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REARRANGEMENT OF FOURIER SERIES AND FOURIER SERIES WHOSE TERMS HAVE RANDOM SIGNS

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1. Introduction

Let us denote by $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$ the one dimensional torus, $L^2 := L^2(\mathbf{T})$, and $C := C(\mathbf{T})$ the sets of square (Lebesgue) integrable functions and continuous functions, resp., and $\varepsilon = (\varepsilon_k)$ a Rademacher system on $I = [0,1]$. In the probability space $(I, \mathcal{L}, P)$ belonging to the Rademacher system ε, $\mathcal{L}$ is the algebra of Lebesgue measurable sets in I and the probability measure P is the Lebesgue measure on I. The expectation with respect to this probability space will be denoted by E throughout the paper.

Following Zygmund [12] we write for the Fourier series of any $f \in L^2$

$$(1) \qquad \begin{cases} f \sim S(f,\cdot) := \sum_{n=0}^{\infty} A_n, \quad S_n := S_n(f,\cdot) := \sum_{j=0}^{n} A_j, \\ A_j(x) := c_j \cos(jx + \Theta_j) := a_j \cos jx + b_j \sin jx. \end{cases}$$

The series coming from f by giving random signs to its terms is

$$(2) \qquad f_\varepsilon \sim \sum_{n=0}^{\infty} \varepsilon_n A_n,$$

and the Pisier-algebra $\mathcal{P}$ is

$$(3) \qquad \mathcal{P} := \{f \in L^2 : \quad P(f_\varepsilon \in C) = 1\}.$$

The characterization of $\mathcal{P}$ was a long-standing problem of the theory of Fourier series initiated by Payley and Zygmund [5] in 1930. For history and development we refer to [3]. The problem was finally solved by Marcus and Pisier [4] in 1978.

Another old but still open problem is the following. Denote $\nu : \mathbf{N} \leftrightarrow \mathbf{N}$ any permutation of $\mathbf{N}$, and introduce for any f with Fourier series (1) the

ν-rearrangement of the series and the corresponding partial sums as

$$(4)\qquad f \underset{\nu}{\sim} {}_{\nu}S(f) := \sum_{n=0}^{\infty} A_{\nu(n)}, \quad {}_{\nu}S_n(f) := \sum_{j=0}^{n} A_{\nu(j)}.$$

The class

$$(5)\qquad U := \{f \in C : \exists \nu : \mathbf{N} \leftrightarrow \mathbf{N}, \quad {}_{\nu}S_n \to f \quad \text{uniformly on} \quad \mathbf{T}\}$$

is a subspace of C. The problem of deciding if $U = C$ or not, was posed already in 1962 by Ulyanov, cf. [10] pp. 58–59, or [9].

In 1986 the following result was proved [6]. For any $f \in C$ there exist a rearrangement ν and a subsequence (n_k) of $\mathbf{N}$ such that ${}_{\nu}S_{n_k} \to f$ $(k \to \infty)$ uniformly on $\mathbf{T}$. On the basis of this and other results the conjecture $U = C$ was formulated and an equivalent finite version was given in [6], where the reader may find more about historical background and motivation of the problem.

The aim of the present paper is to study the connection between the classes $\mathcal{P}$ and U.

THEOREM 1 (Pecherskiĭ [13]). $\mathcal{P} \cap C \subset U$.

THEOREM 2. *There exists* $f \in U$ *with* $f \notin \mathcal{P}$.

As a by-product we obtain several other criteria for an $f \in C$ to belong to U. The results of the paper were obtained in 1987 and form a part (essentially Chapter III.2) of the thesis [7]. The author would like to express his gratitude to Professors G. Halász, B. Kashin and S. Konjagin for useful comments and references. In particular, Professor S. Konjagin called the attention of the author to a recent paper of D. V. Pecherskiĭ [13]. The paper deals with related problems using a key lemma (Lemma 1 in the paper) which is somewhat similar to Chobanjan's Lemma quoted here as Lemma 4. Using his new lemma, Pecherskiĭ proved Theorem 1 of this work as Theorem 2 of [13], cf. p. 25. Hence this result must be attributed to Pecherskiĭ, as his Theorem, and Section 3 of this work describes only a second although independent and different proof for it. Let us mention that this proof was worked out in 1987, before the appearance of [13].

2. Some lemmas

LEMMA 1. *For all* $f \in \mathcal{P}$ *we have*

$$\delta_n := \delta_n(f) := \sup_{m \geqq n} E\|S_m(f_\varepsilon, \cdot) - S_n(f_\varepsilon, \cdot)\|_\infty \to 0 \quad (n \to \infty).$$

PROOF. Well-known, see e.g. [3] Ch. 2, Theorem 4 and Ch. 5, Theorem 3.

Now let us denote the de la Vallée Poussin means of f by

$$V_n := V_n(f) := \sum_{j=0}^{n} A_j + \sum_{j=n+1}^{2n} \left(2 - \frac{j}{n}\right) A_j. \tag{6}$$

We also introduce for any $f \in L^2$ with Fourier series (1) the usual notation

$$s_k := \sqrt{\sum_{n=2^k+1}^{2^{k+1}} c_n^2}. \tag{7}$$

LEMMA 2. *For any $f \in L^2$ and $k > 2$ there exists a 0–1 sequence $\omega = (\omega_i)$ with $i = 2^k + 1, \dots, 2^{k+1}$ such that*

$$\left\| S_n(f) + \sum_{i=n+1}^{2n} \omega_i A_i - V_n(f) \right\|_\infty \leqq 8\sqrt{k} \cdot s_k \quad \text{with} \quad n = 2^k.$$

PROOF. This follows from Lemma 2 of [6].

LEMMA 3. *Let $P(x) = \sum_{i=0}^{N} A_i(x)$ be any trigonometric polynomial of degree not exceeding N. Then we have*

$$E\|P_\varepsilon\|_\infty = E\left\|\sum_0^N \varepsilon_i A_i\right\|_\infty \leqq 2\sqrt{\log N}\,\|P\|_2.$$

PROOF. See [8], (5.1.2) Lemma, p. 290.

LEMMA 4. *Let X be any normed space and $\mathbf{x}_1, \dots, \mathbf{x}_N$ be elements of X. There exists a permutation σ of $\{1, 2, \dots, N\}$ such that*

$$\max_{M \leqq N} \left\| \sum_{i=1}^{M} \mathbf{x}_{\sigma(i)} \right\|_X \leqq 9 \left\| \sum_{i=1}^{N} \mathbf{x}_i \right\|_X + 9E \left\| \sum_{i=1}^{N} \varepsilon_{j_i} \mathbf{x}_i \right\|_X,$$

where $(\varepsilon_{j_i}) \subset \varepsilon$ with $j_i \neq j_k$ $(i \neq k)$ for $i = 1, 2, \dots, N$.

PROOF. This is Corollary 1 on p. 56 of [1].

LEMMA 5. *Suppose that $f \in L^2$ and s_k is non-increasing. Then $f \in \mathcal{P}$ if and only if*

$$\sum_{k=1}^{\infty} s_k < \infty.$$

PROOF. Necessity is proved in [5], and sufficiency is contained in [3], Ch. 7, Theorem 1.

LEMMA 6. *Let c_n be any sequence satisfying the conditions*

i) $\sum_1^{\infty} c_n^2 < \infty$,

ii) $1/c_n$ *is concave,*

iii) c_n *is monotonically decreasing.*

Then there exists an $f \in C$ with Fourier series (1), i.e. for some (Θ_n) the L^2 series described in (1) belongs to a continuous function.

PROOF. This is a well-known result of Salem, cf. [12] Ch V, (10.1) Theorem. We note that the statement is true even if ii) is not supposed, see [2].

3. Proof of Theorem 1

Let us take any $f \in \mathcal{P}$. Since $f \in L^2$, $\sum s_k^2 = \|f\|_2 < \infty$ and for every $\eta > 0$ one can find a $k \in \mathbf{N}$ with $k \cdot s_k^2 < \eta$. That is, we have some $k_j \to \infty$ with

$$s_{k_j}^2 \leqq \frac{\eta_j}{k_j}, \quad \eta_j \to 0 \quad (j \to \infty). \tag{8}$$

We define ν as the composition of two other permutations,

$$\nu = \sigma \circ \pi \tag{9}$$

First we construct π as the disjoint union of permutations π_k on the blocks $[2^k + 1, 2^{k+1}]$. For $k = k_j > 2$ we apply Lemma 2 and define π_k with

$$\pi_k(i) := \begin{cases} \min\{p : \omega_p = 1 \text{ and } p \neq \pi_k(l) \text{ for any } l < p\}, & n < i \leqq m_j \\ \min\{p : p \neq \pi_k(l) \text{ for any } m_j < l < p\}, & m_j < i \leqq 2n, \end{cases} \tag{10}$$

where

$$n = 2^k = 2^{k_j} \quad \text{and} \quad m_j = n + \sum_{i=n+1}^{2n} \omega_i. \tag{11}$$

That is, π_k^{-1} places the indices with $\omega_i = 1$ to the beginning of the interval $[n+1, 2n]$, and places the indices with $\omega_i = 0$ to the other end. So for $k = k_j$ we have by the definition of π_k

$$S_{2^k} + \sum_{i=2^k+1}^{2^{k+1}} \omega_i A_i = S_{2^k} + \sum_{i=2^k+1}^{m_j} A_{\pi_k(i)} \quad (k = k_j). \tag{12}$$

Now for $k \neq k_j$ we choose π_k to be identity and take

$$\pi := \bigcup_{k=0}^{\infty} \pi_k. \tag{13}$$

Trivially

$$_\pi S_{2^k} = S_{2^k} \tag{14}$$

and hence from Lemma 2, (8), (12), (13) and (14) we get

$$\left\| {_\pi S_{m_j}} - V_{2^{k_j}} \right\|_\infty \leqq 8\sqrt{\eta_j}. \tag{15}$$

Next we define σ so that

$$\sigma := \bigcup_{j=0}^{\infty} \sigma_j, \quad \sigma_j : [m_j + 1, m_{j+1}) \leftrightarrow [m_j + 1, m_{j+1}). \tag{16}$$

Here we can take $m_0 := -1$ and σ_0 to be identity on $[0, m_1)$. Consider the polynomial

$$T_j :=_\pi S_{m_{j+1}} -_\pi S_{m_j}. \tag{17}$$

Our goal is to rearrange the order of the terms of T_j by σ_j to ensure small partial sums. We apply Lemma 4 in the Banach space C with the ∞-norm and $N = m_{j+1} - m_j$, $\mathbf{x}_i = A_{\pi(i+m_j)}$, $j_i = \pi(i + m_j)$ $(i = 1, \ldots, N)$. We obtain a certain σ_j with

$$\max_M \left\| \sigma_j(T_j)_M \right\|_\infty \leqq 9\|T_j\|_\infty + 9\left\|(T_j)_\varepsilon\right\|_\infty. \tag{18}$$

Note that $f \in \mathcal{P}$, and Lemma 1 entails

$$E\left\| \left(S_{2^{k_j+1}} - S_{2^{k_j}}\right) \right\|_\infty \to 0 \quad (j \to \infty). \tag{19}$$

Since the left hand side of (12) is exactly ${}_{\pi}S_{m_j}$, we have

$$P_j := \sum_{i=n+1}^{2n} \omega_i A_i = {}_{\pi}S_{m_j} - {}_{\pi}S_n \quad \left(n = 2^{k_j}\right) \tag{20}$$

and also

$$T_j = {}_{\pi}S_{m_{j+1}} - {}_{\pi}S_{m_j} = P_{j+1} - P_j - \left(S_{2^{k_{j+1}}} - S_{2^{k_j}}\right). \tag{21}$$

Hence in view of (19)

$$E\left\|(T_j)_\varepsilon\right\|_\infty \leqq E\left\|(P_j)_\varepsilon\right\|_\infty + E\left\|(P_{j+1})_\varepsilon\right\|_\infty + o(1) \quad (j \to \infty). \tag{22}$$

Now further use of (8), $\|P_j\|_2 \leqq s_{k_j}$, $\deg(P_j) \leqq 2^{k_j}$ and Lemma 3 ensure

$$E\left\|(P_j)_\varepsilon\right\|_\infty \to 0 \quad (j \to \infty), \tag{23}$$

hence from (22) and (23)

$$E\left\|(T_j)_\varepsilon\right\|_\infty \to 0 \quad (j \to \infty). \tag{24}$$

Next we make use of the continuity of f in the form that $V_n(f) \to f$ uniformly on $\mathbf{T}$, cf. [11]. Hence (15) and $f \in C$ entails $\|T_j\|_\infty \to 0$ $(j \to 0)$ and so (18) and (24) imply

$$\max_M \left\|\sigma_j(T_j)_M\right\|_\infty \to 0 \quad (j \to \infty). \tag{25}$$

We obtain from (16), (17) and (25) that

(26)
$$\max_{m_j \leqq M \leqq m_{j+1}} \left\|\sigma_j(T_j)_M\right\|_\infty = \max_{m_j \leqq M \leqq m_{j+1}} \left\|{}_{\sigma \cdot \pi}f_M - {}_{\pi}S_{m_j}\right\|_\infty \to 0 \ (j \to \infty).$$

Since (15) and $f \in C$ entails $\left\|{}_{\pi}S_{m_j} - f\right\|_\infty \to 0$, (26) and (9) concludes the proof of Theorem 1.

4. Further criteria for $f \in U$

THEOREM 3. *If $f \in C$ and s_k is nonincreasing, then $f \in U$.*

To prove this theorem first we note that in view of monotonicity and $f \in C \subset L^2$ we have $s_k = o\left(\frac{1}{\sqrt{k}}\right)$. Hence it suffices to prove the following

THEOREM 4. *If $f \in C$ and $s_k = o\left(\frac{1}{\sqrt{k}}\right)$, then $f \in U$.*

PROOF. The proof is very similar to that of Theorem 1.

If we define $k_j := j$ and $\eta_j := j \cdot s_j^2$, we get (8) with $k_j = j$ according to our assumption on s_k. Now repeating the proof of Theorem 1 with $k_j = j$ the only change is that to prove (19), instead of using Lemma 1, we refer to Lemma 3. Since $f \in \mathcal{P}$ was used only there, this modification proves Theorem 4.

COROLLARY. *If $f \in C$ satisfies the multiplier condition*

$$\sum_{n=1}^{\infty} c_n^2 \cdot \log n < \infty,$$

then $f \in U$.

Professor B. Kashin informed the author that this was conjectured more than ten years ago. It can be compared to the multiplier condition

$$\sum c_n^2 \cdot \log^{1+\varepsilon} n < \infty$$

of Payley and Zygmund to ensure $f \in \mathcal{P}$.

5. Proof of Theorem 2

Let us define

$$c_n := \frac{1}{\sqrt{n} \cdot \log n} \quad (n \geqq 10) \tag{27}$$

and $c_n = c_{10}$ for $n \leqq 10$, say. This sequence satisfies conditions i), ii) and iii) of Lemma 6, hence there exists an $f \in C$ with Fourier series (1), where c_n is defined in (27). Obviously (7) and (27) mean for s_k that

$$\sum s_k^2 < \infty, \quad \sum s_k = \infty \tag{28}$$

and also

(29)
$$s_k^2=\sum_{2^k+1}^{2^{k+1}}\frac{1}{n\log^2 n}\geqq\sum_{2^k+1}^{2^{k+1}}\left(\frac{1}{(2n)\log^2(2n)}+\frac{1}{(2n-1)\log^2(2n-1)}\right)=$$
$$=s_{k+1}^2,$$

i.e. s_k is monotonic. Now for monotonic s_k (28) and Lemma 5 ensure $f\notin\mathcal{P}$, while Theorem 3 gives $f\in U$. This concludes the proof of Theorem 2.

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