frac{1}{I(G)}$ in his language) can be strictly between the two bounds above and improved the upper bound $\chi(G)$ to $\chi_c(G)$, the circular chromatic number of G. He also showed the following.

Theorem 4 *([25])* $h_\square(G) = \lim_{n\to\infty} \chi_f(G^{\square n})$.

That is, while $h_\square(G)$ can be strictly larger than the fractional chromatic number, it always coincides with its corresponding ultimate value.

3.2 Direct powers

Now we turn to direct powers.

Definition 9 *The direct product $G \times H$ of two graphs G and H is defined on the vertex set $V(G\times H) = V(G)\times V(H)$ with edge set $E(G\times H) = \{\{uv, xy\} : \{u,x\} \in E(G)$ and $\{v,y\} \in E(H)\}$. The n^{th} direct power $G^{\times n}$ of G is the n-fold direct product $G\times G\times\ldots\times G$.*

Definition 10 *The ultimate Hall-ratio with respect to the direct power is defined as*

$$h_\times(G) = \lim_{n\to\infty} \rho(G^{\times n}).$$

The existence of the limit follows again by monotonicity and boundedness. Monotonicity is a consequence of $G^{\times i} \subseteq G^{\times(i+1)}$ that can be seen by simply duplicating the last coordinate of each sequence forming a vertex of $G^{\times i}$. A notion analoguous to the ultimate independence ratio was introduced under the name ultimate categorical independence ratio by Brown, Nowakowski, and Rall in [7]. It is defined as $A(G) := \lim_{n\to\infty} i(G^{\times n})$. The relation to $h_\times(G)$, however, is quite different to that we have seen in case of the Cartesian product. The reason of the difference is, that while $i(G^{\square n})$ decreases with n, $i(G^{\times n})$ is increasing. Since G is an induced subgraph of $G^{\times n}$ for all n (take the sequences containing the same letter at each coordinate), we have $h_\times(G) \geq \rho(G) = \frac{1}{i(G)} \geq \frac{1}{A(G)}$, and thus $h_\times(G)A(G) > 1$ whenever any of the two ultimate values differ from the initial values of $\rho(G)$ and $i(G)$, respectively. It is proven in [7] that $i(G) > \frac{1}{2}$ implies $A(G) = 1$. Thus adding $|V(G)|$ isolated vertices to any graph G will lift $A(G)$ to 1, while it will not change the value of $h_\times(G)$. This shows that the parameters $A(G)$ and $h_\times(G)$ are highly independent of each other.

It is quite obvious that $\chi_f(G^{\times n}) = \chi_f(G)$. (A fractional coloring of G naturally extends to a fractional coloring of $G^{\times n}$ by considering only the first coordinates. This implies $\chi_f(G^{\times n}) \leq \chi_f(G)$ while the reverse inequality follows from $G \subseteq G^{\times n}$.) This implies $h_\times(G) \leq \chi_f(G)$. In all the cases we know the corresponding values, there is equality in the previous inequality. It seems plausible to beleive this is always the case but we do not have a proof of this.

Conjecture 1

$$h_\times(G) = \chi_f(G)$$

Let us remark that once we know a finite k for which we have $\rho(G^{\times k}) \geq \chi_f(G)$ that implies $h_\times(G) = \chi_f(G)$. This is because the monotonicity of $\rho(G^{\times i})$ in i then implies $h_\times(G) \geq \chi_f(G)$ and since $\chi_f(G)$ is also an upper bound for $h_\times(G)$, equality follows.

The existence of some finite k with $\rho(G^{\times k}) = \chi_f(G)$ is trivial if $\chi_f(G) = \omega(G)$ or if G is vertex-transitive. (In both cases $k = 1$ will do.) A less trivial but still easy case is when G is an odd wheel. Let the wheel consisting of a cycle of length m and an additional point joint to every vertex of the cycle be denoted by W_m.

Proposition 1 *$h_\times(W_m) = \chi_f(W_m)$ for every $m \geq 3$.*

Proof. The cases when m is even or $m = 3$ are trivial, since then W_m is perfect, thus we are done by the general inequalities $\omega(G) \leq h_\times(G) \leq \chi_f(G) \leq \chi(G)$.

Assume now that $m = 2s+1$ and $s \geq 2$. Then $\chi_f(W_m) = \frac{3s+1}{s}$. Thus it is enough to show a subgraph F of $W_m^{\times k}$ for some k with $\rho(F) \geq [i(F)]^{-1} = \frac{3s+1}{s}$. Let the points of the m-cycle of W_m be $1, 2, \ldots, (2s+1)$ and the additional point be 0. Consider the 2-length sequences in the union of the following sets: $Z := \{00\}$, $A := \{01, 03, 05, \ldots, 0(2s-3)\}$, $B := \{10, 30, 50, \ldots, (2s-3)0\}$, $D := \{22, 44, 66, \ldots, (2s-4)(2s-4)\}$, $L := \{(2s-2)(2s-2), (2s-1)(2s-1), (2s)(2s), (2s+1)(2s+1)\}$. Let F be the subgraph of $W_m^{\times 2}$ induced by the above sequences. We show that $[i(F)]^{-1} = \frac{3s+1}{s}$. Since $|V(F)| = 3s+1$ we have to show $\alpha(F) = s$. Consider a maximal independent set S. Since all vertices in A are adjacent to all vertices in B, at least one of $S \cap A$ and $S \cap B$ is empty. Without loss of generality, we may assume $S \cap B = \emptyset$. If $00 \in S$ then $S \cap (D \cup L) = \emptyset$ and thus $S \subseteq A \cup Z$. But then $|S| \leq |A| + |Z| = (s-1) + 1 = s$ and we are done. Thus we can assume $00 \notin S$. In this case $S \subseteq A \cup D \cup L$. Observe that the subgraph induced by $A \cup D \cup L$ is isomorphic to C_{2s+1}, the cycle of length $2s+1$. But then $|S| \leq \alpha(C_{2s+1}) = s$, so we are done, again. $\square$

Proposition 1 suggests that the following approach might lead to a proof of Conjecture 1. Let G be an arbitrary graph and $f(v)$ be a non-negative function on $V(G)$ defining an optimal fractional clique, i. e., maximizing $\Sigma_{v \in V(G)} f(v)$ under the constraint $\Sigma_{v \in S} f(v) \leq 1$ holding for every independent set S. By the duality theorem of linear programming $\Sigma_{v \in V(G)} f(v)$ is then equal to $\chi_f(G)$. Since $\chi_f(G)$ is rational and all $f(v)$'s can be chosen rational (cf. [22]), there is some integer M such that $Mf(v)$ is integral for every v. Now take $f(v)$ copies of v for each $v \in V(G)$ as "one length sequences". Their total number is $M\chi_f(G)$ and the largest independent set they "induce" has size M, the only problem being that many of our "sequences" are identical. That is, if we could extend these "one length sequences" to longer ones so that all of them become different while the size of the largest independent set they induce in the corresponding power would not increase then we were done. Indeed, our sequences would then define an induced subgraph F of $G^{\times n}$ for some n with $\frac{|V(F)|}{\alpha(F)} = \chi_f(G)$. This would imply $\rho(G^{\times n}) \geq \chi_f(G)$ which by our earlier discussion would prove Conjecture 1. In the proof of Proposition 1 this approach worked with $n = 2$. It might also be the case that choosing $n = M$ and considering all sequences of length M and type P, where $P(v)$ is defined by $P(v) = \frac{f(v)}{\chi_f(G)}$, would give an induced subgraph of $G^{\times M}$ with $[i(F)]^{-1} = \chi_f(G)$. If so, this would prove Conjecture 1.

4 On the lexicographic power

Unlike the other products leading to the graph powers treated in this paper the lexicographic product (often called also substitution) is not commutative.

Definition 11 *The lexicographic product* $G \circ H$ *of two graphs* G *and* H *is defined on the vertex set* $V(G \circ H) = V(G) \times V(H)$ *with edge set* $E(G \circ H) = \{\{uv, xy\}: \{u,x\} \in E(G)$ *or* $u = x$ *and* $\{v, y\} \in E(H)\}$*. The* n^{th} *lexicographic power* $G^{\circ n}$ *of* G *is the* n*-fold lexicographic product* $G \circ G \circ \ldots \circ G$*.*

That is, two sequences of the original vertices are adjacent in the lexicographic power iff they are adjacent in the first coordinate where they differ. It is straightforward from the definitions that $G^{(n)} \subseteq G^{\circ n} \subseteq G^n$.

Definition 12

$$h_\circ(G) := \lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})}.$$

The existence of the limit follows similarly as in the case of the co-normal power using that here we have the analoguous equalities $|V(F_k \circ F_m)| = |V(F_k)| \cdot |V(F_m)|$ and $\alpha(F_k \circ F_m) = \alpha(F_k) \cdot \alpha(F_m)$ that can be applied to the subgraphs F_i of $G^{\circ i}$ achieving $\rho(G^{\circ i})$ $(i = k, m)$. (The first of these is trivial, the latter is also easy to see, cf. Proposition 8.9 in [16], attributed to Geller and Stahl [12].)

The above equalities readily give $\rho(G) \leq h_\circ(G)$, while $R(G) \leq h_\circ(G)$ follows from $G^{(n)} \subseteq G^{\circ n}$ and Corollary 1. We know from the Remark after Corollary 1 that neither of these lower bounds is universally better than the other, thus we write

$$h_\circ(G) \geq \max\{\rho(G), R(G)\}.$$

It is known that the fractional chromatic number behaves multiplicatively with respect to the lexicographic product (this is Theorem 8.40 in [16]), so we have

$$h_\circ(G) \leq \lim_{n\to\infty} \chi_f(G^{\circ n}) = \chi_f(G).$$

It is also a consequence of the above and $G^{\circ n} \subseteq G^n$ that the analogue of Theorem 1 is also true for the lexicographic product, see [14]. Since, unlike the sequence $\sqrt[i]{\chi(G^i)}$, the sequence $\sqrt[i]{\rho(G^i)}, (i = 1, 2, \ldots)$ converges to $\chi_f(G)$ from below, the above relation of the power graphs does not imply an analogue of Theorem 2 for lexicographic powers.

In Section 2 it was central in our arguments that the subgraphs of the power graphs induced by all fixed type sequences, i. e., $G^n[\mathcal{T}_P^n]$ and $G^{(n)}[\mathcal{T}_P^n]$, were vertex-transitive. The analoguous statement is not true here in general because of the non-commutitative nature of the lexicographic product.

Nevertheless, we beleive that the analoguous statement to that of Theorem 2 is true here, but we do not have a proof of this. We finish our discussion by stating it as a conjecture.

Conjecture 2

$$h_\circ(G) = \chi_f(G).$$

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