Local Langlands correspondence for GL_n
and the exterior and symmetric square ε–factors

J. W. Cogdell

(Notes for a talk given at the Rényi Institute, February 2012,
based on joint work with F. Shahidi and T-L. Tsai)

1. Artin ε-factors

In 1923 Artin began to attach L-functions to n-dimensional Galois representations

$$\rho : \mathrm{Gal}(K/k) \to GL_n(\mathbb{C}).$$

For places v of k that are unramified for K and ρ he set

$$L(s, \rho_v) = \det(I - \rho(\mathrm{Frob}_v) q_v^{-s})^{-1}$$

depending only on $\rho_v = \rho|_{D_v}$, the restriction of ρ to the decomposition group at v. This gave him what we would call the partial L-function

$$L^S(s, \rho) = \prod_{v \notin S} L(s, \rho_v).$$

He went on to show that this partial L-function converged, had a meromorphic continuation and functional equation, and was additive and inductive. He used these to formulate (but not prove) the Artin reciprocity law.

In 1930 he extended this work to define

- $L(s, \rho_v)$ for ramified places v
- $\Gamma_\infty(s, \rho)$ the archimedean factors

and so a complete L-function

$$\Lambda(s, \rho) = \Gamma_\infty(s, \rho) L(s, \rho)$$

and showed it had the same properties as the previous partial L-function.

In 1931 he defined the *Artin conductor* $\mathfrak{f}(\rho) = \prod_{v<\infty} \mathfrak{f}(\rho_v)$ as an ideal of $\mathfrak{o}_k$, where $\mathfrak{f}(\rho_v)$ depends on ρ_v restricted to all the higher ramification groups at v. He then showed a functional equation for the Artin L-functions of the shape

$$\Lambda(s, \rho) = \varepsilon(s, \rho) \Lambda(1 - s, \rho^\vee)$$

where now

$$\varepsilon(s, \rho) \sim W(\rho)[|d_k|^n N(\mathfrak{f})]^{-(s-1/2)}$$

with $W(\rho)$ a *global* root of unity, the *Artin root number*.

Only later, in the period 1956–1972, were Dwork, Langlands, and finally Deligne able to implicitly factor this root number and prove the *existence* of local factors $\varepsilon(s, \rho_v, \psi_v)$, with ψ_v an additive character of k_v, such that

$$\varepsilon(s, \rho) = \prod_v \varepsilon(s, \rho_v, \psi_v).$$

This talk is about the subtle local factors $\varepsilon(s,\Lambda^2(\rho_v),\psi_v)$ and $\varepsilon(s,\mathrm{Sym}^2(\rho_v),\psi_v)$.

2. The Langlands Correspondence

One way to try to understand the properties of the Artin L-functions, particularly if you are an analytic number theorist, is to embed

$$\left\{\begin{matrix}\rho : \mathrm{Gal}(\overline{k}/k) \to GL_n(\mathbb{C}) \\ \text{irreducible}\end{matrix}\right\} \quad\hookrightarrow\quad \left\{\begin{matrix}\text{cuspidal representations} \\ \pi \text{ of } GL_n(\mathbb{A}_k)\end{matrix}\right\}$$

in such a way that

$$\Lambda(s,\rho)=\Lambda(s,\pi(\rho)) \quad \text{and} \quad \varepsilon(s,\rho)=\varepsilon(s,\pi(\rho)).$$

Of course, we do not know how to prove this in general.

However, since we now know that everything can be factored into local components, we can now ask about

$$\left\{\begin{matrix}\rho_v : \mathrm{Gal}(\overline{k_v}/k_v) \to GL_n(\mathbb{C}) \\ \text{continuous}\end{matrix}\right\} \quad\hookrightarrow\quad \left\{\begin{matrix}\text{irrreducible admissible} \\ \text{representations } \pi_v \text{ of } GL_n(k_v)\end{matrix}\right\}$$

now in such a way that

$$\Lambda(s,\rho_v)=\Lambda(s,\pi(\rho_v)) \quad \text{and} \quad \varepsilon(s,\rho_v,\psi_v)=\varepsilon(s,\pi(\rho_v),\psi_v).$$

This is the *local Langlands conjecture.* We could replace $\mathrm{Gal}(\overline{k_v}/k_v)$ by any of its other avatars such as the Weil group W_{k_v} or the Weil-Deligne group W'_{k_v}.

This was established in 2000 by Harris and Taylor and then a "simplified" proof was given by Henniaart. Their result says:

Local Langlands Correspondence: *Let F be a non-archimedean local field (characteristic 0) . Then for each $n \geq 1$ there exist a unique bijection*

$$Rep_n(W'_F)=\left\{\begin{matrix}\rho : W'_F \to GL_n(\mathbb{C}) \\ \textit{Frobenius semi-simple}\end{matrix}\right\}_{/\sim} \quad \leftrightarrows \quad \mathcal{A}_n(F)=\left\{\begin{matrix}\textit{irreducible admissible} \\ \textit{representations } \pi_v \textit{of} GL_n(F)\end{matrix}\right\}_{/\sim}$$

denoted $\rho \mapsto \pi(\rho)$ such that

- (i) *for $n=1$ this is local class field theory (suitably normalized)*
- (ii) *for all $\rho \in Rep_n(W'_F)$ and $\rho' \in Rep_m(W'_F)$ we have*

$$L(s,\rho\otimes\rho')=L(s,\pi(\rho)\times\pi(\rho'))$$
$$\varepsilon(s,\rho\otimes\rho',\psi)=\varepsilon(s,\pi(\rho)\times\pi(\rho'),\psi)$$

- (iii) $\pi(\det(\rho))=\omega_{\pi(\rho)}$
- (iv) $\pi(\rho^\vee)=\tilde{\pi}(\rho)$
- (v) *For any character χ of $F^\times \simeq W_F^{ab}$, $\pi(\rho\otimes\chi)=\pi(\rho)\otimes\chi$.*

So the establishment of the LLC was built not just on the equality of the standard L- and ε-factors, but on the twisted L- and ε-factors of pairs.

The LLC should be *robust* when it comes to such factors. It should respect various parallel operations on the arithmetic and analytic sides. This gives us more tools for trying to understand these factors and their arithmetic. As examples we have the exterior and symmetric square operations. If $\rho \in Rep_n(W_F')$, then $\Lambda^2(\rho)$ and $\mathrm{Sym}^2(\rho)$ are again Galois representations of dimension $n(n \mp 1)/2$ and as such have associated L- and ε-factors:

$$L(s, \Lambda^2(\rho)),\ \varepsilon(s, \Lambda^2(\rho), \psi) \quad \text{and} \quad L(s, \mathrm{Sym}^2(\rho)),\ \varepsilon(s, \mathrm{Sym}^2(\rho), \psi).$$

On the analyticc side, Shahidi has given the corresponding operations for $\pi \in \mathcal{A}_n(F)$, namely

$$L(s, \pi, \Lambda^2),\ \varepsilon(s, \pi, \Lambda^2, \psi) \quad \text{and} \quad L(s, \pi, \mathrm{Sym}^2),\ \varepsilon(s, \pi, \mathrm{Sym}^2, \psi).$$

This then leads to the following question:

Question: *If $\rho \mapsto \pi(\rho)$ under the LLC and if we let $R^2 = \Lambda^2$ or Sym^2 do we have*

$$L(s, R^2(\rho)) = L(s, \pi(\rho), R^2) \quad \textit{and} \quad \varepsilon(s, R^2(\rho), \psi) = \varepsilon(s, \pi(\rho), R^2, \psi) \ ?$$

In 2010 Henniart showed that the equality of L-factors holds. Now, with Shahidi and Tsai, we can show

Theorem (C-S-T): *If $\rho \in Rep_n(W_F')$ and $R^2 = \Lambda^2$ or Sym^2, then*

$$\varepsilon(s, R^2(\rho), \psi) = \varepsilon(s, \pi(\rho), R^2, \psi).$$

3. Three reductions

1. Analytically it is easier to deal with the γ-factor instead of the ε-factor, since this occurs naturally in various local functional equations. For $R^2 = \Lambda^2$ or Sym^2,

$$\gamma(s, \pi, R^2, \psi) = \frac{\varepsilon(s, \pi, R^2, \psi) L(1-s, \tilde{\pi}, R^2)}{L(s, \pi, R^2)}.$$

We can then define the corresponding arithmetic factor by a similar formula

$$\gamma(s, R^2(\rho), \psi) = \frac{\varepsilon(s, R^2(\rho), \psi) L(1-s, R^2(\rho^\vee))}{L(s, R^2(\rho))}.$$

Since Henniart has shown that the L-factors agree under the LLC, this reduces us to the consideration of the γ-factors.

2. Since

$$\gamma(s, \pi \times \pi, \psi) = \gamma(s, \pi, \Lambda^2, \psi) \gamma(s, \pi, \mathrm{Sym}^2, \psi)$$
$$\gamma(s, \rho \otimes \rho, \psi) = \gamma(s, \Lambda^2(\rho), \psi) \gamma(s, \mathrm{Sym}^2(\rho), \psi)$$

then by the LLC we reduce to considering just $R^2 = \Lambda^2$.

3. Using structure theory on the two sides, we can reduce to just considering

$$\begin{aligned} \rho &\in Rep_n^0(W_F) = \{\text{irreducible } n\text{-dimensional representations of } W_F\} \\ \pi &\in \mathcal{A}_n^0(F) = \{\text{supercuspidal representations of } GL_n(F)\}. \end{aligned}$$

4. Three techniques

1. **Additivity / multiplicativity**. Additivity of the local factors was built into the existence of the local factors due to Deligne, following Artin. Multiplicativity, the corresponding operation on the analytic side, is due to Shahidi:

$$\begin{aligned} \gamma(s, \Lambda^2(\rho_1 \oplus \rho_2), \psi) &= \gamma(s, \Lambda^2(\rho_1), \psi)\gamma(s, \Lambda^2(\rho_2)\psi)\gamma(s, \rho_1 \otimes \rho_2, \psi) \\ \gamma(s, \mathrm{Ind}(\pi_1 \otimes \pi_2), \Lambda^2, \psi) &= \gamma(s, \pi_1, \Lambda^2, \psi)\gamma(s, \pi_2, \Lambda^2, \psi)\gamma(s, \pi_1 \times \pi_2, \psi). \end{aligned}$$

Note the parallel formalism in the two formulas.

2. **Stability**. Arithmetic stability was one of the conditions that Deligne built into his local factors.

Arithemtic stability (Deligne): *If* $\rho_1,\ \rho_2 \in Rep_n^0(W_F)$ *with* $\det(\rho_1) = \det(\rho_2)$*, then for all sufficiently highly ramified characters of* W_F *we have*

$$\gamma(s, \Lambda^2(\rho_1 \otimes \chi), \psi) = \gamma(s, \Lambda^2(\rho_2 \otimes \chi), \psi).$$

On the analytic side, this is one of our results.

Arithemtic stability (C-S-T) : *If* $\pi_1,\ \pi_2 \in \mathcal{A}_n^0(F)$ *with* $\omega_{\pi_1} = \omega_{\pi_2}$*, then for all sufficiently highly ramified characters of* $F^\times$ *we have*

$$\gamma(s, \pi_1 \otimes \chi, \Lambda^2, \psi) = \gamma(s, \pi_2 \otimes \chi, \Lambda^2, \psi).$$

3. **Globalization and the global functional equation**. Given a local pair $(\rho, \pi = \pi(\rho))$ over the local field F, suppose we can find a global field $\mathbb{F}$ and a global pair $(\Sigma, \Pi = \pi(\Sigma))$ with $\Sigma \in Rep_n^0(W_{\mathbb{F}})$ and $\Pi \in \mathcal{A}_n^0(\mathbb{A}_{\mathbb{F}})$ and a place v_0 of $\mathbb{F}$ so that

$$\mathbb{F}_{v_0} = F, \qquad \Sigma_{v_0} = \rho, \qquad \Pi_{v_0} = \pi.$$

(Even though we don't know the GLC in general, we can do this for certain representations.) Then we have at our disposal the global functional equations

$$\begin{aligned} \Lambda(s, \Lambda^2(\Sigma)) &= \varepsilon(s, \Lambda^2(\Sigma))\Lambda(1 - s, \Lambda^2(\Sigma^\vee)) \\ \Lambda(s, \Pi, \Lambda^2) &= \varepsilon(s, \Pi, \Lambda^2)L(1 - s, \tilde{\Pi}, \Lambda^2) \end{aligned}$$

which we can (essentially) pull apart to give

$$\gamma(s,\Lambda^2(\rho),\psi)^{-1}=\prod_{v|\infty}\gamma(s,\Lambda^2(\Sigma_v),\psi_v)\prod_{\substack{v<\infty\\ unram}}\gamma(s,\Lambda^2(\Sigma_v),\psi_v)\prod_{\substack{v<\infty\\ ramified\\ v\neq v_0}}\gamma(s,\Lambda^2(\Sigma_v),\psi_v)$$

$$\gamma(s,\pi,\Lambda^2,\psi)^{-1}=\prod_{v|\infty}\gamma(s,\Pi_v,\Lambda^2,\psi_v)\prod_{\substack{v<\infty\\ unram}}\gamma(s,\Pi_v,\Lambda^2,\psi_v)\prod_{\substack{v<\infty\\ ramified\\ v\neq v_0}}\gamma(s,\Pi_v,\Lambda^2,\psi_v).$$

(One can't really write the unramified contributions like this, but for exposition this is simplest.) But now

(i) we have equality of the individual archimedean factors buy the archimedean LLC and the archimedean work of Shahidi

(ii) we have the equality of the individual unramified factors by additivity and multiplicativity: by structure theory the unramified Galois representations are sums of unramified characters and on the analytic side we have the induced from the corresponding unramified characters,

(iii) we force equality of the individual ramified factors for $v \neq v_0$ by globally twisting by characters that are highly ramified at these places and then using stability

to conclude that

$$\gamma(s,\Lambda^2(\rho),\psi)=\gamma(s,\pi(\rho)\Lambda^2,\psi).$$

It is (iii) that needs to be understood and checked for each application.

5. Key steps in the proof

1. **Stable Equality**: *Let $\rho \in Rep_n^0(W_F)$. The for all sufficiently highly ramified characters χ of $F^\times \simeq W_F^{ab}$ we have*

$$\gamma(s,\Lambda^2(\rho\otimes\chi),\psi)=\gamma(s,\pi(\rho)\otimes\chi,\Lambda^2,\psi).$$

We prove this by induction with the following

Induction hypothesis: *For all non-archimedean local fields E and all $\rho \in Rep_m^0(W_E)$ with $m<n$ and all sufficiently ramified characters χ of $E^\times \simeq W_E^{ab}$ we have*

$$\gamma(s,\Lambda^2(\rho\otimes\chi),\psi)=\gamma(s,\pi(\rho)\otimes\chi,\Lambda^2,\psi).$$

Here we use our first globalization:

Globalization I: *Let F be a p-adic local field and ω_0 a character of $W_F^{ab} \simeq F^\times$. Then there exists a number field $\mathbb{F}$, a Galois representation $\Sigma \in Rep_n^0(W_{\mathbb{F}})$, and a place v_0 of $\mathbb{F}$ such that*

(i) $\mathbb{F}_{v_0} = F$, $\rho_0 = \Sigma_{v_0} \in Rep_n^0(W_F)$, *and* $\det(\rho_0) = \omega_0$,

(ii) *for all* $v < \infty$, $v \neq v_0$, Σ_v *is reducible,*

(iii) $\Pi = \pi(\Sigma) = \otimes' \pi(\Sigma_v)$ *is cuspidal.*

We prove this by extending ω_0 to $\mathbb{F}$ in an appropriate way and then inducing on the arithmetic side and using automorphic induction on the analytic side.

Then we can applyy our global functional equation technique to the pair $(\rho_0, \pi_0 = \pi(\rho_0))$ and (Σ, Π). At the ramified places other that v_0 we have Σ_v is reducible, so we can now use additivity/multiplicativity combined with stability through our induction hypothesis to obtain equality at these places. Our conclusion is then:

Equality at a base point: *Let F be a p-adic field and ω_0 a character of $F^\times \simeq W_F^{ab}$. Then there exists $\rho_0 \in Rep_n^0(W_F)$ with $\det(\rho_0) = \omega_0$ such that for all characters χ of $F^\times \simeq W_F^{ab}$ we have*

$$\gamma(s, \Lambda^2(\rho_0 \otimes \chi), \psi) = \gamma(s, \pi(\rho_0) \otimes \chi, \Lambda^2, \psi).$$

The stable version of our theorem for general $\rho \in Rep_n^0(W_F)$ then follows from this equality at a base point by applying arithmetic and analytic stability on the two sides.

2. **Equality for Monomial Representations** : *Suppose we have an extension of local fields $F \subset L \subset E$ with $(E : F) = n$ and a character $\chi : \mathrm{Gal}(E/L) \to \mathbb{C}^\times$. Let*

$$\rho = \mathrm{Ind}_{\mathrm{Gal}(E/L)}^{\mathrm{Gal}(E/F)}(\chi).$$

Then

$$\gamma(s, \Lambda^2(\rho), \psi) = \gamma(s, \pi(\rho), \Lambda^2, \psi).$$

This is proved by a second globalization, which was used by Henniart in his proof of the local Langlands correspondence.

Globalization II: *There exists global extensions $\mathbb{F} \subset \mathbb{L} \subset \mathbb{E}$ with $(\mathbb{E} : \mathbb{F}) = n$ and a character $\mathfrak{X} : \mathrm{Gal}(\mathbb{E}/\mathbb{F}) \to \mathbb{C}^\times$ and a place v_0 of $\mathbb{F}$ such that*

(i) $\mathbb{F}_{v_0} = F$, $\mathbb{E}_{w_0} = E$, $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) = \mathrm{Gal}(E/F)$, $\mathbb{L}_{v_0'} = L$, *and* $\mathfrak{X}_{v_0'} = \chi$,

(ii) *If* $\Sigma = \mathrm{Ind}_{\mathrm{Gal}(\mathbb{E}/\mathbb{L})}^{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}(\mathfrak{X})$ *then* $\Sigma_{v_0} = \rho$,

(iii) $\Pi = \pi(\Sigma)$ *is cuspidal automorphic.*

At the ramified primes, we can now use the Stable Equality to match the individual factors.

3. **Endgame: Brauer Induction**: *If* $\rho \in Rep_n^0(\mathrm{Gal}(E/F))$ *then*

$$\rho = \sum n_i \ \mathrm{Ind}_{\mathrm{Gal}(E/L_i)}^{\mathrm{Gal}(E/F)}(\chi_i)$$

with $n_i \in \mathbb{Z}$.

We now use additivity/multiplicativity to deduce the equality of local factors for such ρ from the equality for monomial representations. This, upon twisting by unramified characters, finishes the proof of equality for general $\rho \in Rep_n^0(W_F)$.