Vaught's Conjecture from the Perspective of Algebraic Logic

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Abstract

Vaught's Conjecture states, that if T is a complete first order theory in a countable language such that T has uncountably many pairwise non-isomorphic countably infinite models then T has $2^{\aleph_0}$ many pairwise non-isomorphic countably infinite models.

In this note we prove that if T has at least $\aleph_1$ many countable models which are pairwise separable by critical types (see definitions 1.1 and 1.2 below), then T has continuum many such models, that is, a certain weak version of Vaught's conjecture is true. The proofs are based on the representation theory of Cylindric Algebras (for details see [9] and [10]) and elementary topological properties of the Stone spaces of these Cylindric Algebras.

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1 Introduction

Let T be a complete first order theory in a countable language. Recall, that for any cardinal κ, $I(T,\kappa)$ denotes the number of pairwise non-isomorphic models of T of cardinality κ. Vaught conjectured, that the following is true: if – counting up to isomorphism – T has at least $\aleph_1$ many countable models then T has $2^{\aleph_0}$ many countable models; in symbols:

$$(*) \qquad I(T,\aleph_0) > \aleph_0 \text{ implies } I(T,\aleph_0) = 2^{\aleph_0}.$$

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This conjecture became an important open problem, it was mentioned in practically all monographs on model theory. See, for example

- Hodges [11], on page 339,
- Shelah [21], Problem (8) on page XXI,
- Chang-Keisler [5], Conjecture (6) on page 597,
- Buechler [4], Conjecture 2.3.1 on page 39 and
- Marker [12], on page 155.

The original conjecture had been published in Vaught [22].

Vaught's conjecture stimulated an intensive research; we recall the following related results:

- Morley proved in [13] that $I(T,\aleph_0) > \aleph_1$ implies $I(T,\aleph_0) = 2^{\aleph_0}$ (see also Theorem 4.4.16 of [12]);
- Bouscaren and Lascar proved in [2] that $(*)$ is true for $\aleph_0$-stable T of finite Morley rank;
- Shelah proved in [20] that $(*)$ is true for $\aleph_0$-stable T and
- Buechler proved in [3] that $(*)$ is true for superstable T of finite U-rank.

In these results, some extra assumptions has been made on T and then the corresponding special case of Vaught's conjecture has been settled. In this note we follow a same approach: we will assume that T has a further property: T has at least $\aleph_1$ many countable models which are ,,relatively far from being isomorphic to each other". Then we show, that T in fact, has continuum many countable models which are still, pairwise ,,relatively far from being isomorphic to each other". To be more precise, we need the following definitions.

Definition 1.1 *Let $p \in S_n(T)$ be a type in n variables. We say, that $p' \subseteq p$ is a proper subtype of p iff p' is a type, but the number of free variables occurring in formulas of p' is strictly smaller than n.*

A type $p \in S(T)$ is defined to be critical iff p is nonisolated, but every complete proper subtype $p' \subseteq p$ is isolated.

Definition 1.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two models of T. We say, that $\mathcal{A}$ and $\mathcal{B}$ can be separated by critical types iff there exists a critical type $p \in S(T)$ such that the cardinalities of realizations of p in $\mathcal{A}$ and $\mathcal{B}$ are different.*

The main goal of this paper is to show

Theorem 3.7: if there exists a set $\{\mathcal{A}_i : i < \aleph_1\}$ of countable models of T which are pairwise separable by critical types, then T has continuum many countable models which are pairwise separable by critical types, particularly, $I(T,\aleph_0) = 2^{\aleph_0}$.

In more detail, if we replace "non-isomorphic" by "separable with critical types" in Vaught's original conjecture, then the conjecture becomes true. The assumption, that T has $\aleph_1$ many countable models such that each pair of them can be separated by critical types, is more restrictive than the original assumption of Vaught's conjecture; however, our conclusion, that T has continuum many countable models which are pairwise separable by critical types, is still stronger than the conclusion of Vaught's conjecture.

Recall, that $L_1(T)$ is the smallest fragment of $L_{\omega_1\omega}$ containing $L_{\omega\omega}$ and the formulas $\bigwedge p$ for all types $p \in S_n(T)$ and $n \in \omega$. Martin's conjecture claims, that if T (a complete first-order theory) has fewer than $2^{\aleph_0}$ countable models, then every countable model of T has an $\aleph_0$-categorical theory in $L_1(T)$ (i.e. it is determined, up to isomorphism, by its theory in $L_1(T)$). It is well known, that Vaught's conjecture follows from Martin's conjecture. Our main theorem 3.7 is related to this implication, but an essential difference is the following: we do not assume that the consequence of Martin's conjecture holds for T. We assume only, that T has $\aleph_1$ many countable models having different $L_1(T)$-theories and we do not exclude the possibility that T may have non-isomorphic countable models with same $L_1(T)$-theories. Hence, classical methods, such as investigations on Scott ranks, seem not to be applicable in our case. At the technical level, our proofs are based on the representation theory of Cylindric Algebras, the details will be completely recalled below and can also be found in [9] and in [10].

The paper is organized as follows. At the end of this section we are summing up our system of notation. In Section 2 we recall some standard methods and results of algebraic logic related to representations of Cylindric Algebras. We try to keep this paper self-contained, hence this section is more detailed, than as usual. However, familiarity of [9] or [10] may be an advantage of the reader. Section 2 has a survey character: with the exception of Theorem 2.1 there are no new results in it. It's main goal is to "translate" Vaught's conjecture to a question about non-base-isomorphic representations of Lindenbaum Algebras associated to first order theories. We believe, that translating Vaught's conjecture to algebraic logic may be useful for further related investigations, hence we provide more results in this direction than really needed in later sections. We also note, that further investigations of Vaught's conjecture with methods of algebraic logic are under development, see e.g. [17] and [18]. Finally, in Section 3 we investigate some topological properties of Stone spaces of locally finite dimensional cylindric algebras. Based on these investigations, in Corollary 3.7 we prove Theorem 3.7 which is the main result of the paper. In Section 4 we mention further related results and pose questions that remained open.

Our system of notation is mostly standard, but the following list may be useful. Throughout ω denotes the set of natural numbers and for every $n \in \omega$ we have $n = \{0, 1, ..., n-1\}$. Let A and B be sets. Then ${}^{A}B$ denotes the set of functions whose domain is A and whose range is a subset of B. In addition, $|A|$ denotes the cardinality of A; if κ is a cardinal then $[A]^\kappa$ denotes the set of subsets of A which are of cardinality κ and $\mathcal{P}(A)$ denotes the power set of A, that is, $\mathcal{P}(A)$ consists of all subsets of A. For any distinct elements i, j from a given set U, $[i/j] \in {}^U U$ is the function on U which maps i to j and leaves every other element fixed. Throughout we use function composition in such a way that the rightmost factor acts first. That is, for functions f, g we define $f \circ g(x) = f(g(x))$.

2 Cylindric Algebraic Preliminaries

Algebraic investigations of different logics can be traced back since the middle of the 19th century, beginning with the work of Boole, De Morgan, Peirce, Schröder, Löwenheim and others. In order to provide an algebraic treatment of first order logics, Tarski introduced the classes of Cylindric Algebras in the 1930's. Cylindric algebras are Boolean algebras endowed with certain other operations corresponding to quantifiers and equality. More precisely, for an ordinal α, an algebra $\mathcal{A} = \langle A; \wedge, -, c_i, d_{i,j} \rangle_{i,j \in \alpha}$ is defined to be a cylindric algebra of dimension α if $\langle A, \wedge, - \rangle$ is a Boolean algebra, the c_i's are unary operations, the $d_{i,j}$'s are constants and $\mathcal{A}$ satisfies some equational postulates. These postulates can be found, for example in Definition 1.1.1 of [9]; we do not recall these postulates because we assume the reader is familiar with them. Intuitively, in a first order language modeled by a cylindric algebra, α is the sequence of individual variables and for every $i, j \in \alpha$ the operation c_i corresponds to the quantifier $\exists v_i$ and $d_{i,j}$ corresponds to the formula $v_i = v_j$. As usual, if $\Gamma = \{i_0, ..., i_{n-1}\} \subseteq \alpha$ is finite then the term $c_{i_0} ... c_{i_{n-1}}(x)$ is denoted by $c_{(\Gamma)}(x)$.

Throughout this paper we fix a first order language L in which the sequence of individual variables is $\langle v_i : i \in \omega \rangle$. Moreover, we assume that T is countable and does not contain function- and constant-symbols; the latter assumption can be done without loss of generality.

There are two natural ways how to construct a cylindric algebra. We refer to these two constructions as "algebraizing syntax" and "algebraizing semantics".

Algebraizing Syntax

Let T be a first order theory in L. (Variants of) the algebras obtained below are called the Lindenbaum-algebras of T. Let F be the set of first order formulas of L. Two formulas φ, ψ are defined to be *equivalent mod T* iff $T \models \varphi \Leftrightarrow \psi$. This notion of equivalence is really an equivalence relation; the class of φ is denoted by $\varphi / \equiv_T$. Let $A = \{\varphi / \equiv_T : \varphi \in F\}$. We will define certain operations on A as follows.

$(\varphi / \equiv_T) \cdot (\psi \equiv_T) = (\varphi \wedge \psi) / \equiv_T;$

$-(\varphi/\equiv_T) = (\neg\varphi)/\equiv_T$;
$c_i(\varphi/\equiv_T) = (\exists v_i\varphi)/\equiv_T$;
$d_{i,j} = (v_i = v_j)/\equiv_T$.

It is easy to check that the above operations are well defined on A (i.e., their results do not depend on the choices of the particular representatives of the equivalence classes they are applied for). By Theorem 1.1.10 of [9] the algebra $CA(T) = \langle A; \cdot, -, c_i, d_{i,j}\rangle_{i,j\in\omega}$ is a cylindric algebra.

Algebraizing Semantics

Let U be a set. Then the *full cylindric set algebra* on U of dimension ω is the structure $\langle \mathcal{P}({}^\omega U); \cap, -, C_i, D_{i,j}\rangle_{i,j\in\omega}$ where $\cap$ is set theoretical intersection, $-$ is complementation (w.r.t. ${}^\omega U$) and for any $X \subseteq {}^\omega U$ and $i, j \in \omega$

$C_i(X) = \{s \in {}^\omega U : (\exists z \in X)(s_{|\omega-\{i\}} = z_{|\omega-\{i\}})\}$ and
$D_{i,j} = \{s \in {}^\omega U : s(i) = s(j)\}$.

The class Cs_ω of ω dimensional cylindric set algebras is defined to be the class of all subalgebras of ω dimensional full cylindric set algebras. If $\mathcal{A} \in Cs_\omega$ is a subalgebra of the full set algebra $\langle \mathcal{P}({}^\omega U); \cap, -, C_i, D_{i,j}\rangle_{i,j\in\omega}$ then U is called the *base set* of $\mathcal{A}$. Let $\mathcal{A} = \langle U; R_i\rangle_{R_i\in L}$ be any model for L. For any formula φ of L let $||\varphi||^{\mathcal{A}} = \{s \in {}^\omega U : \mathcal{A} \models \varphi[s]\}$. This can be identified by the relation defined by φ in $\mathcal{A}$. (Strictly speaking, the relation defined by φ has a finite arity: this arity is equal with the number of free variables of φ. In order to establish a "uniform treatment" of definable relations which does not sensitive to the particular arities, in algebraic logic it is a standard practice to consider the ω-dimensional $||\varphi||^{\mathcal{A}}$ as the relation defined by φ in $\mathcal{A}$. We will see below, that using dimension sets, the information about the arities of definable relations can be recovered.)

For each model $\mathcal{A} = \langle U; R\rangle_{R\in L}$ one can associate a cylindric set algebra $Cs_\omega(\mathcal{A})$ defined as follows. $Cs_\omega(\mathcal{A})$ is a subalgebra of the ω dimensional full cylindric set algebra on U whose universe (underlying set) consists of the definable relations of $\mathcal{A}$, that is, the set of elements of $Cs_\omega(\mathcal{A})$ is $\{||\varphi||^{\mathcal{A}} : \varphi$ is a (parameter-free) formula of $L\}$. It is easy to check that $Cs_\omega(\mathcal{A})$ is a cylindric algebra for any model $\mathcal{A}$.

It is also easy to verify that two models $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent iff $Cs_\omega(\mathcal{A})$ and $Cs_\omega(\mathcal{B})$ are isomorphic. An isomorphism $f : Cs_\omega(\mathcal{A}) \to Cs_\omega(\mathcal{B})$ is called a *base isomorphism* iff there exists a bijection g between the universes of $\mathcal{A}$ and $\mathcal{B}$ such that for any $x \in Cs_\omega(\mathcal{A})$ we have $f(x) = \{g \circ s : s \in x\}$. One can easily show that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $Cs_\omega(\mathcal{A})$ and $Cs_\omega(\mathcal{B})$ are base isomorphic.

Following the practice of [9] and [10], cylindric operations in $CA(T)$ will be denoted by c_i and $d_{i,j}$ while in Cylindric Set Algebras these operations will be denoted by C_i and $D_{i,j}$, respectively.

Cylindric Dimension Sets

Let $\mathcal{A}$ be an ω dimensional cylindric algebra and let $x \in A$. Then the dimension set $\Delta(x)$ of x is defined to be $\Delta(x) = \{i \in \omega : c_i(x) \neq x\}$. $\mathcal{A}$ is called *locally finite dimensional* iff every element of $\mathcal{A}$ has a finite dimension set. If T is a theory and $\mathcal{A}$ is a model then both $CA(T)$ and $Cs_\omega(\mathcal{A})$ are locally finite dimensional (basically, because each first order formula contains a finite number of individual variables). The class of ω-dimensional locally finite dimensional cylindric algebras is denoted by LF_ω. Suppose $\mathcal{A}$ is a locally finite dimensional cylindric set algebra with base set U. Then $\mathcal{A}$ is defined to be *regular* iff for every $x \in A$ and every $s, z \in {}^\omega U$

$$s \in x,\ s_{|\Delta(x)} = z_{|\Delta(x)} \ \text{ implies } \ z \in x.$$

Clearly, $Cs_\omega(\mathcal{A})$ is regular for every model $\mathcal{A}$ (because all the definable relations of $\mathcal{A}$ have finite arities). The class of regular elements of Cs_ω will be denoted by Cs_ω^{reg}. In the next paragraph we recall an argument showing, that $Cs_\omega(\mathcal{A})$ does not have any further properties in general. That is, every $LF_\omega \cap Cs_\omega^{reg}$ is of the form $Cs_\omega(\mathcal{A})$ for some model $\mathcal{A}$ for L.

Suppose $\mathcal{A} \in LF_\omega \cap Cs_\omega^{reg}$, suppose $f : L \longrightarrow A$ is a function on relation symbols of L such that for every $R \in L$ the arity of R is equal with $\Delta(f(R))$ and suppose the range of f generates $\mathcal{A}$. Then $\mathcal{A}$ determines a model for L as follows. Let the base set of $\mathcal{A}$ be U and for every $x \in A$ let $x' = \{s_{|\Delta(x)} : s \in x\}$. Then x' is a $\Delta(x)$-ary relation on U. Finally $\mathcal{M} = \langle U; f(R)'\rangle_{R\in L}$ is a model for L and $\langle U; x'\rangle_{x\in A}$ is the collection of all relations definable in $\mathcal{M}$. In addition, $\mathcal{A} = Cs_\omega(\mathcal{M})$, in other words, the identity function on U determines a base isomorphism between $\mathcal{A}$ and $Cs_\omega(\mathcal{M})$.

Substitutions

Substituting individual variables for other individual variables in a formula is a frequently used and natural operation. As explained in Section 1.5 of [9], this kind of substitution can also be expressed in cylindric algebraic terms. For distinct $i, j \in \omega$ let $s_{[i/j]}(x) = c_i(x \wedge d_{i,j})$. Clearly, $s_{[i/j]}$ is a term of the language of cylindric algebras. Moreover, one can easily check, that if ψ is obtained by substituting v_j for v_i in φ then $CA(T) \models \psi/\equiv_T = s_{[i/j]}(\varphi/\equiv_T)$ and analogously, $Cs_\omega(\mathcal{A}) \models ||\psi||^{\mathcal{A}} = S_{[i/j]}(||\varphi||^{\mathcal{A}})$. More general simultaneous substitutions can be described with functions $\tau \in {}^\omega\omega$: for every $i \in \omega$ one should substitute simultaneously $v_{\tau(i)}$ for v_i. As explained in Definitions 1.11.9 and 1.11.13 of [9], these kind of generalized substitutions can be introduced in every LF_ω, the derived function is denoted by s_τ. In fact, if $\mathcal{A} \in LF_\omega$ then for every $x \in A$ there is a cylindric term $t_{\tau,x}$ such that $s_\tau(x) = t_{\tau,x}(x)$. Note, that s_τ is not a term function, because $t_{\tau,x}$ depends on (the dimension set of) x. Moreover, by Remark 1.11.10 of [9] (see page 237 therein), $t_{\tau,x} = s_\varrho$ where $\varrho \in {}^\omega\omega$ is such that $\{n \in \omega : \varrho(n) \neq n\} \subseteq \Delta(x)$. If $\mathcal{A} \in LF_\omega$ is a cylindric set algebra with base set U then for every $x \in A$ and every $\tau \in {}^\omega\omega$ we have $S^{\mathcal{A}}_\tau(x) = \{s \in {}^\omega U : s \circ \tau \in x\}$.

The following facts can also be found in [9] and play an important role in the present note.

Fact 1 (item 1.11.12(iii) of [9]).
For every $\tau \in {}^\omega\omega$ the function s_τ is a Boolean homomorphism in every LF_ω.
Fact 2 (item 1.11.14(ix) of [9]).
For every $\tau, \sigma \in {}^\omega\omega$ one has $s_\tau(s_\sigma(x)) = s_{\tau\circ\sigma}(x)$ in every LF_ω.
Fact 3 (item 1.11.6(i) of [9]).
For every $\mathcal{A} \in LF_\omega$ and $i \in \omega$ one has $c_i(x) = sup\{s_{[i/j]}(x) : j \in \omega - \Delta(x)\}$.
Fact 4 (Remark 2.3.15 of [9]).
If T is a complete theory then $CA(T)$ is a simple algebra
(that is, $CA(T)$ has only trivial congruence relations).
Fact 5 (item 1.11.14 (xi) of [9]).
If $\sigma_{|\Delta(x)} = \tau_{|\Delta(x)}$ then $s_\sigma(x) = s_\tau(x)$.

Representations

Suppose T is a theory and $\mathcal{A} \models T$. Then $Cs_\omega(\mathcal{A})$ is a homomorphic image of $CA(T)$; in more detail, for each $R \in L$ let $f(R/\equiv_T) = ||R||^{\mathcal{A}}$. Since $\{R/\equiv_T: R \in L\}$ generates $CA(T)$, and $\mathcal{A} \models T$, a straightforward induction on the complexity of formulas of L shows that f can be extended to a homomorphism. If T is a complete theory then, according to Fact 4 in the previous list, $CA(T)$ is simple, and hence f is, in fact, an embedding. Conversely, as recalled in the subsection "Algebraizing Semantics" above, an element $\mathcal{A}$ of $LF_\omega \cap Cs^{reg}_\omega$ (together with a distinguished set of its generators) corresponds to a model for L; this model satisfies T iff there is a homomorphism from $CA(T)$ onto $\mathcal{A}$ (mapping the distinguished sets of generators of $CA(T)$ and $\mathcal{A}$ onto each other). Hence, the problem of constructing a model of T is equivalent with the problem of finding a homomorphism from $CA(T)$ into some $\mathcal{A} \in LF_\omega \cap Cs^{reg}_\omega$.

Finding a homomorphism from $CA(T)$ onto some $LF_\omega \cap Cs^{reg}_\omega$ is a special case of the so called *representation problem*. Originally the representation problem had been investigated in order to obtain a purely algebraic proof for the Completeness Theorem of First Order Logic. Some more recent works on the representation problem of cylindric and cylindric-like algebras can be found e.g. in [7], [14], [15], [16], Ferenczi [6], Sayed-Ahmed [19] and the references therein.

In Remark 3.2.9 of [10], there is a method, how to construct homomorphisms from an LF_ω onto an $LF_\omega \cap Cs^{reg}_\omega$. Below we recall this method and show, that every $LF_\omega \cap Cs^{reg}_\omega$ corresponding to a countable model of T can be obtained as a result of this construction. To do this, we need some further preparations.

Let $\mathcal{A}$ be any Boolean algebra. The set of ultrafilters of $\mathcal{A}$ is denoted by $\mathcal{U}(\mathcal{A})$. The Stone topology on this set forms a compact Hausdorff space and it is defined as follows: if $x \in A$ then N_x is defined to be the basic clopen set $N_x = \{\mathcal{F} \in \mathcal{U}(\mathcal{A}) : x \in \mathcal{F}\}$ and $\{N_x : x \in A\}$ forms a basis of the Stone topology on $\mathcal{U}(\mathcal{A})$. Suppose that $x \in A$ and $Y = \{y_i : i \in \omega\} \subseteq A$ are such that $x = sup\{y_i : i \in \omega\}$. We will say, that $\mathcal{F} \in \mathcal{U}(\mathcal{A})$ preserves Y if $x \in \mathcal{F}$ implies the existence of an $i \in \omega$ for which $y_i \in \mathcal{F}$.

Now let $\mathcal{A} \in LF_\omega$. For each $i \in \omega$ and $x \in A$ let

$$\mathcal{U}_{i,x}(\mathcal{A}) = \{\mathcal{F} \in \mathcal{U}(\mathcal{A}) : \mathcal{F} \text{ preserves } \{s_{[i/j]}(x) : j \in \omega - \Delta(x)\}\}$$

and let $\mathcal{H}(\mathcal{A}) = \cap_{i\in\omega, x\in A}\mathcal{U}_{i,x}(\mathcal{A})$. For completeness we note that for each i, x the complement of $\mathcal{U}_{i,x}(\mathcal{A})$ is a nowhere dense subset of the Stone space $\mathcal{U}(\mathcal{A})$, and hence, by Baire's category theorem, $\mathcal{H}(\mathcal{A})$ is non-void (this is easy to check, but we don't include here a proof because we will see directly that $\mathcal{H}(\mathcal{A})$ is non-void and below we don't use the fact that the complement of $\mathcal{U}_{i,x}(\mathcal{A})$ is nowhere dense).

Now we are ready to recall the method of constructing homomorphisms of $\mathcal{A} \in LF_\omega$ into elements of Cs^{reg}_ω (as we mentioned, this construction has been appeared in Remark 3.2.9 of [10], see also [1]). Let $\mathcal{F} \in \mathcal{H}(\mathcal{A})$ and let

$$E = \{\langle i, j\rangle \in \omega \times \omega : d_{i,j} \in \mathcal{F}\}.$$

Then E is an equivalence relation; it will be called the *kernel* of $\mathcal{F}$. For any $\tau \in {}^\omega\omega$ we will denote by τ/E the function satisfying $\tau/E(i) = \tau(i)/E$ for every $i \in \omega$. Finally, for each $x \in A$ let

$$rep_{\mathcal{F}}(x) = \{\tau/E : \tau \in {}^\omega\omega, s_\tau(x) \in \mathcal{F}\}.$$

By Remark 3.2.9 of [10], $rep_{\mathcal{F}}$ is a homomorphism onto some Cs^{reg}_ω with countable base (f preserves the c_i's because for every $i \in \omega$ and $x \in A$ we have $\mathcal{F} \in \mathcal{U}_{i,x}(\mathcal{A})$ and by Fact 3 above, $c_i(x) = sup\{s_{[i/j]}(x) : j \in \omega - \Delta(x)\}$). Our next goal in Theorem 2.1 is to show, that the converse is also true: each homomorphism of $\mathcal{A}$ onto some base-countable Cs^{reg}_ω is of the form $rep_{\mathcal{F}}$ for some $\mathcal{F} \in \mathcal{H}(\mathcal{A})$.

Let $h : \omega \to \omega$ be a function. As usual, $ker(h)$ is defined to be the equivalence relation $ker(h) = \{\langle n, m\rangle \in \omega \times \omega : \; h(n) = h(m)\}$. Throughout, id_ω denotes the identity function on ω.

Theorem 2.1 *Let $\mathcal{A} \in LF_\omega$, $\mathcal{B} \in Cs^{reg}_\omega$ with countably infinite base set and let f be a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Let Θ be an equivalence relation on ω such that Θ has infinitely many equivalence classes and each equivalence class is infinite. Then* *there is an ultrafilter $\mathcal{F} \in \mathcal{H}(\mathcal{A})$ such that $f = rep_{\mathcal{F}}$ and the kernel of $\mathcal{F}$ is Θ.*

Proof. Let $h : \omega \to \omega$ be a surjective function such that $ker(h) = \Theta$. Since the base set of $\mathcal{B}$ is countable, we may assume that this base set is $\omega/ker(h)$. Now let

$$\mathcal{F} = \{x \in A : id_\omega/ker(h) \in f(x)\}.$$

It is easy to check that $\mathcal{F}$ is an ultrafilter of $\mathcal{A}$. Our goal is to show that $\mathcal{F} \in \mathcal{H}(\mathcal{A})$, the kernel of $\mathcal{F}$ is Θ and $f = rep_{\mathcal{F}}$.

Let $x \in A, i \in \omega$ be arbitrary; we show that $\mathcal{F} \in \mathcal{U}_{i,x}(\mathcal{A})$. So suppose $c_i(x) \in \mathcal{F}$. Then $id_\omega/ker(h) \in f(c_i(x))$, that is, $id_\omega/ker(h) \in C_i^{\mathcal{B}}(f(x))$. Hence, there is a function $s \in {}^\omega\omega$ for which

$$(*) \qquad (id_\omega)_{|\omega-\{i\}} = s_{|\omega-\{i\}}$$

and $s/ker(h) \in f(x)$. Since $s(i)/ker(h)$ is infinite, there exists a $j \in s(i)/ker(h) - (\{i\} \cup \Delta(x))$. By $(*)$ we have $s(j) = j$, hence $\langle s(i), s(j)\rangle \in ker(h)$. Then $s/ker(h) \in f(x) \cap D_{i,j}^{\mathcal{B}}$. Therefore, again by $(*)$,

$$id_\omega/ker(h) \in C_i^{\mathcal{B}}(f(x) \cap D_{i,j}^{\mathcal{B}}) = f(c_i(x) \wedge d_{i,j}) = f(s_{[i/j]}(x));$$

consequently $s_{[i/j]}(x) \in \mathcal{F}$, as desired. Since x and i were chosen arbitrarily, it follows that $\mathcal{F} \in \mathcal{H}(\mathcal{A})$.

Next, we show that Θ is the kernel of $\mathcal{F}$. Let $E = \{\langle n, m\rangle \in \omega \times \omega : d_{n,m} \in \mathcal{F}\}$ be the the kernel of $\mathcal{F}$ (recall, that it is an equivalence relation). Then

$$\begin{aligned}
&\langle i, j\rangle \in E \Leftrightarrow\\
&d_{i,j} \in \mathcal{F} \Leftrightarrow\\
&id_\omega/ker(h) \in D_{i,j}^{\mathcal{B}} \Leftrightarrow\\
&i/ker(h) = j/ker(h) \Leftrightarrow\\
&h(i) = h(j) \Leftrightarrow\\
&\langle i, j\rangle \in ker(h).
\end{aligned}$$

Hence $E = ker(h) = \Theta$. Finally we show $f = rep_{\mathcal{F}}$. Let $x \in A$ and $\tau \in {}^\omega\omega$ be arbitrary. Then

$$\begin{aligned}
&\tau/E \in rep_{\mathcal{F}}(x) \Leftrightarrow\\
&s_\tau(x) \in \mathcal{F} \Leftrightarrow\\
&id_\omega/ker(h) \in f(s_\tau(x)) \Leftrightarrow\\
&id_\omega/ker(h) \in S_\tau^{\mathcal{B}}(f(x)) \Leftrightarrow\\
&(id_\omega/ker(h)) \circ \tau \in f(x) \Leftrightarrow\\
&(id_\omega \circ \tau)/ker(h) \in f(x) \Leftrightarrow\\
&\tau/E \in f(x).
\end{aligned}$$

So $rep_{\mathcal{F}}(x) = f(x)$. This completes the proof. ∎

3 On Non-isomorphic Representations

By Theorem 2.1 if $\mathcal{A} \models T$ is countable then there is an ultrafilter $\mathcal{F} \in \mathcal{H}(CA(T))$ for which $rep_{\mathcal{F}}$ is an isomorphism between $CA(T)$ and $Cs_\omega(\mathcal{A})$. Thus, instead of countable models of T, one can study $\mathcal{H}(CA(T))$. We will do so in the present section.

The first difficulty with this approach is that for different $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{H}(CA(T))$ the images of $rep_{\mathcal{F}_0}$ and $rep_{\mathcal{F}_1}$ may determine isomorphic models of T. Theorem 3.2 below will be useful to handle this problem. Before stating it, we need further preparations.

Definition 3.1 *Suppose $\varrho : \omega \to \omega$ and $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{H}(CA(T))$. Suppose in addition, that $rep_{\mathcal{F}_0} : CA(T) \to \mathcal{B}_0$ and $rep_{\mathcal{F}_1} : CA(T) \to \mathcal{B}_1$ are surjective homomorphisms. We say, that ϱ induces a base isomorphism between $\mathcal{B}_0$ and $\mathcal{B}_1$ iff the following stipulations hold for any $i, j \in \omega$:*

(1) if $\langle i, j\rangle \in E_0$ then $\langle \varrho(i), \varrho(j)\rangle \in E_1$;

(2) the function $\hat{\varrho} : \omega/E_0 \to \omega/E_1$ defined by $\hat{\varrho}(i/E_0) = \varrho(i)/E_1$ is a bijection between ω/E_0 and ω/E_1;

(3) the function $f : B_0 \to B_1$ defined by $f(x) = \{\hat{\varrho} \circ (s/E_0) : s/E_0 \in x\}$ satisfies $f \circ rep_{\mathcal{F}_0} = rep_{\mathcal{F}_1}$ (that is, $\hat{\varrho}$ is a base isomorphism between $\mathcal{B}_0$ and $\mathcal{B}_1$).

Note, that if (1) of the above definition is satisfied, then $\hat{\varrho}$ is well defined.

Theorem 3.2 *Let T be a countable, consistent theory. Let $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{H}(CA(T))$ and suppose $rep_{\mathcal{F}_0} : CA(T) \to \mathcal{B}_0$ and $rep_{\mathcal{F}_1} : CA(T) \to \mathcal{B}_1$ are homomorphisms where $\mathcal{B}_0$ and $\mathcal{B}_1$ having base sets $\omega/E_0, \omega/E_1$, respectively. Then the following assertions are equivalent.*

(1) $\varrho : \omega \to \omega$ induces a base isomorphism between $\mathcal{B}_0$ and $\mathcal{B}_1$;

(2) $\mathcal{F}_0 = s_\varrho^{-1}(\mathcal{F}_1)$ and the range of ϱ meets every equivalence class of E_1;

(3) If K is a generating set of $CA(T)$ such that $d_{0,1} \in K$ then for every $x \in K$ and every $\tau \in {}^\omega\omega$ we have $s_\tau(x) \in \mathcal{F}_0$ iff $s_\varrho(s_\tau(x)) \in \mathcal{F}_1$. In addition, the range of ϱ meets every equivalence class of E_1.

Proof. Suppose ϱ induces a base isomorphism between $\mathcal{B}_0$ and $\mathcal{B}_1$. Then, by (2) of Definition 3.1, the range of ϱ meets every equivalence class of E_1. Moreover, for any $x \in CA(T)$ we have

$$x \in \mathcal{F}_0 \Leftrightarrow$$
$$s_{id_\omega}(x) \in \mathcal{F}_0 \Leftrightarrow$$

$$\begin{aligned}&id_\omega/E_0 \in rep_{\mathcal{F}_0}(x) \Leftrightarrow\\ &\hat{\varrho}\circ(id_\omega/E_0) \in rep_{\mathcal{F}_1}(x) \Leftrightarrow\\ &\varrho/E_1 \in rep_{\mathcal{F}_1}(x) \Leftrightarrow\\ &s_\varrho(x) \in \mathcal{F}_1.\end{aligned}$$

Thus, (1) implies (2). Clearly, (2) implies (3). Finally assume (3). Then, for any $x \in K$ and $\tau \in {}^\omega\omega$ we have

$$(*) \qquad \begin{aligned}&\tau/E_0 \in rep_{\mathcal{F}_0}(x) \Leftrightarrow\\ &s_\tau(x) \in \mathcal{F}_0 \Leftrightarrow\\ &s_\varrho(s_\tau(x)) \in \mathcal{F}_1 \Leftrightarrow\\ &s_{\varrho\circ\tau}(x) \in \mathcal{F}_1 \Leftrightarrow\\ &(\varrho\circ\tau)/E_1 \in rep_{\mathcal{F}_1}(x).\end{aligned}$$

Next, we show, that (1) of Definition 3.1 and its converse have been satisfied. Let $i, j \in \omega$. Choose an arbitrary function $\tau \in {}^\omega\omega$ such that $\tau(0) = i$ and $\tau(1) = j$. Then

$$\begin{aligned}&\langle i, j\rangle \in E_0 \Leftrightarrow\\ &\tau/E_0 \in D^{\mathcal{B}_0}_{0,1} \Leftrightarrow\\ &\tau/E_0 \in rep_{\mathcal{F}_0}(d_{0,1}) \overset{(by\ *)}{\Leftrightarrow}\\ &(\varrho\circ\tau)/E_1 \in rep_{\mathcal{F}_1}(d_{0,1}) \Leftrightarrow\\ &(\varrho\circ\tau)/E_1 \in D^{\mathcal{B}_1}_{0,1} \Leftrightarrow\\ &\langle \varrho(\tau(0)), \varrho(\tau(1))\rangle \in E_1 \Leftrightarrow\\ &\langle \varrho(i), \varrho(j)\rangle \in E_1.\end{aligned}$$

Thus, (1) of Definition 3.1 holds and hence $\hat{\varrho}$ is well defined. In addition, because of the converse of 3.1(1), it follows, that $\hat{\varrho}$ is an injective function. Since, by assumption, the range of ϱ meets every equivalence class of E_1, it follows, that $\hat{\varrho}$ is surjective. Thus, (2) of Definition 3.1 has been satisfied, as well.

Finally, observe, that for any $x \in K$ and $\tau \in {}^\omega\omega$ we have

$$\tau/E_0 \in rep_{\mathcal{F}_0}(x) \quad \overset{(by\ *)}{\Leftrightarrow} \quad (\varrho\circ\tau)/E_1 \in rep_{\mathcal{F}_1}(x) \quad \Leftrightarrow \quad \hat{\varrho}\circ(\tau/E_0) \in rep_{\mathcal{F}_1}(x).$$

This, combined with surjectivity of $\hat{\varrho}$, implies that, the function f defined in 3.1(3), satisfies $f \circ rep_{\mathcal{F}_0|K} = rep_{\mathcal{F}_1|K}$. Finally, because of K is a generating set of $CA(T)$, it follows that ϱ induces a base isomorphism between $\mathcal{B}_0$ and $\mathcal{B}_1$. ∎

In the next series of lemmas we are presenting technical ingredients needed to prove the main theorem of the paper.

Lemma 3.3 *Suppose $\mathcal{A} \in LF_\omega$, $a, b \in A - \{0\}$, $j \in \omega - \Delta(a)$ and $j \neq i, i \in \omega$. Then*

(a) $a \wedge d_{i,j} \neq 0$*;*
(b) $a \wedge -d_{i,j} \neq 0$ *and*
(c) if $\Delta(a) \cap \Delta(b) = \emptyset$ *and* $\mathcal{A}$ *is simple, then* $a \wedge b \neq 0$.

Proof. Observe, that

$$c_j(d_{i,j} \wedge a) = c_j(d_{i,j} \wedge c_j(a)) = c_j(d_{i,j}) \wedge c_j(a) = c_j(a) \geq a > 0.$$

Hence $d_{i,j} \wedge a \neq 0$, this proves (a) and (b) can be shown similarly. In fact, the proof of (c) is also rather similar:

$c_{\Delta(a)}c_{\Delta(b)}(a \wedge b) = c_{\Delta(a)}c_{\Delta(b)}(c_{\Delta(b)}(a) \wedge c_{\Delta(a)}(b)) = c_{\Delta(a)}(c_{\Delta(b)}(a) \wedge c_{\Delta(b)}c_{\Delta(a)}(b)) = c_{\Delta(a)}(c_{\Delta(b)}(a) \wedge c_{\Delta(a)}c_{\Delta(b)}(b)) = c_{\Delta(a)}c_{\Delta(b)}(a) \wedge c_{\Delta(a)}c_{\Delta(b)}(b)$

in addition, $c_{\Delta(a)}c_{\Delta(b)}(a)$ and $c_{\Delta(a)}c_{\Delta(b)}(b))$ are 0-dimensional and non-zero elements (a and b respectively, are nonzero lower bounds for them). Since $\mathcal{A}$ is assumed to be simple, it follows, that the last term in the previous computation is equal to $1^{\mathcal{A}}$. Consequently, $a \wedge b \neq 0$, as desired. ∎

Recall, that $L_1(T)$ is the smallest fragment of $L_{\omega_1\omega}$ containing $L_{\omega\omega}$ and the formulas $\bigwedge p$ for all types $p \in S_n(T)$ and $n < \omega$.

Lemma 3.4 *Let* $0 \neq a \in CA(L_1(T))$*. Then* $|N_a \cap \mathcal{H}(CA(L_1(T)))| = 2^{\aleph_0}$.

Proof. Let $\langle u_i : i \in \omega\rangle$ be an enumeration of the set $\{\langle x, k\rangle : x \in CA(T), k \in \omega\}$. By recursion we will construct a tree $\langle t_s : s \in {}^{<\omega}2\rangle$ such that the following stipulations hold for any $s, z \in {}^{<\omega}2$:

(a) $t_s \in L_1(CA(T)) - \{0\}$;
(b) if $s \subseteq z$ then $t_z \leq t_s$;
(c) $t_{s\frown 0} \wedge t_{s\frown 1} = 0$;
(d) $N_{t_s} \subseteq \mathcal{U}_{u_{|s|-1}}(CA(L_1(T)))$.

Let $t_{\langle\rangle} = a$ and assume, that t_s has already been defined such that (a)-(d) holds. Assume $u_{|s|} = \langle x, k\rangle$. Throughout this proof, we will work in $CA(L_1(T))$. If $t_s \wedge -c_k(x) \neq 0$, then let $b = t_s \wedge -c_k(x)$.

Next, suppose $t_s \wedge -c_k(x) = 0$, particularly $0 \neq t_s \leq c_k(x)$. We show, that there exists $n \in \omega$ such that $t_s \wedge s_{[k/n]}(x) \neq 0$. To do so, seeking a contradiction, assume that for all $n \in \omega$ we have $t_s \wedge s_{[k/n]}(x) = 0$. It follows, that $s_{[k/n]}(x) \leq -t_s$ for all $n \in \omega$. Since $CA(L_1(T))$ is an LF_ω, by fact 3 (that is, by 1.11.6(i) of [9]) we have $c_k(x) = sup\{s_{[k/j]}(x) : j \in \omega\}$. Consequently, $c_k(x) \leq -t_s$. Combining this with the first sentence of this paragraph, we obtain $t_s \leq c_k(x) =\leq -t_s$, which is impossible, because by (a) we have $t_s \neq 0$. So, there exists $n \in \omega$ such that $t_s \wedge s_{[k/n]}(x) \neq 0$; let $b = t_s \wedge s_{[k/n]}(x)$. Let i, j be distinct elements from $\omega - (\Delta(t_s) \cup \Delta(b))$ and let $t_{s^\frown 0} = b \wedge d_{i,j}$ and $t_{s^\frown 1} = b \wedge -d_{i,j}$. Then, by Lemma 3.3, $t_{s^\frown 0} \neq 0$ and $t_{s^\frown 1} \neq 0$, so (a) remains true. In addition, (b) and (d) remain true because $t_{s^\frown 0}, t_{s^\frown 1} \leq b$. Finally, (c) clearly remains true. In this way, $\langle t_s : s \in {}^{<\omega}2\rangle$ can be completely built up.

Now, because of $\mathcal{U}(CA(L_1(T)))$ is compact, for every $f \in {}^{\omega}2$ there exists an ultrafilter $\mathcal{F}_f$ in $\cap_{i\in\omega} N_{t_{f|_i}}$. It follows from (c), that $f \neq g$ implies $\mathcal{F}_g \neq \mathcal{F}_g$. Thus $\{\mathcal{F}_f : f \in {}^{\omega}2\} \subseteq N_a \cap \mathcal{H}(CA(L_1(T)))$ verifies the statement. ∎

Lemma 3.5 *Assume $0 \neq y \in CA(T)$ and $a_0, ..., a_{n-1} \in CA(L_1(T))$ are (more precisely, correspond to) critical types such that for each distinct $i, j < n$ we have $\Delta(a_i) \cap \Delta(a_j) = \emptyset$. Assume $\gamma = \Delta(y)$ is not a superset of $\Delta(a_i)$, for every $i < n$. Let $b_i \in CA(T)$ be an element that isolates $a_i|_\gamma := \{x \in CA(T) : a_i \leq x, \Delta(x) \subseteq \gamma \cap \Delta(a_i)\}$. If*

$$(*) \qquad a_0 \wedge ... \wedge a_{n-1} \wedge y = 0$$

then

$$b_0 \wedge ... \wedge b_{n-1} \wedge y = 0.$$

Proof. First observe, that for each $i < n$ there exists $a_i \leq b'_i \in CA(T)$ with $\Delta(b'_i) \subseteq \Delta(a_i)$ such that $b'_0 \wedge ... \wedge b'_{n-1} \wedge y = 0$ (otherwise the set

$$\{y\} \cup \{x \in CA(T) : (\exists i < n)(a_i \leq x \text{ and } \Delta(x) \subseteq \Delta(a_i)\}$$

would have the finite intersection property, hence it could be extended to a type i.e., a non-zero element of $CA(L_1(T))$ which would below y and a_i for all $i < n$; this would contradict to $(*)$). Next we show, that for all $i < n$ we have

$$(**) \qquad b_i \leq c_{(\Delta(a_i)-\gamma)}(b'_i).$$

Indeed, by assumption, $a_i \neq 0$. Hence

$$0 \quad < \quad b'_i \wedge a_i \quad \leq \quad c_{(\Delta(a_i)-\gamma)}(b'_i) \wedge a_i.$$

This means, that $c_{(\Delta(a_i)-\gamma)}(b'_i)$ is an at most $(\Delta(a_i) \cap \gamma)$-dimensional element of $CA(T)$ which is consistent with a_i. Since a_i corresponds to a type, it follows, that $a_i \leq c_{(\Delta(a_i)-\gamma)}(b'_i)$, and again, since this last element is at most $(\Delta(a_i) \cap \gamma)$-dimensional, we have $b_i \leq c_{(\Delta(a_i)-\gamma)}(b'_i)$, as desired. Hence $(**)$ has been established.

Let $\delta = \cup_{i<n}\Delta(a_i)$. Now, by the first paragraph of this proof, we have

$$
\begin{aligned}
0 = c_{(\delta-\gamma)}(b'_0 \wedge ... \wedge b'_{n-1} \wedge y) = \\
& c_{(\delta-\gamma)} c_{(\Delta(a_0)-\gamma)}(b'_0 \wedge c_{(\Delta(a_0)-\gamma)}(b'_1 \wedge ... \wedge b'_{n-1} \wedge y)) = \\
& c_{(\delta-\gamma)}(c_{(\Delta(a_0)-\gamma)}(b'_0) \wedge c_{(\Delta(a_0)-\gamma)}(b'_1 \wedge ... \wedge b'_{n-1} \wedge y)) \overset{(**)}{\geq} \\
& c_{(\delta-\gamma)}(b_0 \wedge c_{(\Delta(a_0)-\gamma)}(b'_1 \wedge ... \wedge b'_{n-1} \wedge y)) = \\
& c_{(\delta-\gamma)}(b_0 \wedge b'_1 \wedge ... \wedge b'_{n-1} \wedge y) = \\
& c_{(\delta-\gamma)}(b'_1 \wedge b_0 \wedge b'_2 \wedge ... \wedge b'_{n-1} \wedge y) = \\
& c_{(\delta-\gamma)} c_{(\Delta(a_1)-\gamma)}(b'_1 \wedge c_{(\Delta(a_1)-\gamma)}(b_0 \wedge b'_2 \wedge ... \wedge b'_{n-1} \wedge y)) = \\
& c_{(\delta-\gamma)}(c_{(\Delta(a_1)-\gamma)}(b'_1) \wedge c_{(\Delta(a_1)-\gamma)}(b_0 \wedge b'_2 \wedge ... \wedge b'_{n-1} \wedge y)) \overset{(**)}{\geq} \\
& c_{(\delta-\gamma)}(b_1 \wedge c_{(\Delta(a_1)-\gamma)}(b_0 \wedge b'_2 \wedge ... \wedge b'_{n-1} \wedge y)) = \\
& c_{(\delta-\gamma)}(b_1 \wedge b_0 \wedge b'_2 \wedge ... \wedge b'_{n-1} \wedge y) = ... \\
& c_{(\delta-\gamma)}(b_0 \wedge b_1 \wedge b_2 \wedge ... \wedge b_{n-1} \wedge y).
\end{aligned}
$$

Consequently, $b_0 \wedge b_1 \wedge b_2 \wedge ... \wedge b_{n-1} \wedge y = 0$, as desired. ∎

Now we present the last technical lemma which is the cornerstone in our investigations.

Theorem 3.6 *Let T be a countable, complete theory. If T has infinitely many critical types then there exists a set $\{\mathcal{F}_i : i < 2^{\aleph_0}\} \subseteq \mathcal{H}(CA(T))$ such that for any $i \neq j < 2^{\aleph_0}$ and $\varrho \in sym(\omega)$ we have $\mathcal{F}_i \neq s_\varrho^{-1}(\mathcal{F}_j)$ (in fact, there is a critical type contained in either $\mathcal{F}_i$ or $\mathcal{F}_j$, and omitted by the other). In addition, for any $i < 2^{\aleph_0}$, each equivalence class of the kernel of $\mathcal{F}_i$ is infinite.*

Proof. Let $F = \{f : f$ is an injective function, $dom(f) \subseteq \omega$ is finite, $ran(f) \subseteq \omega\}$. Let $\{\langle j_m, \sigma_m\rangle : m \in \omega\}$ be an $\aleph_0$-abundant enumeration of $\omega \times F$ (that is, each element of $\omega \times F$ occurs infinitely many times in the enumeration). In addition, let $\langle s_i : i \in \omega\rangle$ be an enumeration of the set $\{\langle x, k\rangle : x \in CA(T), k \in \omega\}$ and let $g : \omega \to \omega$ be such that $g^{-1}(j)$ is infinite for any $j \in \omega$.

We will construct trees $\langle t_p : p \in {}^{<\omega}2\rangle$, $\langle t'_p : p \in {}^{<\omega}2\rangle$ and $\langle t''_p : p \in {}^{<\omega}2\rangle$ such that the following stipulations hold for every $p, q \in {}^{<\omega}2$:

(a) $t_p \neq \emptyset$;

(b) t'_p is a critical type in $CA(L_1(T))$ or it is 1, in addition,
$t'_{p\frown 0}$ is always a critical type and $t''_p \in CA(T)$;

(c) if $p \subseteq q$ then $t_q \subseteq t_p$, in fact, for $\varepsilon \in \{0, 1\}$ we have $t_{p\frown\varepsilon} = t_p \cap N_{t'_{p\frown\varepsilon}} \cap N_{t''_{p\frown\varepsilon}}$;

(d) $t_p \subseteq \mathcal{U}_{s_{|p|-1}}(CA(L_1(T)))$;

(e) if $p \neq q$ and t'_p, t'_q are critical types then $\Delta(t'_p) \cap \Delta(t'_q) = \emptyset$ and $s_\varrho(t'_p) \neq t'_q$ for any $\varrho \in sym(\omega)$;

(f) $|\{j < |p| : g(j) = g(|p|-1)\}| \leq |\{j \in \omega : t_p \subseteq N_{d_{j,g(|p|-1)}}\}|$;

(g) if $j_{|p|-1} < |p|$ and $p(j_{|p|-1}) = 1$ then $s_{\sigma_{|p|-1}}(t'_{(p|_{j_{|p|-1}})^\frown 0}) \wedge t''_p = 0$.

Let $t_{\langle\rangle} = \mathcal{U}(CA(L_1(T)))$ and let $t'_{\langle\rangle} = t''_{\langle\rangle} = 1^{CA(L_1(T))}$. Next, suppose that $i \in \omega$ and t_p, t'_p, t''_p have already been defined for every $p \in {}^{<i}2$, such that (a)-(g) hold. Let $p \in {}^{i-1}2$ and suppose $s_{i-1} = \langle x, k\rangle$. We will split our construction to several steps.

Step 1. By (a) and Lemma 3.4 there exists $\mathcal{F} \in t_p \cap \mathcal{H}(CA(L_1(T)))$. Hence, there exists $z_p \in \mathcal{F} \cap (\{-c_k(x)\} \cup \{s_{[k/n]}(x) : n \in \omega\})$. Clearly, $\mathcal{F} \in t_p \cap N_{z_p}$, particularly, the latter is not empty. In addition, $N_{z_p} \subseteq \mathcal{U}_{s_{i-i}}(CA(L_1(T)))$. We will choose $t''_{p^\frown 0}$ and $t''_{p^\frown 1}$ below z_p hence (d) will remain true. Next, we concentrate to (g).

Step 2. If $j_{|p|} > |p|$ or $p(j_{|p|}) = 0$ then let $z'_p = z_p$. Now suppose $j_{|p|} \leq |p|$ and $p(j_{|p|}) = 1$. Let $\gamma = \sigma_{|p|}(\Delta(t'_{(p|_{j_{|p|}})^\frown 0}))$. Let $I = \{i \leq |p| : t'_{p|i}$ is a critical type $\}$. We distinguish two cases.

Case 1. First suppose, that there exists $i \in I$ such that $\Delta(t'_{p|i}) = \gamma$. Then, by (e) we have $t'_{p|i} \neq s_{\sigma_{|p|}}(t'_{(p|_{j_{|p|}})^\frown 0})$, hence, there exists $y \in CA(T)$ with $y \wedge s_{\sigma_{|p|}}(t'_{(p|_{j_{|p|}})^\frown 0}) = 0$ and $t'_{p|i} \leq y$. Since $0 \neq N_{z_p} \cap t_p \subseteq N_{t'_{p|i}}$, it follows, that $t_p \cap N_y \cap N_{z_p} \neq \emptyset$. In this case let $z'_p = z_p \wedge y$.

Case 2. Next, suppose, that for all $i \in I$ we have $\Delta(t'_{p|i}) \neq \gamma$. Then, for every $i \in I$ we have $\Delta(t'_{p|i}) \not\subseteq \gamma$, because otherwise, the critical type $s_{\sigma_{|p|}}(t'_{(p|_{j_{|p|}})^\frown 0})$ would properly contain the nonisolated type $t'_{p|i}$ as a subtype with strictly smaller dimension set. Hence, for every $i \in I$, the set

$$A_i := \{a \in CA(T) : \ t'_{p|i} \leq a, \ \Delta(a) \subseteq \gamma \cap \Delta(t'_{p|i})\}$$

is an isolated type (because $t'_{p|i}$ is critical). Let $b_i \in CA(T)$ be such that b_i isolates A_i. Let $a = z_p \wedge_{i \leq |p|} t''_{p|i}$ and let $b = a \wedge_{i \in I} b_i$. Observe, that $b = z_p \wedge \wedge_{i \leq |p|} t''_{p|i} \wedge \wedge_{i \in I} b_i \neq 0$ by step 1, and by our construction in step 2. Since b does not isolate $s_{\sigma_{|p|}}(t'_{(p|_{j_{|p|}})^\frown 0})$, there exists $0 \neq y \in CA(T)$ such that $y \leq b$, $\Delta(y) \subseteq \gamma$ and $y \wedge s_{\sigma_{|p|}}(t'_{(p|_{j_{|p|}})^\frown 0}) = 0$. We will show that

$$(*) \qquad t_p \cap N_y \cap N_{z_p} \neq \emptyset.$$

To do so, assume, seeking a contradiction, that $t_p \cap N_y \cap N_{z_p} = \emptyset$. By (b) and (c), we have $t_p = \wedge_{i \in I} t'_{p|i} \wedge \wedge_{i \leq |p|} t''_{p|i}$. Hence our indirect assumption may be rephrased as

$$(\#) \qquad \wedge_{i\in I}\, t'_{p|i} \wedge \wedge_{i\leq |p|} t''_{p|i} \wedge y \wedge z_p = 0.$$

By construction, $y \leq b \leq a = z_p \wedge \wedge_{i\leq |p|} t''_{p|i}$. Hence, $(\#)$ implies $\wedge_{i\in I} t'_{p|i} \wedge y = 0$. Thus, Lemma 3.5 may be applied to y and $a_i = t'_{p|i}$ $(i \in I)$, whence we obtain $b_0 \wedge ... \wedge b_{n-1} \wedge y = 0$. However, according to our construction, $0 \neq y \leq b \leq b_0 \wedge ... \wedge b_{n-1}$, contradicting to the previous sentence. So $(*)$ has been established. In this case let $z'_p = z_p \wedge y$. In the next step we are dealing with (e).

Step 3. Let $t'_{p^\frown 1} = 1$. Next, since p is finite, and T has infinitely many critical types, there exists $r \in CA(L_1(T))$ corresponding to a critical type, such that:

- for any $q \in {}^{\leq |p|}2$, if t'_q is a critical type and $\varrho \in sym(\omega)$ is arbitrary, then $r \neq s_\varrho(t'_q)$, and
- $\Delta(r)$ is disjoint from $\Delta(z'_p)$ and from $\Delta(t'_q) \cup \Delta(t''_q)$ for any $q \in {}^{\leq |p|}2$.

If $a, b \in CA(T)$ are such that $\Delta(a) \subseteq \Delta(t_p \wedge z'_p)$, $\Delta(b) \subseteq \Delta(r)$ and $t_p \wedge z'_p \leq a, r \leq b$, then, by Lemma 3.3 (c) we have $a \wedge b \neq 0$. Hence, the set

$$D := \{a \wedge b : \Delta(a) \subseteq \Delta(t_p),\, \Delta(b) \subseteq \Delta(r), t_p \wedge z'_p \leq a, r \leq b\}$$

has the finite intersection property. Consequently, D can be extended to a type containing $t_p \wedge z'_p$ and r. It follows, that $t_p \cap N_{z'_p} \cap N_r \neq \emptyset$. In this case let $t'_{p^\frown 1} = r$. Finally, we concentrate to (f).

Step 4. Choose $j \in \omega - (\Delta(z'_p) \cup \Delta(t'_{p^\frown 0}) \cup_{q\in {}^{\leq|p|}2} (\Delta(t'_q) \cup \Delta(t''_q)))$. (Such a j exists, because we are intending to omit the dimension sets of finitely many finite dimensional elements). Let $t''_{p^\frown 0} = t''_{p^\frown 1} = z'_p \wedge d_{j,g(|p|)}$. Then, by Lemma 3.3 we have $N_{t''_{p^\frown\varepsilon}} \cap N_{t'_{p^\frown\varepsilon}} \cap t_p \neq \emptyset$, for any $\varepsilon \in \{0,1\}$.

Finally, for $\varepsilon \in \{0,1\}$, let $t_{p^\frown\varepsilon} = t'_{p^\frown\varepsilon} \wedge t''_{p^\frown\varepsilon}$. Then it is immediate, that (a), (b) and (c) remain true. Moreover, (d) remains true by Step 1, (e) remains true by Step 3, (f) remains true (by induction and) Step 4, and finally, (g) remains true by Step 2. In this way $\langle t_p : p \in {}^{<\omega}2\rangle$, $\langle t'_p : p \in {}^{<\omega}2\rangle$ and $\langle t''_p : p \in {}^{<\omega}2\rangle$ can be completely built up.

Now, for every $f \in {}^\omega 2$ the sequence $\langle t_{f|_n} : n \in \omega\rangle$ is decreasing because of (c); in addition, by (a) and (b) consists of non-empty clopen sets in $\mathcal{U}(CA(L_1(T)))$; consequently, it has the finite intersection property. Since $\mathcal{U}(CA(L_1(T)))$ is a compact space, it follows, that there exists $\mathcal{F}(f)' \in \cap_{n\in\omega} t_{f|_n}$. Let $\mathcal{F}(f) = \mathcal{F}(f)' \cap CA(T)$. Since $\langle s_n : n \in \omega\rangle$ is an enumeration of the set $\{\langle x, i\rangle :\; x \in CA(T), i \in \omega\}$, (d) implies that $\mathcal{F}(f) \in \mathcal{H}(CA(T))$. The last sentence of the statement of the theorem is true because of (g) and because of the choice of the function g.

Finally, we claim, that if $f, g \in {}^\omega 2$ are different, and $\varrho \in sym(\omega)$ then $\mathcal{F}(f) \neq s_\varrho^{-1}(\mathcal{F}(g))$. To verify this, let $j \in \omega$ be the smallest number for which $f(j) \neq g(j)$; we may assume $f(j) = 0$ and $g(j) = 1$. By (b), $t'_{f|_j}$ is a critical type. Let $\sigma = \varrho|_{\Delta(t'_{f|_j})}$. Since the enumeration of $\omega \times F$ fixed at the beginning of the proof is $\aleph_0$-abundant, there exists $m \geq j$ such that $\langle j_m, \sigma_m\rangle = \langle j, \sigma\rangle$. Let $p = g|_{m+1}$. Now observe that

- $j_{|p|-1} = j_m = j \leq m < m+1 = |p|$ and

- $p(j_{|p|-1}) = p(j_m) = p(j) = g(j) = 1$.

Hence, by (g), we have

$$(***) \qquad 0 = s_{\sigma_{|p|-1}}(t'_{(p|_{j_{|p|-1}})^\frown 0}) \wedge t''_p = s_{\sigma_m}(t'_{(p|_j)^\frown 0}) \wedge t''_p = s_\sigma(t'_{(f|_j)^\frown 0}) \wedge t''_p.$$

By (b) we have, that $s_\sigma(t'_{(f|_j)^\frown 0})$ is a critical type and $t''_p \in CA(T)$. Then there exists $a \in CA(T)$ with $s_\sigma(t'_{(f|_j)^\frown 0}) \leq a, \Delta(a) \subseteq \Delta(s_\sigma(t'_{(f|_j)^\frown 0}))$ and $a \wedge t''_p = 0$ (otherwise the set

$$\{a \wedge t''_p \in CA(T) : s_\sigma(t'_{(f|_j)^\frown 0}) \leq a, \Delta(a) \subseteq \Delta(s_\sigma(t'_{(f|_j)^\frown 0}))\}$$

would have the finite intersection property, hence it could be extended to a type containing $s_\sigma(t'_{(f|_j)^\frown 0})$ and t''_p, contradicting to $(***)$). Now $s_\sigma^{-1}(a) \in \mathcal{F}(f)$, because $t'_{(f|_j)^\frown 0} \in \mathcal{F}(f)'$. In addition, since $t_{g|_{m+1}} \leq t''_p \in CA(T)$, it follows, that $t''_p \in \mathcal{F}(g)$. By the choice of a, we have $a \wedge t''_p = 0$, so $a \notin \mathcal{F}(g)$, consequently, $s_\sigma^{-1}(a) \notin s_\sigma^{-1}(\mathcal{F}(g))$. This observation implies $\mathcal{F}(f) \neq s_\sigma^{-1}(\mathcal{F}(g))$, as desired. In addition, it is easy to see, that $\mathcal{F}(f)$ contains the critical type $t'_{(f|_j)^\frown 0}$ and $\mathcal{F}(g)$ omits it. ∎

Now we are presenting our main result.

Theorem 3.7 *(Main theorem of the paper.)*

(1) If T is a complete first order theory having infinitely many critical types, then T has continuum many countable models, which are paorwise separable by critical types. Particularly, $I(T, \aleph_0) = 2^{\aleph_0}$.

(2) If there exists a set $\{\mathcal{A}_i : i < \aleph_1\}$ of countable models of T which are pairwise separable by critical types, then T has continuum many countable models, which are paorwise separable by critical types. Particularly, $I(T, \aleph_0) = 2^{\aleph_0}$.

Proof. We start by prove (1).

By Theorem 3.6 there exists a set $\{\mathcal{F}_i : i < 2^{\aleph_0}\} \subseteq \mathcal{H}(CA(T))$ such that for any $i \neq j < 2^{\aleph_0}$ and $\varrho \in sym(\omega)$ we have $\mathcal{F}_i \neq s_\varrho^{-1}(\mathcal{F}_j)$. In addition, for any $i < 2^{\aleph_0}$, each equivalence class of the kernel of $\mathcal{F}_i$ is infinite. For each $i < 2^{\aleph_0}$ let $\mathcal{B}_i$ be the image of $CA(T)$ under $rep_{\mathcal{F}_i}$. Suppose $i \neq j < 2^{\aleph_0}$ and assume, seeking a contradiction, that $\varrho : \omega \to \omega$ induces a base isomorphism between $\mathcal{B}_i$ and $\mathcal{B}_j$. Since the kernels of $\mathcal{F}_i$ and $\mathcal{F}_j$ consist of infinitely many infinite equivalence classes, we may assume, that ϱ is a bijection. Hence, by Theorem 3.2 $\mathcal{F}_i = s_\varrho^{-1}(\mathcal{F}_j)$ would follow, contradicting to the first sentence of this paragraph. So $CA(T)$ has continuum many set algebras as homomorphic images that are pairwise not base isomorphic and each of which has a countable base set. This is equivalent with $I(T, \aleph_0) = 2^{\aleph_0}$. Theorem 3.6 also implies, that the countable models corresponding to $\{\mathcal{B}_i : i < 2^{\aleph_0}\}$ are pairwise separable by critical types.

Now we turn to (2). Assume $\{\mathcal{A}_i : i < \aleph_1\}$ is a set of countable models of T

which are pairwise separable by critical types. To complete the proof, it is enough to show, that (1) can be applied: in this case T has infinitely many critical types.

Suppose, seeking a contradiction, that there are finitely many critical types only, and enumerate them as $\{p_n : i < N\}$. For each $i < \aleph_1$ and $n < N$ let $\nu_i(n)$ be the cardinality of realizations of p_n in $\mathcal{A}_i$. Then $\nu_i : N \to \omega + 1$ is a function, and if $i \neq j$, then $\nu_i \neq \nu_j$ because $\mathcal{A}_i$ and $\mathcal{A}_j$ can be separated by critical types. But $\{\nu_i : i < \aleph_1\} \subseteq {}^N(\omega + 1)$; the first set is of cardinality $\aleph_1$, while the cardinality of the second one is $\aleph_0$, only. This contradiction completes the proof. ∎

4 Concluding Remarks

In this section we present some further questions and research directions which remained open. We believe, that the methods of this paper may be further developed to handle them.

It would also be interesting to study Vaught's conjecture for first order languages without equality. The algebraic counterparts of these logics are the so called diagonal-free cylindric algebras. They also have a well established representation theory, but substitution operations no longer term-definable in them. Hence, one should add them to the signature and study the so called polyadic and substitution algebras (see e.g. [16]). It seems plausible, that further study of different versions of substitution algebras would be useful in order to adapt the method of this paper for the case of logics without equality.

Suppose T is not a complete theory. Instead of taking countably infinite models of T, one then can take finite models as well. It also would be interesting to say something about the number of pairwise non-isomorphic models of T of a fixed finite cardinality. Our methods also seems to be applicable. For completeness, we mention, that with a completely different technique we already started to study some extensions of Morley's categoricity theorem to the finite: if T is $\aleph_1$-categorical (but not necessarily $\aleph_0$-categorical), then, under some further technical conditions, large enough finite portions of T have unique n-element models for any large enough finite $n \in \omega$. For more details we refer to [8].

Finally, turning back to the original problem, we concluded to the following conjecture: if T has uncountably many countable models which are pairwise non elementary embedable into each other, then there are $2^{\aleph_0}$ many countable models of T which are pairwise non elementarily embedable into each other. It seems, that probably this question may also be answered in the affirmative by further developing the techniques presented in this paper. In [17] related investigations have been started. In this paper we are dealing with the case of equation-free logics mentioned in the second paragraph in this section, as well, as the number of countable models which are pairwise not elementarily embeddable into each other. Endow ω with the discrete topology and ${}^\omega\omega$ with the product topology. In [17] we show, that if

$G \subseteq {}^\omega\omega$ is a σ-compact monoid (that is, G is a monoid whose universe is a union of countably many compact subsets of ${}^\omega\omega$) and T has $\aleph_1$ many countable models which are pairwise not elementarily embeddable into each other by elements of G, then T has continuum many such models. Since we assume, that G is a monoid only (and not a group), classical results of descriptive set theory on actions of Polish groups are no longer available; however a cylindric algebraic approach is working in that setting, too.

Further related investigations on Vaught's conjecture on the basis of algebraic logic may also be found in [18]. There, among others, we are giving a cylindric algebraic proof of Morley's celebrated theorem: $I(T,\aleph_0) > \aleph_1$ implies $I(T,\aleph_0) = 2^{\aleph_0}$.

Finally, we mention a further research direction which remained open. We are asking if further properties of the spectrum function $I(T,\kappa)$ could also be established with cylindric algebraic methods. We are particularly interested in estimates for $I(T,\kappa)$, for uncountable κ.

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