L-FUNCTIONS, MODULARITY, AND FUNCTORIALITY

J.W. COGDELL

1. Arithmetic L-functions

• The prototypical L-function of arithmetic is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}.$$

It begins life defined for $Re(s) > 1$ but then has an analytic continuation with a simple pole at $s = 1$ and satisfies a functional equation

$$\xi(s) = \pi^{-s/2} \gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(1-s).$$

The arithmetic content of this function is well known:
– non-vanishing along $Re(s) = 1$ implies the *Prime Number Theorem*
– the location of the non-trivial zeros on $Re(s) = \frac{1}{2}$ is the *Riemann Hypothesis*, which has the honor of being both a Hilbert problem and a Clay problem.

• If $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ is a Dirichlet character then we have the Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \chi(p) p^{-s}\right)^{-1}.$$

These also converge for $Re(s) > 1$, now have an entire continuation to the complex plane, and satisfy a similar functional equation. As for arithmetic:
– the non-vanishing of $L(1,\chi)$ implies Dirichlet's theorem on primes in arithmetic progressions: for $(a,m) = 1$ there are an infinite number of primes p of the form $p = a + mk$.

• In 1923 Artin, in thinking about class field theory, attached to every finite dimensional representation of a Galois group, say

$$\rho : Gal(k/\mathbb{Q}) \to GL_n(\mathbb{C})$$

an L-function defined by an Euler product of degree n only:

$$L(s,\rho) = \prod_p \det\left(I - \rho(\sigma_p) p^{-s}\right)^{-1}$$

where σ_p is the Frobenius substitution attached to a prime p by

$$\sigma_p(\alpha) \equiv \alpha^p \pmod{\mathfrak{P}} \qquad \text{for all} \quad \alpha \in \mathfrak{o}_K.$$

He proved
– convergence for $Re(s) > 1$

– some power had a meromorphic continuation (power can be one by Brauer)
– it satisfied a functional equation

$$\Lambda(s,\rho) = L_\infty(s,\rho)L(s,\rho) = \varepsilon(s,\rho)\Lambda(1-s,\rho)$$

and he *conjectured*
– $L(s,\rho)$ if ρ is irreducible and non-trivial (the *Artin Conjecture*).

As for the arithmetic, through these L-functions he was led to formulate and later prove:
– Artin's General Reciprocity Law
which is a broad generalization of the law of quadratic reciprocity of classical number theory which implies all other reciprocity laws (Hilbert's 12^{th} problem).

- Today, if M is a "motive" then to M is attached an L-function

$$L(s,M) = \prod_p L_p(s,M)$$

which converges for $Re(s) >> 0$ and is *conjectured* to be "nice":
– meromorphic or entire continuation
– bounded in vertical strips
– functional equation $\Lambda(s,M) = L_\infty(s,M)L(s,M) = \varepsilon(s,M)\Lambda(1-s,M^\vee)$.

For example, M could correspond to an elliptic curve, then $L(s,M) = L(s,E)$, the L-function that figured into Wile's proof of Fermat and whose behavior at $s=1$ is the content of the Birch and Swinnerton-Dyer Conjecture.

2. Automorphic L-functions

Classical Theory: $f : \mathfrak{H} \to \mathbb{C}$ is a modular form of weight k for $\Gamma \subset SL_2(\mathbb{Z})$ if

(1) f is holomorphic;
(2) for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z);$$

(3) f is holomorphic at the cusps of Γ.

Examples:

(1) $\theta_q(z)$ the theta series attached to a quadratic form $q(x)$;
(2) $\Delta(z)$ the discriminant function from the theory of elliptic modular functions.

We will restrict to $\Gamma = SL_2(\mathbb{Z})$ for simplicity of exposition.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ then $f(z+1) = f(z)$ and we have the Fourier expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

Then $f(z)$ is cuspidal, or a cusp form, if $a_0 = \int_0^1 f(z+x)\, dx = 0$, i.e.,

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

These Fourier coefficients often carry interesting arithmetic information:

(1) If $f(z) = \theta_q(z)$, then $a_n = r(n,q)$ counts the number of times n is represented by the quadratic form q.
(2) If $f(z) = \Delta(z)$, then $a_n = \tau(n)$ is Ramanujan's τ–function.

Hecke attached to each cusp form a complex analytic invariant – its L-function given by the Dirichlet series:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

This L-function was connected to f by an integral representation, namely the Mellin transform:

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^{\infty} f(iy) y^s d^{\times} y$$

Through this integral representation, Hecke was able to prove

Theorem *$L(s,f)$ is* NICE*: entire, BVS, and satisfies a functional equation $\Lambda(s,f) = i^k\Lambda(k-s,f)$.*

The functional equation comes from the modular transformation law under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ sending $z \mapsto -1/z$.

Since the Mellin transform has an inverse integral transform, Hecke was able to prove the CONVERSE to this THEOREM.

Theorem *If $D(s) = \sum a_n/n^s$ is* NICE *with the correct functional equation as above then $f(z) = \sum a_n e^{2\pi i n z}$ is a cusp form of weight k for $SL_2(\mathbb{Z})$ and $D(s) = L(s,f)$.*

The modularity of $f(z)$ essentially comes from the Fourier expansion and the functional equation.

Note that in 1967 Weil proved a corresponding Converse Theorem for $\Gamma_0(N)$ by using the functional equation not just for $L(s,f)$ but also for

$$L(s,f,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a_n}{n^s}$$

with Dirichlet characters χ of conductor prime to the level N.

In 1936 Hecke also introduced an algebra of operators $\mathcal{H} = \{T_n\}$, indexed by the integers, acting on modular forms and proved that if $f(\tau)$ was a simultaneous eigen-function for the operators in $\mathcal{H}$ then $L(s,f)$ possessed an Euler product of degree 2:

$$L(s,f) = \prod_p (1 - a_p p^{-s} + p^{-2s})^{-1}.$$

Note the following dichotomy:

Arithmetic L-functions $L(s,M)$:
– defined by Euler products
– analytic properties are conjectural
– arithmetic is clear.

Automorphic L-functions $L(s,f)$:
– defined by a Dirichlet series
– analytic properties are clear and in fact characterize them
– Euler product and arithmetic are more mysterious.

Surprisingly, even though Artin and Hecke were colleagues at Hamburg for 15 years, there is no indication that they thought these to be linked. This was left to Langlands.

2.1. **Modern theory:** The modern theory of automorphic L-functions lets us address parts of this dichotomy. In the modern theory, Hecke's modular form $f(\tau)$ is replaces by an automorphic representation of $GL_2(\mathbb{A})$ or even $GL_n(\mathbb{A})$.

This is an adelic theory (beginning with Artin's student Tate). Let

$$\mathbb{A} = \mathbb{R} \times {\prod}' \mathbb{Q}_p$$

this is a locally compact topological ring having $\mathbb{Q}$ embedded diagonally as a canonical discrete group with $\mathbb{A}/\mathbb{Q}$ compact. Then

$$GL_n(\mathbb{A}) = GL_n(\mathbb{R}) \times {\prod}' GL_n(\mathbb{Q}_p)$$

with $GL_n(\mathbb{Q})$ embedded diagonally as a canonical discrete subgroup with finite co-volume (modulo the center). Then this group $GL_n(\mathbb{A})$ acts by right translation on the space of *cuspidal automorphic forms*:

$$\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$$

which then decomposes as

$$\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})) = \oplus\, m(\pi)V_\pi$$

with the (infinite dimensional) constituents being the *cuspidal automorphic representations*. As $GL_n(\mathbb{A})$ decomposes as a (restricted) product, so do the representations:

$$\pi \simeq \pi_\infty \otimes (\otimes' \pi_p) = \otimes' \pi_v$$

with (π_v, V_{π_v}) an irreducible admissible representation of $GL_n(\mathbb{Q}_v)$.

The theory of L-functions now takes the shape

$$\begin{aligned}
\pi_\infty &\longrightarrow L(s,\pi_\infty) \longleftrightarrow \Gamma(s)\\
\pi_p &\longrightarrow L(s,\pi_p) = Q_p(p^{-s})^{-1} \text{ with } Q_p(X) \in \mathbb{C}[X] \text{ of degree } \leq n\\
\pi &\longrightarrow \Lambda(s,\pi) = L(s,\pi_\infty)L(s,\pi) = L(s,\pi_\infty)\prod_p L(s,\pi_p) \quad Re(s) >> 0.
\end{aligned}$$

So each local factor π_v gives us an Euler factor much as in Tate's thesis and together these give us an Euler product definition of Hecke's L-functions, as in the arithmetic case. However we still have a global analogue of Hecke's integral representation that lets us prove:

Theorem [J,P-S,S] *$L(s,\pi)$ is* NICE*: entire, BVS and satisfies a functional equation*

$$\Lambda(s,\pi) = \varepsilon(s,\pi)\Lambda(1-s,\widetilde{\pi}).$$

And moreover we are still able to invert!

Theorem[C,P-S] *Let $\pi = \otimes'\pi_v$ be an irreducible admissible representation of $GL_n(\mathbb{A})$. (Think of this as a collection of local data.) Suppose that the formal L-function*

$$L(s,\pi) := \prod_v L(s,\pi_v)$$

converges for some $Re(s) >> 0$ and has a automorphic central character. Suppose that for every $\pi' \in \mathcal{T}_0$, an appropriate cuspidal automorphic twisting set, we have that all $L(s, \pi \times \pi')$ are NICE*. Then π is in fact cuspidal automorphic.*

Moral: *All* NICE *degree n L-functions are modular, i.e., associated to a cuspidal automorphic representation π of $GL_n(\mathbb{A})$.*

3. Modularity

Now return to Artin's L-functions $L(s,\rho)$ attached to degree n representations of $\mathcal{G}_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ from before. These are conjectured to be nice, so that one view the moral result as giving credence to:

Langlands' Conjecture: *There should be an injection of n-dimensional Galois representations into automorphic representations of $GL_n(\mathbb{A})$*

$$\{\rho : \mathcal{G}_{\mathbb{Q}} \to GL_n(\mathbb{C})\} \hookrightarrow \{\textit{automorphic representations } \pi \textit{ of } GL_n(\mathbb{A})\}$$

such that $L(s,\rho) = L(s,\pi)$ – Modularity of Galois Representations.

Since we now have local versions of both sides, we can also formulate:

Local Langlands' Conjecture: *There should be a an injection of n-dimensional local Galois representations into admissible representations of $GL_n(\mathbb{Q}_v)$*

$$\{\rho_v : \mathcal{G}_{\mathbb{Q}_v} \to GL_n(\mathbb{C})\} \leftrightarrow \{\textit{admissible representations } \pi \textit{ of } GL_n(\mathbb{Q}_v)\}$$

such that $L(s,\rho) = L(s,\pi)$ – Modularity of Local Galois Representations.

Historical note: Langlands made these conjectures in 1967 based on Euler products that arose in the theory of Eisenstein series. He did not restrict himself to $\mathbb{Q}$ nor to GL_n. The converse theorem as we gave it wasn't established until the 1990's although it was believed to be true before.

Locally, this injection can be extended to a bijection if we replace $\mathcal{G}_{\mathbb{Q}_v}$ by certain extensions, due to Weil and Deligne respectively:

$$\mathcal{G}_{\mathbb{Q}_v} \longrightarrow W_{\mathbb{Q}_v} \longrightarrow W'_{\mathbb{Q}_v}.$$

Local Langlands' Conjecture: *There should be a natural bijection*

$$\{\rho_v : W'_{\mathbb{Q}_v} \to GL_n(\mathbb{C})\} \leftrightarrow \{\textit{admissible representations } \pi \textit{ of } GL_n(\mathbb{Q}_v)\}$$

such that $L(s,\rho_v) = L(s,\pi_v)$ along with a list of other compatibility conditions.

This is now a **Theorem** due to Harris-Taylor and then simplified by Henniart! So *local Galois representations are modular.*

Note: While there is a global version of the Weil group $W_{\mathbb{Q}}$ there is no global Weil-Deligne group. Instead all one can hope for at the moment is a type of Hasse principle: local/global compatibility.

$$\rho \to \{\rho_v = \rho|_{D_v}\} \to \{\pi_v\} \to \pi = \otimes \pi_v$$

and ask that π be automorphic or modular.

Can we say anything about Gloabl Modularity? What we have to offer is a avatar of this modularity: Global Functoriality.

4. Functoriality

Langlands of course, being Langlands, made versions of these conjectures for other groups, like symplectic groups $H = Sp_{2n}$ corresponding to the theory of classical Siegel modular forms, or orthogonal groups $H = SO_n$ which arise in the study of quadratic forms. To each he associated a complex analytic dual group:

H	${}^LH$
GL_n	$GL_n(\mathbb{C})$
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$
SO_{2n}	$SO_{2n}(\mathbb{C})$

Local Langlands Conjecture for H: *There should be a correspondence*

$$\{\phi_v : W'_{\mathbb{Q}_v} \to {}^L H\} \leftrightarrow \{\textit{admissible representations } \pi \textit{ of } H(\mathbb{Q}_v)\}$$

such that $L(s, \phi_v) = L(s, \pi_v)$ *along with a list of other compatibility conditions.*

One should think of the left hand side here as a special type of Galois representations. This has been proved for certain families of π_v, such as those for $v = \infty$ by Langlands and for π_v unramified by Satake.

To go from special Galois representations into ${}^LH$ and automorphic representations of $H(\mathbb{A})$ we still rely on a local/global compatibility.

Langlands Principle of Functoriality is a is both a local and global avatar of this conjecture viewed as giving a arithmetic parameterization of analytic data. For a special case, note that we have natural embeddings of say $H = SP_{2n}(\mathbb{C})$ or $SO_{2n}(\mathbb{C})$ into $GL_{2n}(\mathbb{C})$ and $H = SO_{2n+1}(\mathbb{C})$ into $GL_{2n+1}(\mathbb{C})$. This then gives an opportunity to use the arithmetic data to transfer analytic data:

Local Functoriality: *If* π_v *is an irreducible admissible representation of* $H(\mathbb{Q}_v)$ *then we can obtain an irreducible admissible representation* Π_v *of* $GL_N(\mathbb{Q}_v)$ *by following the diagram*

$$\begin{tikzcd}
 & {}^LH \arrow[rr, "u"] & & {}^LGL_N & \\
\pi_v \arrow[r, mapsto] & \phi_v & & \Phi_v \arrow[r, mapsto] & \Pi_v. \\
 & & W'_{\mathbb{Q}_v} \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

and this should satisfy

$$L(s, \pi_v) = L(s, \phi_v) = L(s, \Phi_v) = L(s, \Pi_v)$$

along with a list of other compatibility conditions

and then globally:

Global Functoriality Conjecture: *If $\pi = \otimes'\pi_v$ is a cuspidal automorphic representation of $H(\mathbb{A})$ then the representation $\Pi = \otimes'\Pi_v$ of $GL_N(\mathbb{A})$ we obtain by following the diagram*

$$\begin{tikzcd}
 & & {}^L H \arrow[rr, "u"] & & {}^L GL_N & & \\
\pi = \otimes'\pi_v & \pi_v \arrow[r, mapsto] & \phi_v & & \Phi_v \arrow[r, mapsto] & \Pi_v & \Pi = \otimes'\Pi_v. \\
 & & & W'_{\mathbb{Q}_v} \arrow[uul] \arrow[uur] & & &
\end{tikzcd}$$

should be automorphic and moreover should satisfy

$$L(s,\pi) = \prod_v L(s,\pi_v) = \prod_v L(s,\Pi_v) = L(s,\Pi)$$

along with similar equalities for twisted versions and for ε-factors.

If we had Global Modularity, we could write down a global diagram much like the local one. Thus establishment of instances of Global Functoriality can be considered as evidence of

• Global Modularity: The existence of a global version of the Weil-Deligne group, often called the Langlands group, to mediate a global diagram.

• Specific *global modularity* of certain orthogonal or symplectic Galois representations.

• The strong Artin conjecture.

We (Kim, Cogdell, P-S, Shahidi, followed by others) have been able to establish many of instances of *Global Functoriality* recently by using the Converse Theorem, i.e., our *Moral Principle* in a practical way. The outline of the program is as follows:

1. One begins with a (globally generic) cuspidal representation $\pi = \otimes\pi_v$ of a (quasi-split) group $H(\mathbb{A})$

2. Candidate lift: One applies the local functoriality diagram to each component π_v to construct a local representations Π_v of $GL_N(\mathbb{Q}_v)$. (One must be able to finesse a finite number of places v.) Put these together to form $\Pi = \otimes\Pi_v$ on $GL_N(\mathbb{A})$. This may or may not be automorphic.

3. Control of L-functions: Since π is automorphic, you have analytic control of the invariants $L(s,\pi)$ (by the Langlands-Shahidi method) which you transfer over to $L(s,\Pi)$.

4. Apply the Converse Theorem. Then you conclude that Π (or at least something very close to it) is automorphic!

This is Global Functoriality!

This is more than just a moral victory. Many interesting applications in the realm of what is called analytic number theory (bounds towards Ramanujan, etc) have followed, but that would be the topic of another talk.