Transitive colorings of tournaments

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This is an updated version of my 2006 Memphis seminar notes.

An edge coloring of a tournament is *transitive* if the edges in each color class define a transitively oriented digraph. How restricted are these tournaments? It is easy to observe that only transitive tournaments have transitive 2-colorings. What can one say about tournaments having transitive 3-colorings? Two of the three color classes can be quite complicated (in a certain sense): split the vertices into two parts, A, B. Color class 1 can be an arbitrary bipartite graph oriented from A to B. The bipartite complement, oriented from B to A, can be color class 2. Then color class 3 is "tame", the union of two disjoint transitive subtournaments on A, B. (One can easily argue that bipartite graphs are "tame too".)

Is it true that transitively 3-colored tournaments have dominating sets of bounded size, say bounded by 2011? Or, for the optimists, bounded by 3?

A vertex set S of a tournament is dominating if for every vertex $w \notin S$ there exists some $v \in S$ such that vw is an edge of the tournament. A well-known fact (an illustration of the probability method and also of Paley tournaments) is that for any positive integer k there are tournaments without dominating sets of size k, sometimes these tournaments are called k-paradoxical. The question is whether transitively 3-colored tournaments can be 2001-paradoxical? I give below some motivations (and related questions), the general form is Problem 5 below.

I. Domination with monochromatic paths. The following questions were asked by Sands, Sauer and Woodrow, [6]:

Problem 1. For each positive integer k is there a (least) integer $f(k)$, such that every tournament whose edges are colored with k colors, contains a set S of $f(k)$ vertices with the property: every vertex not in S can be reached by a monochromatic directed path with starting point in S.

Problem 2. (a.)Does $f(3)$ exist? (b.)In particular, $f(3)=3$?

Problem 3. Let T be a tournament whose edges are colored with three colors so that every cyclic triangle is colored with at most two colors. Does there exist a vertex in T from which every other vertex can be reached by a monochromatic directed path?

Remarks. It was proved in [6] that $f(2)=1$ (in fact in a more general form), $f(3)\geq 3$, and mentioned that Problems 1,2 were also asked by Paul Erdős. Problem 3 has an affirmative answer if the condition (no cyclic 3-colored triangle) on the coloring is restricted further, requiring:
1. There is no transitive 3-colored triangle [7].
2. The tournament becomes transitive upon deleting a vertex or reversing a "sink-source" path [8].
3. At any vertex v there are at most two colors on the incoming edges and if there are exactly two then only the third color is used on the outgoing edges [10].
4. At any vertex v there are at most two colors on the edges incident to v [4], [5].

II. Covering by boxes. I proposed two problems in 1985, the first was answered by the following theorem of Bárány and Lehel [2].

Theorem. For each d there exists a (least) $g(d)$ such that every finite $T\subset R^d$ contains a subset $S\subset T$ of at most $g(d)$ points with the property: every point in $T\setminus S$ is "between" two points of S. Here "between" means: between in each coordinate.

Remarks. A linear set has a smallest and a largest element so $g(1)=2$ is obvious. A planar set has leftmost, rightmost, highest, lowest vertices, thus $g(2)\leq 4$ is easy. Similarly, a set in R^3 has at most 6 extreme points by selecting the two extremes in three coordinates, leading to $g(3)\leq 6$? No, $g(3)>16$....In fact, no reasonable upper bound is is known for $g(3)$ (it seems that presently $g(3)\leq 3^{14}$ from [2] is the best). The upper bound of [2],

$$(2d^{2^d}+1)^{d2^d}$$

was improved to $2^{2^{d+2}}$ by Pach [9] applying the main result of Ding, Seymour and Winkler [3]. The best known upper bound is

$$g(d)\leq 2^{2^d+d+\log(d)+\log\log(d)+O(1)}$$

by Alon, Brightwell, Kierstead, Kostochka, Winkler in [1]. (In fact, the background of the improvements is the Vapnik-Chernovenkis, Haussler-Welzl bound on τ in terms of τ^* and the VC-dimension.)

The lower bound $g(d)>2^{2^{d-1}}$ comes from many sources, in fact related to Ramsey problems as well, see [2]: it is possible to define $2^{2^{d-1}}$ points in R^d so that none of them is beetween two others. In fact, this is sharp: in any set of $2^{2^{d-1}}+1$ points of R^d there is a point between two others. This nice proposition comes easily from iterating the Erdős - Szekeres monotone sequence lemma. (A proof is in [9] while [1] refers to this proposition as an unpublished result of N. G. de Bruijn).

My second problem (a possible extension of Bárány-Lehel theorem) is still unsolved:

Problem 4. For each positive integer k is there a (least) $h(k)$ such that every tournament whose edges are transitively colored with k colors contains a set S of $h(k)$ vertices with the property: for every vertex $t \notin S$ there are $x, y \in S$ such that x, t, y is a monochromatic directed path. It is easy to see that $h(2) = 2$ but the existence of $h(3)$ is not known.

III. k-majority tournaments. A tournament is called a k-majority tournament if it can be defined by $2k - 1$ linear orders on a set T, orienting each pair of T as the majority of the linear orders do. It was proved in [1] that there is a (least) $m(k)$ such that k-majority tournaments can be dominated by at most $m(k)$ vertices, $m(2) = 3$, $c_1 k / \log(k) \leq m(k) \leq c_2 k \log(k)$.

The problem I propose is a common subproblem of Problems 1 and 4.

Problem 5. For each positive integer k is there a (least) $p(k)$ such that every tournament whose edges are transitively colored with k colors contains a dominating set S of at most $p(k)$ vertices. It is easy to see that $p(2) = 1$ but the existence of $p(3)$ is not known. In fact, the example showing $f(3) \geq 3$ also shows $p(3) \geq 3$: let T be a cyclic triangle whose edges are colored with three colors and replace each vertex of T by T. Stéphan Thomassé [11] showed me another possibility: the Paley tournament on seven vertices can be partitioned into three copies of the transitive digraph with edges $12, 32, 34, 45, 35, 65, 67$.

Final remark. The existence of $p(k)$ implies the existence of $m(k)$. Indeed, if T is defined by the majority rule with linear orders $L_1, \dots \dots, L_{2k-1}$ then one can color each edge $xy \in T$ by the set of indices i for which $x <_{L_i} y$. It is easily seen that this coloring is transitive (using at most $\sum_{i=k}^{2k-1} \binom{2k-1}{i}$ colors).

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