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On the Convergence of Fourier Series of U.A.P. Functions*

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In [9] we investigated the problem of finding convergent rearrangements to Fourier series of continuous periodic functions. For motivation and history we refer to that paper, and here work out only the most important steps of the generalization to the multidimensional case. Our goal is not to provide the detailed investigation of this generalization itself, but to arrive at a convergent rearrangement theorem for uniformly almost periodic functions. These are abbreviated as u.a.p. functions, following the classical monograph of Besicovitch [1], which provides the general reference throughout. In the same book on pages 51 and 52 we can find two theorems on absolute convergence of the Fourier series of a u.a.p. function. Since in general absolute convergence does not hold, convergence depends on the ordering of the—in general unordered—spectrum of f, i.e.,

$$\Lambda(f) := \{\lambda \in \mathbb{R} : c(\lambda) \neq 0\} \tag{1}$$

where

$$c(\lambda) := M_x(f(x)e^{-i\lambda x}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)e^{-i\lambda x}\,dx.$$

The Fourier series of any u.a.p. function f is the formal, unordered sum of the corresponding terms, that is,

$$\sum_{\lambda\in\Lambda(f)} c(\lambda)e^{i\lambda x},$$

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and an ordering of the spectrum, say $\nu: \mathbb{N} \leftrightarrow \Lambda(f)$, defines the partial sums

$$_{\nu}S_n(f, x) = S_n(x) = \sum_{j=1}^{n} c_j e^{i\lambda_j x} \qquad (c_j = c(\lambda_j), \lambda_j = \nu(j)). \tag{2}$$

We shall prove that for any f there exists an ordering of the spectrum, for which some subsequence $n_k \to \infty$ of $\mathbb{N}$ satisfies $S_{n_k} \to f$ uniformly on $\mathbb{R}$. In this paper $\to$ means uniform convergence and $\|\cdot\|_\infty$ is for uniform norm.

In this respect we may note that Bohr (see [1, p. 46]) used the process of ordering the spectrum and choosing some subsequence of the corresponding partial sums in order to obtain the corresponding theorem in the special case of limit periodic functions satisfying, e.g., Lipschitz conditions.

On the other hand a considerable effort has been spent in the direction of a strict generalisation of the periodic case, i.e., handling the Fourier series by taking the "natural ordering" of the spectrum. In this way partial sums of the form

$$S_\omega(f, x) := \sum_{\substack{\lambda \in \Lambda(f) \\ |\lambda| \leqslant \omega}} c(\lambda) e^{i\lambda x}$$

occur which may contain an infinity of terms and so even the existence and almost periodicity of $S_\omega(f)$ is a problem. This latter problem was avoided by Bohr by supposing that the points of the spectrum are well spaced and under this assumption he deduced a convergence theorem ([1, pp. 40–42]). Later Bochner [2] improved this by considering the more general case when the spectrum has no finite accumulation point, and obtained convergence theorems under assumptions on the growth of the spectrum sequence or the coefficients.

Levitan [6] extended the investigations to the case of the existence of one limit point of the spectrum. Bredihina devoted several papers to the question and obtained optimal criteria of convergence. For her work see [5] and the references there.

So there are two approaches, both originating in Bohr's works. The first considers some suitable ordering of the spectrum while the second uses some natural ordering. The second is technically hard and generalizes the well-known criteria for the periodic case. But in general even the existence of partial sums is not clear and despite all the success mentioned, the state of the matter is far from giving results in the full generality of u.a.p. functions. The first, up to now, was much less developed but here it turns out that by this approach every u.a.p. function can be expressed as the the limit of its partial sums. On the other hand it breaks the rule of ordering according to modulus and so specializing to the periodic case leads only to the theorem in [9] dealing with the rearrangements of the classical Fourier series of a continuous periodic function of $\mathbb{R}$. For details concerning this special case see Section 4 and [9].

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LEMMA 1 (Bernstein's Inequality). *If Ω is a probability space and $X: \Omega \to \mathbb{R}$ is a random variable, then for any $\varepsilon > 0$, $\lambda > 0$ we have*

$$P(|x| > \varepsilon) \leqslant e^{-\lambda\varepsilon}\{E(e^{\lambda x}) + E(e^{-\lambda x})\}.$$

Proof.

$$P(x > \varepsilon) = \int_{\{x > \varepsilon\}} dP \leqslant e^{-\lambda\varepsilon} \int_{\{x > \varepsilon\}} e^{\lambda x}\, dP \leqslant e^{-\lambda\varepsilon} E(e^{\lambda x}).$$

Adding this and the same estimate for $-X$ in place of X we get Lemma 1.

LEMMA 2. *For any $z \in \mathbb{C}$ and $0 \leqslant \alpha \leqslant 1$ we have*

$$|\alpha e^{(1-\alpha)z} + (1-\alpha)e^{-\alpha z}| \leqslant e^{|z|^2}.$$

Proof. For $|z| > 1$ trivially and for $|z| \leqslant 1$ use the Taylor expansion.

Let $T^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$ be the d-dimensional torus and $C(T)^d$ the space of continuous functions on it. We write $\mathbf{v}$ for $(v_1, ..., v_d)$ in $\mathbb{R}^d$, T^d, or $\mathbb{Z}^d$, and for $\mathbf{v} \in \mathbb{R}^d$ or $\mathbb{Z}^d$ put $\|\mathbf{v}\| = \max\{v_l : l = 1, ..., d\}$. We may use the inner product $\langle \mathbf{v}, \mathbf{u} \rangle = \sum_{l=1}^d v_l u_l$ for $\mathbf{v} \in \mathbb{Z}^d$ and $\mathbf{u} \in T^d$, in which case $\langle \mathbf{v}, \mathbf{u} \rangle$ is defined mod 2π and $e^{i\langle \mathbf{v}, \mathbf{u} \rangle}$ is uniquely determined.

We call a trigonometric polynomial

$$T(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} a(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} \tag{3}$$

of degree n, if $\max\{\|\mathbf{k}\| : \mathbf{k} \in \mathbb{Z}^d,\ a(\mathbf{k}) \neq 0\} = n$. For any $g \in C(T^d)$ the Fourier series of g is

$$g(\mathbf{x}) \sim \sum_{\mathbf{k} \in \mathbb{Z}^d} b(\mathbf{k}) e^{i\langle k, \mathbf{x} \rangle} \left(b(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{T^d} g(\mathbf{x}) e^{-i\langle \mathbf{k}, \mathbf{x} \rangle}\, d\mathbf{x} \right). \tag{4}$$

If $n \in \mathbb{N}$ and $\mathbf{n} = (n_1, ..., n_d) \in \mathbb{N}^d$, the $\mathbf{n}$th Fejér and the nth De la Vallée Poussin means are

$$\sigma_{\mathbf{n}}(g, x) = \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ |k_j| \leqslant n_j (j = 1, ..., d)}} \left(1 - \frac{|k_1|}{n_1}\right) \cdot \cdots \cdot \left(1 - \frac{|k_d|}{n_d}\right) b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle}$$

and

$$V_n(g, x) = \sum_{\mathbf{e} \in \{0,1\}^d} (-1)^{d+e_1+\cdots+e_d} \sigma_{((1+e_1)n, \ldots, (1+e_d)n)}(g, x)$$
$$= \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ |k_j| \leqslant 2n (j=1,\ldots,d)}} p_n(\mathbf{k}) b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle}, \tag{5}$$

where

$$p_n(\mathbf{k}) = \prod_{l=1}^{d} p_n(k_l), \qquad p_n(m) = \begin{cases} 1, & |m| \leqslant n \\ 2 - |m|/n, & n < |m| \leqslant 2n \end{cases}$$

so that $0 \leqslant p_n(\mathbf{k}) \leqslant 1$ and $V_n \to g$ $(n \to \infty)$ according to the first expression in (5) and $\sigma_{\mathbf{n}} \to g$ $(n_1, \ldots, n_d \to \infty)$. See, e.g., [7, Sect. 22].

LEMMA 3 (Bernstein's Inequality for T^d). *If a trigonometric polynomial T of d variables is of degree N, then for $1 \leqslant l \leqslant d$*

$$\left\| \frac{\partial T}{\partial x_l} \right\|_\infty \leqslant N \|T\|_\infty.$$

Proof. When $d = 1$ this is just the theorem of (3.13) in Chapter X of [10]. Varying only x_l and fixing the other variables as parameters the general case follows immediately.

LEMMA 4. *For a T as in (3) of degree N, and*

$$\mathbf{x}(\mathbf{m}) = \left(\frac{m_1 \pi}{M}, \ldots, \frac{m_d \pi}{M} \right) \qquad (\|\mathbf{m}\| \leqslant M, M > \pi dN) \tag{7}$$

we have the inequality

$$\|T\|_\infty \leqslant 2 \max_{\|\mathbf{m}\| \leqslant M} |T(\mathbf{x}(\mathbf{m}))|.$$

Proof. For any $\mathbf{x} \in T^d$ there is an $\mathbf{x}(\mathbf{m})$ for which

$$|x_l - x(\mathbf{m})_l| \leqslant \frac{\pi}{2M} \qquad (l = 1, \ldots, d).$$

Therefore with this $\mathbf{x}(\mathbf{m})$ by Lemma 3

$$|T(\mathbf{x}) - T(\mathbf{x}(\mathbf{m}))| \leqslant \frac{d\pi}{2M} \max_{l=1,\ldots,d} \left\| \frac{\partial T}{\partial x_l} \right\|_\infty < \frac{1}{2} \|T\|_\infty,$$

whence our statement.

LEMMA 5. *If φ is a nonnegative finite Borel measure on $[1, \infty)$, then there are $n_k \to \infty$ for which $n_{k+1} \geqslant 2n_k$ and*

$$\int_{n_k}^{2n_k} d\varphi = o\left(\frac{1}{\log n_k}\right).$$

Proof. Otherwise there would be $\delta > 0$ and n_0 for which

$$\int_{n_0}^{\infty} d\varphi = \sum_{j=0}^{\infty} \int_{2^j n_0}^{2^{j+1} n_0} d\varphi > \sum_{j=0}^{\infty} \frac{\delta}{\log(2^j n_0)} > \frac{\delta}{\log n_0} \sum_{j=0}^{\infty} \frac{1}{j+1} = \infty,$$

a contradiction.

THEOREM 1. *Let $g \in C(T^d)$ with Fourier series (4). Then there exists a reordering $\sigma: \mathbb{Z}^d \leftrightarrow \mathbb{N}$ for which the partial sums*

$$_\sigma S_n(g, \mathbf{x}) = \sum_{\sigma(\mathbf{k})=1}^{n} b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} \tag{8}$$

of the σ-rearranged Fourier series of g have a subsequence

$$_\sigma S_{N_m}(g, \quad) \to g \quad (N_m \to \infty).$$

Proof. Let $\varphi([1, t]) := \sum_{\|\mathbf{k}\| \leqslant t} |b^2(\mathbf{k})|$. Then Lemma 5 applies, and so with some $\eta_m \to 0$, $n_m \to \infty$ we have

$$\sum_{n_m \leqslant \|\mathbf{k}\| < 2n_m} |b^2(\mathbf{k})| < \frac{\eta_m}{\log n_m}. \tag{9}$$

We define σ to map $\{\|\mathbf{k}\| \leqslant n_m\}$ to $[1, (2n_m+1)^d]$ and to satisfy for a certain $H_m \subset \{\mathbf{k} \in \mathbb{Z}^d : n_m < \|\mathbf{k}\| < 2n_m\} = K_m$ the relation $\sigma(\mathbf{k}) < \sigma(\mathbf{k}')$ if $\mathbf{k} \in H_m$, $\mathbf{k}' \in K_m \setminus H_m$. That is, σ counts first the elements of H_m, and only thereafter the remaining indices of K_m. We define $N_m = (2n_m + 1)^d + |H_m|$, where H_m is still to be chosen. We have thus

$$_\sigma S_{N_m}(g, \mathbf{x}) = \sum_{\|\mathbf{k}\| \leqslant n_m} b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} + \sum_{\mathbf{k} \in H_m} b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} \tag{10}$$

Now, since $n_{m+1} \geqslant 2n_m$ and $V_{n_m}(g) \to g$, we see by (5), (6), (9), and (10) that it suffices to show the following

LEMMA 6. *If for some $\eta > 0$ and $n > 28d$ we have*

$$\sum_{n < \|\mathbf{k}\| < 2n} |b(\mathbf{k})|^2 < \frac{\eta}{\log n}; \tag{11}$$

then for some characteristic sequence $\omega_{\mathbf{k}} \in \{0, 1\}$ $(n < \|\mathbf{k}\| < 2n)$

$$\left| V_n(g, \mathbf{x}) - \sum_{\|\mathbf{k}\| \leqslant n} b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} - \sum_{n < \|\mathbf{k}\| < 2n} \omega_{\mathbf{k}} b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} \right| < 8d\sqrt{\eta} \tag{12}$$

holds for all $\mathbf{x} \in T^d$.

Proof. Take the probability space

$$\Omega = \{0, 1\}^{\{\mathbf{k} : \|\mathbf{k}\| < 2n\}} = \{\omega = (..., \omega_{\mathbf{k}}, ...), \omega_{\mathbf{k}} \in \{0, 1\}, \|\mathbf{k}\| < 2n\} \tag{13}$$

and define the probability measure

$$P(\omega) := \prod_{\|\mathbf{k}\| < 2n} P_{\mathbf{k}}(\omega), \qquad P_{\mathbf{k}}(\omega) := P_{\mathbf{k}}(\omega_{\mathbf{k}}),$$
$$P_{\mathbf{k}}(1) = p_n(\mathbf{k}). \tag{14}$$

This measure is clearly defined so that the coordinate functions or projections

$$X_{\mathbf{k}}(\omega) := \omega_{\mathbf{k}}(\in \{0, 1\}) \tag{15}$$

are independent random variables on Ω and so for any fixed $\mathbf{x} \in T^d$ the random variables

$$Y_{\mathbf{k}}(\mathbf{x}, \omega) := X_{\mathbf{k}}(\omega) b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} = \omega_{\mathbf{k}} b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} \tag{16}$$

are independent random variables on Ω. For the sum

$$Y(\mathbf{x}, \omega) = \sum_{\|\mathbf{k}\| < 2n} Y_{\mathbf{k}}(\mathbf{x}, \omega) \tag{17}$$

the definitions (13), (14), (16), (17), and (5), (6) immediately give the expectation

$$E(Y(\mathbf{x}, \omega)) = \sum_{\|\mathbf{k}\| < 2n} E(Y_{\mathbf{k}}(\mathbf{x}, \omega))$$
$$= \sum_{\mathbf{k}} p_n(\mathbf{k}) b(\mathbf{k}) e^{i\langle \mathbf{k}, \mathbf{x} \rangle} = V_n(g, \mathbf{x}). \tag{18}$$

So our aim is to find some $\omega \in \Omega$ for which the random variable (17) is uniformly close to its expectation (18). If that happens with a positive probability, then we are done, since for $\|\mathbf{k}\| \leqslant n$, $p_n(\mathbf{k}) = 1$ and so $P(\omega_{\mathbf{k}} = 0) = 0$. For any fixed $\mathbf{x} \in T^d$, Lemma 1 implies

$$P(|Y(\mathbf{x}, \omega) - V_n(\mathbf{x})| \geqslant \varepsilon) < e^{-\lambda\varepsilon}\{E(e^{\lambda(Y(\mathbf{x}, \omega) - V_n(\mathbf{x}))})$$
$$+ E(e^{\lambda(V_n(\mathbf{x}) - Y(\mathbf{x}, \omega))})\}. \tag{19}$$

Using independence we obtain

$$E(e^{\lambda(Y(\mathbf{x},\omega)-V_n(\mathbf{x}))}) = \prod_{\|\mathbf{k}\|<2n} E(e^{\lambda(Y_{\mathbf{k}}(\mathbf{x},\omega)-p_n(\mathbf{k})b(\mathbf{k})e^{i\langle \mathbf{k},\mathbf{x}\rangle})})$$
$$= \prod_{\|\mathbf{k}\|<2n} \{p_n(\mathbf{k})e^{(1-p_n(\mathbf{k}))\lambda b(\mathbf{k})e^{i\langle \mathbf{k},\mathbf{x}\rangle}}$$
$$+(1-p_n(\mathbf{k}))e^{-p_n(\mathbf{k})\lambda b(\mathbf{k})e^{i\langle \mathbf{k},\mathbf{x}\rangle}}\}.$$

For $\|\mathbf{k}\| \leqslant n$, $p_n(\mathbf{k})=1$ shows that the factor in $\{\ \}$ is 1, and for $n<\|\mathbf{k}\|<2n$ applying Lemma 2 with $\alpha=p_n(\mathbf{k})$, $z=\lambda b(\mathbf{k})e^{i\langle \mathbf{k},\mathbf{x}\rangle}$ we get

$$E(^{\lambda(Y(\mathbf{x},\omega)-V_n(\mathbf{x}))}) \leqslant \prod_{n<\|\mathbf{k}\|<2n} e^{\lambda^2|b^2(\mathbf{k})|}$$
$$=\exp\left(\lambda^2 \sum_{n<\|\mathbf{k}\|<2n} |b^2(\mathbf{k})|\right).$$

A similar calculation for the last term in (19) proves

$$P(|Y(\mathbf{x},\omega)-V_n(\mathbf{x})| \geqslant \varepsilon) < 2\exp\left(\lambda^2 \sum_{n<\|\mathbf{k}\|<2n} |b(\mathbf{k})|^2 - \lambda\varepsilon\right). \tag{20}$$

If we use this for L different points, (11) and (20) imply

$$P(|Y(\mathbf{x},\omega)-V_n(\mathbf{x})|<\varepsilon \text{ for all the } L \text{ points})$$
$$>1-2L\exp\left(\frac{\lambda^2\eta}{\log n}-\lambda\varepsilon\right). \tag{21}$$

Choose $M=14\,dn$, and the $L=(2M+1)^d$ points as in (7), and apply Lemma 3 with $N=2n$ and $T=Y(\ ,\omega)-V_n$ to infer

$$P(\|Y(\ ,\omega)-V_n\|_\infty<2\varepsilon)>1-2(14\,dn)^d\exp\left(\frac{\lambda^2\eta}{\log n}-\lambda\varepsilon\right). \tag{22}$$

Taking

$$\lambda=\frac{\log n}{\sqrt{\eta}}, \qquad \varepsilon=4d\sqrt{\eta}, \qquad n>28d, \tag{23}$$

(12) is proved with the probability not less than

$$1-\exp(2d\log n)\exp(\log n-4d\log n)=1-\frac{1}{n^{2d-1}}>0.$$

3

Now we turn to a u.a.p. function f. Recall the definitions of homogeneous sets of u.a.p. functions [1, p. 43], base of f (p. 34), Bochner–Fejér polynomials (p. 47), limit periodic functions (p. 35), and the connection between convergence in mean and uniform convergence in the case of a homogeneous set (p. 43). Note that if A is a homogeneous set, then so is its uniform closure, since if $f_n \in A$, $f_n \to f$, then $E\{\varepsilon, f\} \supset E\{\varepsilon, A\}$.

LEMMA 7. *For any f, u.a.p. on $\mathbb{R}$, there is a sequence $\{\beta_1^{(n)}, ..., \beta_{d_n}^{(n)}\} = B_n$ of linearly independent reals such that the functions*

$$g_n(x) := \sum_{\lambda \in \langle B_n \rangle \cap \Lambda(f)} c(\lambda) e^{i\lambda x}$$

$$(\langle B_n \rangle := \{r_1 \beta_1^{(n)} + \cdots + r_d \beta_d^{(n)} : r_l \in \mathbb{Z}, l = 1, ..., d\}$$

exist, are u.a.p. functions, and $g_n \to f$.

Proof. The set A of f and all of its Bochner–Fejér polynomials are homogeneous, hence so is its closure, and whatever B_n is, g_n can be represented as the mean—whence as the uniform—limit of a suitable chosen sequence of Bochner–Fejér polynimials from A. So g_n is equicontinuous and equialmost-periodic with the set A, proving that g_n is u.a.p. Moreover, for any choice of B_n satisfying

$$\langle B_{n+1} \rangle \supset B_n \qquad \text{and} \qquad \Lambda(f) \subset \bigcup_1^\infty \langle B_n \rangle \tag{24}$$

we see that g_n tends to f in mean—whence, by homogeneity, uniformly, too.

LEMMA 8. *For any $\varepsilon > 0$ and h of the form*

$$h(x) = \sum_{\lambda \in \Lambda(f) \cap \langle B \rangle} c(\lambda) e^{i\lambda x} \qquad (B = \{\beta_1, ..., \beta_d\} \text{ are linearly independent})$$

there exists a finite subset $H \subset \Lambda(f) \cap \langle B \rangle$ for which the polynomial

$$P(x) = \sum_{\lambda \in H} c(\lambda) e^{i\lambda x}$$

satisfies

$$\|h - P\|_\infty < \varepsilon.$$

Proof. Let $\mathbf{r} = (r_1, ..., r_d) \in \mathbb{Z}^d$, $\boldsymbol{\beta} = (\beta_1, ..., \beta_d)$, and $\lambda = r_1\beta_1 + \cdots + r_d\beta_d = \langle \mathbf{r}, \beta \rangle$. We denote by $b(\mathbf{r})$ the given $c(\langle \mathbf{r}, \boldsymbol{\beta} \rangle) = c(\lambda)$, and introduce

$$g(\mathbf{x}) = \sum_{r \in \mathbb{Z}^d} b(r) e^{i\langle r, \mathbf{x} \rangle} \qquad (\mathbf{x} \in T^d). \tag{25}$$

Since h is the uniform limit of its Bochner–Fejér polynomials, the representing Bochner–Fejér polynomial sequence is uniformly Cauchy and tends to h formally[1], too. Transforming these polynomials in the same way as g is obtained from h, we obtain a polynomial sequence on T^d. The transformed sequence tends formally to g and (see [1, p. 36]) is uniformly Cauchy, that is, $g \in C(T^d)$, and is the uniform limit of this transformed sequence (which is, in fact, a Fejér polynomial sequence of g). Now an application of Theorem 1 and the inverse transformation furnishes the required P. (This inverse transformation sends a function $\varphi: T^d \to \mathbb{C}$ to $\psi: \mathbb{R} \to \mathbb{C}$ defined by $\psi(x) = \varphi(\beta_1 x, ..., \beta_d x)$; i.e., we only take the "diagonal function.")

We are now in the position to prove

THEOREM 2. *For any u.a.p. function f there exists an ordering ν of the spectrum* (1) *and a sequence $M_k \to \infty$ for which the ν-ordered partial sums defined in* (2) *satisfy ${}_\nu S_{M_k}(f) \to f$.*

Proof. First choose B_n and g_n as in Lemma 7. Assume next that $M_0 = 0$, $M_1, ..., M_k$ are already defined, and so are the values of ν on the interval $[1, M_k]$. Pick an n_k so that $\langle B_{n_k} \rangle \supset \{\lambda : \nu^{-1}(\lambda) \leqslant M_k\}$. If $\varepsilon = \varepsilon_k$ is small enough, the spectrum H_k of the polynomial $P = P_k$ provided by Lemma 8 must contain all the numbers λ_j with $j = \nu^{-1}(\lambda_j) \leqslant M_k$, since $\|g_{n_k} - P_k\|_\infty \geqslant |c(\lambda)|$ for all $\lambda \in \Lambda(g_{n_k}) \setminus H_k$. Let M_{k+1} be the cardinality of H_k, and define ν on $[M_k + 1, M_{k+1}]$ to be an arbitary bijection onto $H_k \setminus \{\lambda : \nu^{-1}(\lambda) \leqslant M_k\}$. If $\varepsilon_k \to 0$, then this procedure clearly defines a correct ν, since ${}_\nu S_{M_k}(f) = P_k$ tends to f.

4

Let us now mention some problems.

Problem 1. Is it true that for some proper ordering of the spectrum even $S_n(f) \to f$ holds?

[1] That is Fourier-coefficientwise.

Problem 2. Does there exist a constant C such that for every polynomial $f(x)=\sum_{\lambda\in\Lambda(f)} c(\lambda)e^{i\lambda x}$, $|\Lambda(f)|=N<\infty$ there exists $\sigma\colon \Lambda(f)\leftrightarrow[1,N]$ for which with $\lambda_j=\sigma^{-1}(j)$ we have

$$\max_{1\leqslant n\leqslant N}\left\|\sum_{j=1}^{n} c(\lambda_j)e^{i\lambda_j x}\right\|_\infty \leqslant C\,\|f\|_\infty ?$$

Problem 3. Does there exist for every $d\in\mathbb{N}$ a constant $C(d)$ with the property that given any $K\in\mathbb{N}$ and any $P(\mathbf{x})\in C(T^d)$, $P(\mathbf{x})=\sum_{\|\mathbf{k}\|_\infty\leqslant K} a(\mathbf{k})e^{i\langle\mathbf{k},\mathbf{x}\rangle}$, there is $\sigma\colon[-K,K]^d\leftrightarrow[1,(2K+1)^d]$ for which

$$\max_{1\leqslant n\leqslant(2K+1)^d}\left\|\sum_{j=1}^{n} a(\sigma^{-1}(j))e^{i\langle\sigma^{-1}(j),\mathbf{x}\rangle}\right\|_\infty < C(d)\,\|P\|_\infty ?$$

Comments. Using Theorem 2 it can be shown analogously to the considerations in Section 4 of [9] that Problems 1 and 2 are equivalent. Another equivalent problem is to state Problem 2 for every u.a.p. function, not only for polynomials. The seemingly weaker assertion of Problem 3 may be hard too. Even the important special case $d=1$ is open (see [9]).

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