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A FEJÉR TYPE EXTREMAL PROBLEM

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1. Let us denote by $\mathcal{T}_n$ the set of trigonometric polynomials with degree $\leqq n$ and $\mathcal{T} = \bigcup_1^\infty \mathcal{T}_n$. C. Caratheodory and L. Fejér investigated several extremal problems concerning nonnegative trigonometric polynomials. One useful result of Fejér answers the following question: "How large can the coefficient of $\cos x$ be in a nonnegative polynomial of $\mathcal{T}_k$ with constant term 1?" Formally, we define

$$\mathcal{S}_k := \left\{g \in \mathcal{T}_k:\ g \geqq 0,\ g(x) = 1 + \sum_1^k b_n \cos nx\right\}$$

and ask for

$$\omega(k) := \sup \left\{b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos x\, dx:\ g \in \mathcal{S}_k\right\}.$$

The assumption that g is a pure cosine polynomial does not restrict generality and will be assumed in the sequel. Fejér obtained in [2] (see also in [3] I p. 869—870 or [6] II Ex. VI.52)

$$\omega(k) = 2 \cos \frac{\pi}{k+2}. \tag{1.1}$$

In the present paper we calculate the following companion of the above problem of Fejér. Let

$$\alpha(k) := \sup \left\{a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x dx:\ f \in \mathcal{F}_k\right\}, \tag{1.2}$$

where

$$\mathcal{F}_k := \mathcal{F} := \left\{f \in \mathcal{T}:\ f \geqq 0, f(x) = 1 + a_1 \cos x + \sum_{k+1}^\infty a_n \cos nx\right\}. \tag{1.3}$$

If $f \in \mathcal{F}$ and $g \in \mathcal{S}_k$ we obtain in view of nonnegativity

$$0 \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x+\pi)\, dx = 2 - a_1 b_1,$$

and so taking supremum we obtain from (1.1)

$$\alpha(k) \leqq 2/\omega(k) = 1/\cos \frac{\pi}{k+2}. \tag{1.4}$$

Our result shows that this estimate is sharp.

THEOREM. *We have* $\alpha(k) = \dfrac{1}{\cos \dfrac{\pi}{k+2}}$.

The proof uses results of Caratheodory and Fejér. Finally we use some linear algebra to calculate $\alpha(k)$. Computations will be marked out and omitted.

2. Let $f \in \mathscr{F}$ and F be defined by $F(0)=1$, $F \in \mathbf{R}[z]$ and $\operatorname{Re} F(e^{ix}) = f(x)$, that is with f in (1.3) we put

(2.1) $$F(z) := 1 + a_1 z + \sum_{k+1}^{\infty} a_n z^n \in \mathbf{R}[z].$$

The condition $f \geqq 0$ is equivalent to $\operatorname{Re} F \geqq 0$ in $|z| \leqq 1$. Therefore

(2.2) $$G(z) := \frac{1-F(-z)}{1+F(-z)} \in \mathbf{R}(z)$$

is regular for $|z| \leqq 1$ and $f \geqq 0$ is equivalent to

(2.3) $$|G(z)| \leqq 1 \qquad (|z| \leqq 1).$$

In a sufficiently small neighbourhood of 0, but then for all $|z|<1$ we have with some $H \in R(z)$

(2.4) $$G(z) = \frac{1}{1 - \dfrac{1-F(-z)}{2}} - 1 = bz + b^2 z^2 + \ldots + b^k z^k + H(z) z^{k+1} \quad (b := a_1/2).$$

Denote the set of regular functions on some domain D by $\mathcal{O}(D)$. Put

(2.5) $$\alpha'(k) := \max\{b\colon \exists G, H \in \mathcal{O}(|z|<1),\ |G|<1,\ G(z) = bz + \ldots + b^k z^k + H(z) z^{k+1}\},$$

which exists in view of the Vitali—Montel theorem. It is easy to observe that

(2.6) $$\alpha'(k) = \frac{1}{2} \alpha(k).$$

Clearly, if $\alpha'(k) = r$, then for any corresponding extremal function G $\sup\limits_{|z|<1} |G| = 1$, and so the value of the Caratheodory—Fejér type extremal quantity

(2.7) $$\mu(r) := \inf\Big\{\sup_{|z|<1} |g|\colon g(z) = rz + \ldots + r^k z^k + h(z) z^{k+1},\ g, h \in \mathcal{O}(|z|<1)\Big\}$$

is exactly 1.

Now we can apply the theorem of Caratheodory and Fejér, cf. [1], [4] and [3] II. p. 186, to the above special case. We obtain

(2.8) $$1 = \mu(r) = \max\{|\lambda|\colon \det(C_r - \lambda I) = 0\},$$

where

$$C_r = \begin{pmatrix} r^k & \dots & r^2 & r & 0 \\ r^{k-1} & \dots & r & 0 & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ r & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Taking into account that (2.9) has only real eigenvalues, (2.6)—(2.8) entails

$$\frac{1}{2}\alpha(k) \geqq r_0 := \min\{r > 0\colon \det(C_r - I)\det(C_r + I) = 0\}. \tag{2.10}$$

3. Now we determine the value of r_0. Denote

$$P_k(r) := -\det(C_r - I), \quad Q_k(r) := \det(C_r + I), \tag{3.1}$$

and note that since

$$\frac{1}{2}\alpha(k) \geqq r_0 = \min\{r > 0\colon P_k(r)Q_k(r) = 0\}, \tag{3.2}$$

we always have $1/2 < r_0 < 1$. First we compute P_k and Q_k for $k = 0, 1, 2, 3$ and 4.

$$\begin{cases} P_0(r) = 1, \; P_1(r) = r - 1, \; P_2(r) = 1 - 2r^2, \\ Q_0(r) = 1, \; Q_1(r) = r + 1, \; Q_2(r) = 1, \\ P_3(r) = r^2 + r - 1, \; P_4(r) = 1 - 3r^2, \\ Q_3(r) = 1 + r - r^2, \; Q_4(r) = 1 - r^2. \end{cases} \tag{3.3}$$

When calculating $\det(C_r - \lambda I)$ we can subtract from each column the next (starting from the left) and obtain

$$\det(C_r - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & \dots & r & 0 \\ r & -\lambda & 0 & \dots & 0 & 0 \\ 0 & r & -\lambda & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ r & 0 & 0 & \dots & -\lambda & 0 \\ 0 & 0 & 0 & \dots & r & -\lambda \end{vmatrix}.$$

Therefore, expanding by the last column, we obtain for the polynomials (3.1) the determinant representations of order k below.

$$P_k(r) = \begin{vmatrix} -1 & 0 & \dots & 0 & r \\ r & -1 & \dots & r & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & r & \dots & -1 & 0 \\ r & 0 & \dots & r & -1 \end{vmatrix}, \quad Q_k(r) = \begin{vmatrix} 1 & 0 & \dots & 0 & r \\ -r & 1 & \dots & r & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & r & \dots & 1 & 0 \\ r & 0 & \dots & -r & 1 \end{vmatrix}. \tag{3.4}$$

LEMMA 1. *The polynomials* (3.1) *satisfy the recursive relations*

$$P_k(r) = P_{k-2}(r) - r^2 P_{k-4}(r), \quad Q_k(r) = Q_{k-2}(r) - r^2 Q_{k-4}(r).$$

Using (3.3) and (3.4) Lemma 1 can be deduced by elementary determinant transformations — we leave the details to the reader. Now for any particular $1/2 < r < 1$ denote

$$\alpha := \frac{1}{2} + i\sqrt{r^2 - \frac{1}{4}}, \quad \beta := \bar{\alpha} = \frac{1}{2} - i\sqrt{r^2 - \frac{1}{4}}. \tag{3.5}$$

A recursive recurrence relation

$$x_{n+1} = x_n - r^2 x_{n-1} \quad (n = 1, 2, \ldots) \tag{3.6}$$

determines the sequence x_n (see e.g. [5], Ch. V.4) as

$$x_n = A\alpha^n + B\beta^n \quad (n = 1, 2, \ldots), \tag{3.7}$$

where A and B are the unique solution of the system

$$\begin{cases} A + B = x_0 \\ A\alpha + B\beta = x_1 \end{cases}. \tag{3.8}$$

In particular, if x_0 and x_1 are real, then $B = \bar{A}$ is immediate. Denote

$$\varphi := \arg\alpha = \arctan\sqrt{4r^2 - 1} \in \left(0, \frac{\pi}{2}\right). \tag{3.9}$$

Lemma 1 and (3.3)—(3.9) give that with some A_1, A_2, A_3 and A_4 with

$$\arg A_1 = \varphi, \quad \arg A_2 = \varphi/2, \quad \arg A_3 = \varphi - \pi/2, \quad \arg A_4 = \frac{\varphi - \pi}{2} \tag{3.10}$$

we have

$$\begin{cases} P_{2n}(r) = A_1\alpha^n + \bar{A}_1\bar{\alpha}^n = 2\,\mathrm{Re}\,(A_1\alpha^n), \quad P_{2n-1}(r) = 2\,\mathrm{Re}\,(A_2\alpha^n), \\ Q_{2n}(r) = 2\,\mathrm{Re}\,(A_3\alpha^n), \quad Q_{2n-1}(r) = 2\,\mathrm{Re}\,(A_4\alpha^n). \end{cases} \tag{3.11}$$

Since $\mathrm{Re}\,(z) = 0$ is identical with $\arg(z) = \pi/2 + m\pi$ $(m \in \mathbf{Z})$, we are led to the equations

$$\begin{cases} P_{2n}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{2m+1}{2n+2}\pi \quad (m \in \mathbf{Z}), \\ P_{2n-1}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{2m+1}{2n+1}\pi \quad (m \in \mathbf{Z}), \\ Q_{2n}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{m}{n+1}\pi \quad (m \in \mathbf{Z}), \\ Q_{2n-1}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{2m}{2n+1}\pi \quad (m \in \mathbf{Z}). \end{cases} \tag{3.12}$$

Summing up, since $0 < \varphi < \pi/2$, we get

LEMMA 2. $P_k(r)Q_k(r)=0$ *if and only if* $\arctan\sqrt{4r^2-1}\frac{j\pi}{k+2}$ *with* $1\leqq j\leqq\frac{k+1}{2}$.

In view of Lemma 2 the roots of the polynomial $P_k(r)Q_k(r)$ are

$$\pm\frac{1}{2\cos\frac{j\pi}{k+2}}\quad\left(j=1,2,\dots,\left[\frac{k+1}{2}\right]\right). \tag{3.13}$$

According to (3.2) and (1.4) this proves the theorem.

References

[1] C. Caratheodory—L. Fejér, Über den Zusammenhang ..., *Rendiconti di Palermo*, **32** (1911), 218—239.

[2] L. Fejér, Über trigonometrische Polynome, *J. angew. Math.*, **146** (1915), 53—82.

[3] L. Fejér, *Gesammelte Arbeiten I—II*, Akadémiai Kiadó (Budapest, 1970).

[4] L. Fejér, Über gewisse Minimumprobleme der Funktionentheorie, *Math. Ann.*, **97** (1927), 104—123.

[5] A. O. Gelfond, *Calculation of finite differences* (in Russian) (Moscow—Leningrad, 1952).

[6] G. Pólya—G. Szegő, *Problems and Theorems in Analysis I—II*, Springer Verlag (Berlin—Heidelberg, 1972).

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