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DECOMPOSITIONS INTO THE SUM OF PERIODIC FUNCTIONS BELONGING TO A GIVEN BANACH SPACE

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1. Introduction

Let $f\colon \mathbf{R}\to\mathbf{R}$ be a real function and suppose that f is the sum of finitely many periodic functions: $f=f_1+\ldots+f_n$, where f_i is periodic modulo a_i $(i=1, \ldots, n)$. Let Δ_a denote the difference operator:

$$\Delta_a g(x) = g(x+a)-g(x) \quad (g\colon \mathbf{R}\to\mathbf{R},\ a\in\mathbf{R}).$$

Then $\Delta_{a_i}f_i=0$ for every i and, as the difference operators commute, we have $\Delta_{a_1}\ldots\Delta_{a_n}f=0$.

Let $\mathcal{F}$ be a class of real functions. We say that $\mathcal{F}$ has the decomposition property (d. pr.) if, for every $f\in\mathcal{F}$ and for every finite system of real numbers $a_1, \ldots, a_n$, the condition $\Delta_{a_1}\ldots\Delta_{a_n}f=0$ implies that there are functions $f_1, \ldots, f_n\in\mathcal{F}$ such that $f=f_1+\ldots+f_n$ and f_i is periodic modulo a_i for every $i=1, \ldots, n$.

A natural generalization of this notion is the following. Let A be a non-empty set and let T be a map of A into itself. A function $g\colon A\to\mathbf{R}$ is said to be T-periodic, if $g\circ T=g$ or, equivalently, if $\Delta_T g=0$, where $\Delta_T g=g\circ T-g$. Now let $T_1, \ldots, T_n$ be maps of A into itself and let $f=f_1+\ldots+f_n$, where f_i is T_i-periodic for every $i=1, \ldots, n$. If the maps T_i commute i.e. if $T_i\circ T_j=T_j\circ T_i$ for every $i, j=1, \ldots, n$ then the operators Δ_{T_i} also commute, and we have

$$\Delta_{T_1}\ldots\Delta_{T_n}f=0. \tag{1}$$

Let $\mathcal{F}$ be a class of real valued functions defined on A. We say that $\mathcal{F}$ has the decomposition property with respect to the maps $T_1, \ldots, T_n$ if for every $f\in\mathcal{F}$, condition (1) implies that there are functions $f_1, \ldots, f_n\in\mathcal{F}$ such that $f=f_1+\ldots+f_n$ and f_i is T_i-periodic for every $i=1, \ldots, n$. (Therefore a class of real functions has the d.pr. if it has the d.pr.w.r.t. every finite system of translations.)

In this paper we show that some Banach spaces of functions have the d.pr. with respect to "reasonable" mappings. Thus the L_p classes for $1\leq p<\infty$ possess the d.pr.w.r.t. commuting measurable maps which do not decrease measure (Theorem 4.1), and in σ-finite (or, more generally, in localizable) spaces L_∞ has the d.pr.w.r.t. commuting measurable maps which do not map sets of positive measure into sets of measure zero (Theorem 4.3, Corollary 4.6). Also, the class of bounded functions defined on an arbitrary set has the d.pr.w.r.t. any commuting system of maps (Corollary 4.7).

As for classes of real functions, we show that $L_\infty(\mathbf{R})$, the class of bounded Lipschitz functions and some other Banach spaces of real functions possess the d.pr. (Theorem 5.2).

We remark that there are "nice" classes of real functions without having the decomposition property. The simplest example is provided by $C(\mathbf{R})$, the class of continuous functions defined on $\mathbf{R}$. In fact, the identity function $f(x)=x$ satisfies $\Delta_a\Delta_b f=0$ for every $a, b\in\mathbf{R}$. On the other hand, f is not the sum of two periodic continuous functions, since f is not bounded.

In Section 2 we give sufficient conditions under which the null space of a product of operators is generated by the null spaces of the operators. In Section 3 we construct a Banach space $\mathcal{F}$ of functions defined on a set A, and two commuting bijections T_1, T_2 of A onto itself such that $f\mapsto f\circ T_1$, $f\mapsto f\circ T_2$ are isometries of $\mathcal{F}$ onto itself, but $\mathcal{F}$ does not have the d.pr.w.r.t. T_1 and T_2. Applying the results of Section 2 and also the ergodic theorem, we investigate the d.pr. of some Banach spaces of functions in Sections 4 and 5.

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2. Null spaces of products of operators

2.1. THEOREM. *Let X be a topological vector space and $B_1, \dots, B_n$ be commuting continuous linear operators on X. Assume that for every $x\in X$ and $1\leq j\leq n$ the closed convex hull of the set $\{B_j^m x\colon m\in\mathbf{N}\}$ contains a fixed point of B_j. Then the null space* $\operatorname{Ker}[(I-B_1)\cdot\ldots\cdot(I-B_n)]$ *is the linear span of the null spaces* $\operatorname{Ker}(I-B_j)$, $j=1, \dots, n$.

PROOF. We prove by induction on n. For $n=1$ the statement is obvious. Suppose that $n>1$ and the assertion holds for $n-1$ operators. Denote $A_j=I-B_j$ $(j=1, \dots, n)$. As the operators commute, we have $\operatorname{Ker} A_j\subset\operatorname{Ker}(A_1\dots A_n)$. Hence it suffices to show that for any $x\in\operatorname{Ker}(A_1\dots A_n)$, $x=x_1+\dots+x_n$, where $x_j\in\operatorname{Ker} A_j$. First we prove that

$$A_1\dots A_{n-1}(I-B_n^k)\,x=0 \tag{2}$$

holds for every $x\in\operatorname{Ker}(A_1\dots A_n)$ and $k\in\mathbf{N}$. Indeed, $I-(I-A_n)^k=A_nC$, where C is a polynomial of A_n. Therefore, by commutativity, the left hand side of (2) equals $CA_1\dots A_n x=0$, since $x\in\operatorname{Ker}(A_1\dots A_n)$.

By (2), $A_1\dots A_{n-1}B_n^k x=A_1\dots A_{n-1}x$ holds for every k and hence

$$A_1\dots A_{n-1}y=A_1\dots A_{n-1}x \tag{3}$$

holds whenever y belongs to the convex hull of the set $H_n=\{B_n^k x\colon k\in\mathbf{N}\}$. As the maps A_j are continuous, (3) is also valid if y is in the closed convex hull of H_n. By assumption, this closed convex hull contains an element x_n with $B_n x_n=x_n$, i.e. $x_n\in\operatorname{Ker} A_n$. Applying (3) with $y=x_n$ we obtain $A_1\dots A_{n-1}(x-x_n)=0$. Since $x_n\in\operatorname{Ker} A_n$, and the induction hypothesis applies for $x'=x-x_n\in\operatorname{Ker}(A_1\dots A_{n-1})$, the theorem is proved.

2.2. REMARK. Let $L_j(x)$ denote the closed subspace generated by

$$\{(B_j^m-I)x\colon m\in\mathbf{N}\}\quad (j=1, \dots, n;\ x\in X).$$

The same proof applies if B_j has a fixed point in $x+L_j(x)$ for every $j=1, \dots, n$ and $x\in X$.

2.3. COROLLARY. *Let X be a linear space over the reals, let $\mathcal{T}$ be a vector topology on X, and let $\|\cdot\|$ be a norm on X such that $S=\{x\in X\colon \|x\|\leqq 1\}$ is $\mathcal{T}$-compact. Let $A_1, \ldots, A_n$ be linear maps of X into itself such that*

(i) $A_i A_j = A_j A_i$ $(i, j=1, \ldots, n)$,

(ii) $\sup\limits_k \|(A_i-I)^k\| < \infty$ $\quad (i = 1, \ldots, n)$,

and

(iii) *each A_i is $\mathcal{T}-\mathcal{T}$ continuous.*

Then $\operatorname{Ker}(A_1 \ldots A_n)$, *as a linear subspace of X, is spanned by the null spaces* $\operatorname{Ker} A_j$, $j=1, \ldots, n$.

PROOF. Let $B_j=I-A_j$ and $H_j(x)=\{B_j^m x\colon m\in\mathbf{N}\}$ $(j=1, \ldots, n;\ x\in X)$. Condition (ii) implies that for every j and x there is a $c>0$ such that $H_j(x)\subset c\cdot S$. Since $c\cdot S$ is convex and compact, the closed convex hull $K_j(x)$ of $H_j(x)$ is also compact. Since B_j is linear, $\mathcal{T}-\mathcal{T}$ continuous, and maps $H_j(x)$ into itself, B_j also maps $K_j(x)$ into itself. Therefore, by the Markov—Kakutani theorem ([2], V. 6, p. 456), B_j has a fixed point in $K_j(x)$ and we can apply 2.1.

2.4. REMARK. Suppose that $\|A_j-I\|\leqq 1$ for every $j=1, \ldots, n$, and let $x\in\operatorname{Ker}(A_1 \ldots A_n)$. It is easy to check that our proof gives a decomposition $x=x_1+\ldots+x_n$ with $x_j=A_j x_j$ and $\|x_j\|\leqq 2^{j-1}\|x\|$ $(j=1, \ldots, n)$.

QUESTION. Can we get a representation with $\|x_j\|\leqq\|x\|$ $(j=1, \ldots, n)$?

2.5. COROLLARY. *Let X be a real Banach space with norm $\|\cdot\|$ and suppose that X is the dual of the normed linear space Y. Let $A_1, \ldots, A_n$ be commuting linear operators of X satisfying* (ii) *of* 2.3. *If each A_j is continuous as a mapping of X into itself in the weak* topology w^* of X, then* $\operatorname{Ker}(A_1 \ldots A_n)$ *is the linear span of the null spaces* $\operatorname{Ker} A_j$ $(j=1, \ldots, n)$.

PROOF. Since the unit ball of X is w^*-compact by the Banach—Alaoglu theorem, we may apply 2.3 with $\mathcal{T}=w^*$.

2.6. COROLLARY. *Let X be a reflexive Banach space and let $A_1, \ldots, A_n$ be commuting linear operators of X satisfying* (ii) *of* 2.3. *Then* $\operatorname{Ker}(A_1 \ldots A_n)$ *is the linear span of the null spaces* $\operatorname{Ker} A_j$ $(j=1, \ldots, n)$.

PROOF. Condition (ii) of 2.3 implies that $A_1, \ldots, A_n$ are bounded operators. Therefore, they are continuous in the weak topology ([2], V. 15, p. 422). Since X is reflexive, the weak topology coincides with w^* and we may apply 2.5.

3. A counter-example

In this section we are going to show that the condition of reflexivity cannot be omitted from the conditions of 2.6. Let X be a family of bounded, real valued functions defined on a set A such that X is a Banach space with the sup norm. Let T_1, T_2 be commuting bijections of A onto itself such that $f\circ T_i\in X$ for every $f\in X$ and $i=1, 2$, let the operators B_1, B_2 be defined by $B_i f=f\circ T_i$ $(f\in X,\ i=1, 2)$, and let $A_i=I-B_i$ $(i=1,2)$. Then B_1, B_2 are norm-preserving and hence A_1, A_2 are continuous operators with $\|A_i-I\|=1$ $(i=1,2)$.

It is easy to see that $f\in\operatorname{Ker} A_i$ if and only if f is T_i-periodic and $f\in\operatorname{Ker} A_1A_2$ if and only if $\Delta_{T_1}\Delta_{T_2}f=0$. Consequently, $\operatorname{Ker} A_1A_2$ is the linear span of $\operatorname{Ker} A_1$ and $\operatorname{Ker} A_2$ if and only if X has the decomposition property w.r.t. T_1 and T_2.

Let $A=\mathbf{Z}\times\mathbf{Z}$, where $\mathbf{Z}$ denotes the set of integers, and let the maps T_1, T_2 be defined by

$$T_1(n,k)=(n+1,k) \quad\text{and}\quad T_2(n,k)=(n,k+1) \quad (n,k\in\mathbf{Z}).$$

3.1. THEOREM. *There is a family X of bounded real valued functions defined on A such that*

(i) *X is a Banach space with the sup norm,*

(ii) *$f\in X$ implies $f\circ T_1, f\circ T_2\in X$,*

(iii) *X does not have the decomposition property w.r.t. T_1 and T_2.*

PROOF. Let $h\colon \mathbf{Z}\to\mathbf{R}$ be a fixed bounded function to be specified later. We define X as the uniform closure of the family of finite linear combinations of the functions $g_{a,b}$ defined by

$$g_{a,b}(n,k)=h(n+a)+h(k+b) \quad (n,k,a,b\in\mathbf{Z}).$$

Then (i) and (ii) are satisfied. We have to show that h can be chosen in such a way that (iii) is also true.

Let $H_1, H_2, \ldots$ be an enumeration of the finite subsets of $\mathbf{Z}$. We can find two sequences of integers x_m, y_m such that the sets H_n+x_n, H_m+y_m $(n,m=1,2,\ldots)$ are pairwise disjoint. (We denote $H+x=\{a+x\colon a\in H\}$.) We define h as the characteristic function of the set $\bigcup_{n=1}^{\infty}(H_n+x_n)$.

We show that if $\varepsilon, c_1, \ldots, c_N, d\in\mathbf{R}$, $a_1,\ldots,a_N\in\mathbf{Z}$, and

$$\Big|\sum_{i=1}^{N} c_i h(n+a_i)-d\Big|<\varepsilon \quad\text{for every}\quad n\in\mathbf{Z}, \tag{4}$$

then

$$\Big|\sum_{i=1}^{N} c_i\Big|<2\varepsilon. \tag{5}$$

Indeed, let $\{a_1,\ldots,a_N\}=H_m$. Then $h(y_m+a_i)=0$ for every $i=1,\ldots,N$ by the definition of h and hence, by (4), $|d|<\varepsilon$. Putting $n=x_m$ in (4) we obtain $\Big|\sum_{i=1}^{N} c_i-d\Big|<\varepsilon$ and (5) follows. Let f be defined by $f(n,k)=h(n)+h(k)$ $(n,k\in\mathbf{Z})$. Then $f\in X$ and $\Delta_{T_1}\Delta_{T_2}f=0$. We prove that there is no decomposition $f=f_1+f_2$ such that $f_i\in X$ and $f_i\circ T_i=f_i$ $(i=1,2)$. Suppose that $f=f_1+f_2$ is such a decomposition. By $f_1\in X$, there are numbers $c_i\in\mathbf{R}$ and $a_i, b_i\in\mathbf{Z}$ $(i=1,\ldots,N)$ such that

$$\Big|f_1(n,k)-\sum_{i=1}^{N} c_i\big(h(n+a_i)+h(k+b_i)\big)\Big|<1/8$$

for every $n, k \in \mathbf{Z}$. Putting $k=0$ and taking into consideration that f_1 does not depend on n, we obtain

$$\Big|\sum_{i=1}^{N} c_i h(n+a_i) - d\Big| < 1/8 \quad (n\in \mathbf{Z}),$$

where $d=f_1(0,0)-\sum_{i=1}^{N} c_i h(b_i)$. As we proved above, this implies

$$(6) \qquad \Big|\sum_{i=1}^{N} c_i\Big| < 1/4.$$

We can show similarly that there are numbers $c'_j \in \mathbf{R}$, $a'_j, b'_j \in \mathbf{Z}$ $(j=1, \dots, M)$ such that

$$(7) \qquad \Big|\sum_{j=1}^{M} c'_j\Big| < 1/4$$

and

$$\Big|f_2(n,k)-\sum_{j=1}^{M} c'_j\big(h(n+a'_j)+h(k+b'_j)\big)\Big| < 1/8$$

for every $n, k \in \mathbf{Z}$. Since $f=f_1+f_2$ and $f(n,k)=h(n)+h(k)$, we obtain, for every n and k,

$$\Big|h(n)+h(k)-\sum_{i=1}^{N} c_i\big(h(n+a_i)+h(k+b_i)\big)-\sum_{j=1}^{M} c'_j\big(h(n+a'_j)+h(k+b'_j)\big)\Big| < 1/4.$$

Putting $k=0$ we obtain, for every n,

$$\Big|h(n)-\sum_{i=1}^{N} c_i h(n+a_i)-\sum_{j=1}^{M} c'_j h(n+a'_j)-e\Big| < 1/4$$

with a constant e. Since (4) implies (5), this gives

$$\Big|1-\sum_{i=1}^{N} c_i-\sum_{j=1}^{M} c'_j\Big| < 1/2$$

which contradicts (6) and (7). This contradiction completes the proof.

4. The decomposition property of the L_p classes

4.1. THEOREM. *Let (X, S, μ) be a measure space and let $T_1, \dots, T_n$ be commuting measurable maps of X into itself such that $\mu(T_i^{-1}(H)) \leqq \mu(H)$ for every $H \in S$ and $i=1, \dots, n$. Then for every $1 \leqq p < \infty$, $L_p(X)$ has the d.pr.w.r.t. $T_1, \dots, T_n$.*

PROOF. Corollary 2.6 can be applied for the case $p>1$, but not for $p=1$. We can apply, however, the pointwise ergodic theorem for both cases. We shall prove by induction on n using an argument similar to that of 2.1. Suppose that the statement of the theorem is true for $n-1$ maps and let $T_1, \dots, T_n$ be as in the theorem. Let $f \in L_p(X)$ be such that $\Delta_{T_1} \dots \Delta_{T_n} f=0$. Let the operators A_i, B_i be defined by $B_j g = g \circ T_j$ and $A_j = I - B_j$ $(j=1, \ldots, n;\ g \in L_p(X))$. Then $f \in \operatorname{Ker}(A_1 \ldots A_n)$ and this implies, as in the proof of 2.1, that

$$A_1 \ldots A_{n-1} f = A_1 \ldots A_{n-1} B_n^k f \tag{8}$$

for every $k \in \mathbf{N}$. Let

$$f_k = \frac{1}{k} \sum_{i=0}^{k-1} B_n^i f \quad (k = 1, 2, \ldots).$$

The condition on the map T_n easily implies that B_n is a linear operator in $L_1(X)$ satisfying $\|B_n\|_\infty \leqq 1$ and $\|B_n\|_1 \leqq 1$. Therefore, by the ergodic theorem ([2], VIII. 6.6, p. 675), f_k converges almost everywhere to a function g. It is easy to check that $\|f_k\|_p \leqq \|f\|_p$ for every k and hence, by Fatou's lemma, $g \in L_p(X)$. It is also easy to see that $B_n g = g$. It follows from (8) that

$$A_1 \ldots A_{n-1} f_k = A_1 \ldots A_{n-1} f$$

holds for every k. Since the inverse image of a null set by any of the maps T_j is a null set, the sequence $A_1 \ldots A_{n-1} f_k$ converges almost everywhere to $A_1 \ldots A_{n-1} g$, and hence we have

$$A_1 \ldots A_{n-1} f = A_1 \ldots A_{n-1} g.$$

Then we can apply the induction hypothesis for the function $f - g \in \operatorname{Ker}(A_1 \ldots A_{n-1})$ and, as $g \in \operatorname{Ker} A_n$, this completes the proof.

In the following we shall use the notion of conditional measure spaces. Given a measure space (X, S, μ), the family $S_0 \subset S$ of sets having finite measure and the restriction $\mu_0 = \mu | S_0$ form a conditional measure space (X, S_0, μ_0). I. E. Segal introduced the notion of localizable spaces using the term "measure space" for conditional measure spaces in [5] (Definition 2.6, p. 279). In the same work (Theorem 5.1, p. 301) Segal proves the equivalence of several important properties of measure spaces. Namely, (X, S_0, μ_0) is localizable if and only if $L_1^* = L_\infty$ holds, and this is equivalent to the following weak form of the Radon—Nikodym theorem: if ν is another (conditional) measure on X with the property that there exists a finite constant c such that

$$\nu(E) \leqq c \cdot \mu(E) \quad (E \in S_0), \tag{9}$$

then there is a function $f \in L_1(X)$ such that

$$\nu(E) = \int_E f \, d\mu \quad (E \in S_0).$$

Note that (9) is more restrictive than absolute continuity since "$\nu(E)=0$ whenever $\mu(E)=0$" does not imply (9) in general. In order to apply Segal's results we shall need the fact that in localizable spaces the Radon—Nikodym theorem holds for finite measures satisfying absolute continuity instead of (9). To simplify the terminology we shall say that (X, S, μ) is localizable, if (X, S_0, μ_0) is localizable according to Segal's definition.

4.2. LEMMA. *Let (X, S, μ) be a localizable measure space. If ν is a finite signed measure on S and ν is absolutely continuous w.r.t. μ, then there exists a function $f\in L_1(X)$ such that $\nu(E)=\int_E f\,d\mu$ for every $E\in S$.*

PROOF. We may assume that ν is a measure. Consider the signed measure $\sigma=\mu-\nu$. Since ν is finite, σ is bounded from below and hence, by Hahn's decomposition theorem, there is a set $N\in S$ such that $\sigma(N\cap E)\leqq 0$ for every $E\in S$ and $\sigma(P\cap E)\geqq 0$ for every $E\in S$, where $P=X\setminus N$. Since $\mu(N)\leqq\nu(N)$ is finite and ν is absolutely continuous w.r.t. μ, we may apply the Radon—Nikodym theorem in N and obtain a function $f_1\in L_1(X)$ such that $f_1(x)=0$ for every $x\in P$ and

$$\nu(E)=\int_E f_1\,d\mu \quad (E\in S, E\subset N).$$

On the other hand, $\sigma(P\cap E)\geqq 0$ for every $E\in S$ and hence $\nu(P\cap E)\leqq\mu(P\cap E)$ $(E\in S)$. Let ν_+ be defined by $\nu_+(E)=\nu(P\cap E)$ $(E\in S)$, then ν_+ is a measure on S such that $\nu_+\leqq\mu$. Since (X, S, μ) is localizable, Segal's theorem entails the existence of an $f_2\in L_1(X)$ such that

$$\nu_+(E)=\int_E f_2\,d\mu \quad (E\in S).$$

It is easy to check that $f=f_1+f_2$ satisfies the conditions of the lemma.

4.3. THEOREM. *Let (X, S, μ) be a localizable measure space and let $T_1, \dots, T_n$ be commuting measurable maps of X into itself such that for every $i=1, \dots, n$*

$$H\in S,\quad \mu(H)=0 \quad \textit{implies} \quad \mu(T_i^{-1}(H))=0. \tag{10}$$

Then $L_\infty(X)$ has the d.pr.w.r.t. $T_1, \dots, T_n$.

PROOF. According to Segal's theorem mentioned above, $L_\infty(X)$ is the dual of $L_1(X)$ and so we can use Corollary 2.5 with $B_i f=f\circ T_i$ $(f\in L_\infty(X))$ and $A_i=I-B_i$ $(i=1, \dots, n)$. By (10), $\|B_i\|\leqq 1$ and, as the T_i's are commuting, so are the B_i's. Hence the only condition to check is the continuity of B_i in the weak* topology of L_∞. It is enough to prove that for any given $g\in L_1(X)$ there exists an $h\in L_1(X)$ such that for every $f\in L_\infty(X)$

$$\Big|\int_X (f\circ T_i)\cdot g\,d\mu\Big|<1 \quad \text{whenever} \quad \Big|\int_X f\cdot h\,d\mu\Big|<1. \tag{11}$$

With the given $g\in L_1(X)$ define the signed measure

$$\nu(E)=\int_{T_i^{-1}(E)} g\,d\mu \quad (E\in S).$$

It follows from (10) that ν is absolutely continuous w.r.t. μ and, since $g\in L_1(X)$, ν is finite. By 4.2, there is a function $h\in L_1(X)$ such that

$$\nu(E)=\int_E h\,d\mu \quad (E\in S).$$

This implies $\int_X f\,d\nu=\int_X f\cdot h\,d\mu$ for every $f\in L_\infty(X)$. It follows from the definition of ν that

$$\int_X (f\circ T_i)\,g\,d\mu = \int_X f\,d\nu \quad (f\in L_\infty(X))$$

and hence (11) holds. This completes the proof.

4.4. QUESTION. Can the condition of localizability be dropped from the conditions of 4.3?

4.5. COROLLARY. *If X is a locally compact topological group and μ is the left (right) Haar measure of X then $L_\infty(X)$ has the d.pr.w.r.t. any system of commuting measurable maps $T_1, \dots, T_n$: $X\to X$ satisfying* (10). *In particular, $T_1, \dots, T_n$ can be translations by commuting elements of X.*

PROOF. (X, S, μ) is localizable by [5], Corollary 5.2, p. 305.

4.6. COROLLARY. *If (X, S, μ) is a σ-finite measure space and $T_1, \dots, T_n$: $X\to X$ are commuting measurable maps satisfying* (10) *then $L_\infty(X)$ has the d.pr.w.r.t. $T_1, \dots, T_n$.*

PROOF. (X, S, μ) is localizable by [5], Corollary 3.2.1, p. 284.

4.7. COROLLARY. *Let X be a non-empty set and let $B(X)$ denote the set of bounded real valued functions defined on X. Then $B(X)$ has the d.pr.w.r.t. any system of commuting maps $T_1, \dots, T_n$: $X\to X$.*

PROOF. Define μ as the counting measure on X and let $S=2^X$. Clearly, $\mu(H)=0$ implies $H=\emptyset$ and hence (10) is satisfied. Since (X, S, μ) is the direct sum of finite measure spaces (supported by one-element sets), it follows from [5], Lemma 3.2.2, p. 284 that (X, S, μ) is localizable. Therefore 4.3 applies.

4.8. REMARK. Our first proof of 4.7 used 2.3 with the product topology on $B(X)\subset R^X$. In this proof the compactness of the unit ball is guaranteed by Tychonoff's theorem. Later another simple proof was communicated to us by Professor Zbigniew Gajda from Katowice. His proof used Banach-limits. The proof given above was suggested by the referee.

5. Classes of real functions with the decomposition property

It follows from 4.6. and 4.7 that the class of bounded real functions and $L_\infty(\mathbf{R})$ both have the d.pr. In the next theorem we consider the uniqueness of decompositions in $L_\infty(\mathbf{R})$.

5.1. THEOREM. *Let $f\in L_\infty(\mathbf{R})$ and suppose that $\Delta_{a_1}\dots\Delta_{a_n}f=0$, where a_i/a_j is irrational for every $1\leqq i<j\leqq n$. Then the decomposition*

(12) $\quad f=f_1+\dots+f_n \quad (f_i\in L_\infty(\mathbf{R})$ *and* $\Delta_{a_i}f=0$ *for every* $i=1, \dots, n)$

is unique, apart from additive constants. In addition, if

$$F_k^{(i)}(x) = \frac{1}{k} \sum_{j=1}^{k} f(x+ja_i) \quad (x\in \mathbf{R},\ i = 1, \ldots, n),$$

then the limits

$$F_i(x) = \lim_{k\to\infty} F_k^{(i)}(x) \quad (i = 1, \ldots, n)$$

exist for almost every x, and for every decomposition (12) $f_i - F_i$ *is constant for every* $i=1, \ldots, n$.

PROOF. By 4.6, there are functions $f_1, \ldots, f_n$ with (12). Let

$$g_k^{(i)}(x) = \frac{1}{k} \sum_{j=1}^{k} f_i(x+ja_1) \quad (x\in \mathbf{R},\ i = 2, \ldots, n),$$

then, by (12), we have $F_k^{(1)} = f_1 + g_k^{(2)} + \ldots + g_k^{(n)}$ for every $k=1, 2, \ldots$. Now $T_i(x) = x+a_1 \pmod{a_i}$ is a measure-preserving transformation of $[0, a_i)$ onto itself, and f_i is a bounded measurable function periodic mod a_i. Hence, by the pointwise ergodic theorem, $g_k^{(i)}$ converges to a function c_i almost everywhere, and c_i is invariant under T_i. Since a_i/a_1 is irrational for $i\geqq 2$, T_i is ergodic ([4], p. 26), and hence c_i is constant. Therefore $F_1 = f_1 + c_2 + \ldots + c_n$ almost everywhere. Similar argument shows that $F_i = f_i + \text{constant}$ for every $i=2, \ldots, n$.

In the next theorem we present some other classes of real functions with the decomposition property.

5.2. THEOREM. *Each of the following classes has the decomposition property.*

(i) b-BV¹: *the class of those bounded functions* $f\colon \mathbf{R}\to\mathbf{R}$ *for which*

$$\sup_x V(f;\ [x, x+1]) < \infty.$$

(ii) b-Lip: *the class of bounded Lipschitz functions on* **R**.

(iii) b-Lip k: *the class of those bounded functions* $f\colon \mathbf{R}\to\mathbf{R}$ *for which* $f^{(k-1)}$ *exists everywhere and is Lipschitz.*

For the proof we shall need the following result.

5.3. LEMMA. *Let* $\mathcal{F}$ *be a class of continuous, real valued functions defined on* **R** *such that if* $f\in\mathcal{F}$ *and c is constant then* $f+c\in\mathcal{F}$. *Let k be a positive integer and let* $\mathcal{G}$ *denote the class of all bounded and k times differentiable functions g with* $g^{(k)}\in\mathcal{F}$.

If $\mathcal{F}$ *has the decomposition property then so does* $\mathcal{G}$.

PROOF. Let $g\in\mathcal{G}$ and suppose that $\Delta_{a_1}\ldots\Delta_{a_n} g = 0$ holds. Taking the k^{th} derivative we obtain $\Delta_{a_1}\ldots\Delta_{a_n} g^{(k)} = 0$. Since $g^{(k)}\in\mathcal{F}$ and $\mathcal{F}$ has the d.pr., there are functions $f_1, \ldots, f_n\in\mathcal{F}$ such that

$$g^{(k)} = f_1 + \ldots + f_n \quad \text{and} \quad \Delta_{a_i} f_i = 0 \quad (i = 1, \ldots, n).$$

Let

$$c_i = (1/a_i) \int_0^{a_i} f_i\, dx, \quad h_i = f_i - c_i \quad (i = 1, \ldots, n), \quad \text{and} \quad c = \sum_{i=1}^{n} c_i.$$

We prove that if $h\colon \mathbf{R} \to \mathbf{R}$ is continuous, periodic modulo a and $\int_0^a h\, dx = 0$, then for every m there is an m times differentiable function G_m such that $G_m^{(m)} = h$ and G_m is periodic modulo a. Indeed, we can put $G_0 = h$. If $m > 0$ and G_{m-1} has been found, then we put $d = (1/a) \int_0^a G_{m-1}\, dx$ and

$$G_m(x) = \int_0^x \big(G_{m-1}(t) - d\big)\, dt.$$

It is easy to check that G_m has the desired properties.

Hence we can find k times differentiable functions $g_1, \ldots, g_n$ such that $g_i^{(k)} = h_i$ and $\Delta_{a_i} g_i = 0$ for every $i = 1, \ldots, n$. Let

$$P = g - (g_1 + \ldots + g_n) - \frac{c}{k!} x^k.$$

Then

$$P^{(k)} = g^{(k)} - (h_1 + \ldots + h_n) - c = 0,$$

and thus P is a polynomial. Let $p = P + \frac{c}{k!} x^k$, then

$$g = g_1 + \ldots + g_{n-1} + (g_n + p).$$

The functions g_i are continuous and periodic, hence bounded, and g is bounded by assumption. Thus p is bounded as well and hence it is constant. Therefore, as g_i and $g_n + p$ are in $\mathcal{G}$, the proof is complete.

PROOF OF THEOREM 5.2. We denote $\|f\|_u = \sup_x |f(x)|$ and

$$\|f\|_{\mathrm{Lip}} = \inf \{K\colon\ |f(x) - f(y)| \leqq K|x - y| \ \text{for every}\ x, y \in \mathbf{R}\}.$$

It is easy to see that b-BV[1] and b-Lip are Banach spaces with the norms $\|f\| = \|f\|_u + \sup_x V\big(f;\ [x, x+1]\big)$ and $\|f\| = \|f\|_u + \|f\|_{\mathrm{Lip}}$, respectively. We denote by $\mathcal{T}$ the product topology of $\mathbf{R}^{\mathbf{R}}$, and also the corresponding subspace topologies for b-BV[1] and b-Lip. It is easy to check that the unit balls of these spaces are $\mathcal{T}$-closed subsets of the $\mathcal{T}$-compact set $S = \{f\colon \mathbf{R} \to \mathbf{R};\ \|f\|_u \leqq 1\}$. Hence these unit balls are compact in $\mathcal{T}$. Now the statement that b-BV[1] and b-Lip both have the d.pr. easily follows from Corollary 2.3.

Next we show that $f^{(k-1)}$ is bounded for every $f \in$ b-Lip k. We say that a function $g\colon \mathbf{R} \to \mathbf{R}$ is *big* if there are arbitrarily long intervals on which either $g \geqq 1$ or $g \leqq -1$ holds. Suppose that $f \in$ b-Lip k and $f^{(k-1)}$ is not bounded. Since $f^{(k-1)}$ is Lipschitz, this easily implies that $f^{(k-1)}$ is big. Now, if g is differentiable and g' is big then so is g. Hence f' is big, which clearly contradicts the fact that f is bounded.

Therefore $f \in$ b-Lip k implies $f^{(k-1)} \in$ b-Lip, that is $f \in$ b-Lip k if and only if f is bounded and $f^{(k-1)} \in$ b-Lip. Hence, by Lemma 5.3, b-Lip k has the d.pr.

5.4. REMARK. If we delete the condition of boundedness, the corresponding classes BV^1, Lip, and Lip k will not have the d.pr. since each of these classes contains the identity function.

5.5. REMARK. By Theorem 4.1 all the classes $L_p(\mathbf{R})$ $(p \geqq 1)$ have the d.pr. Since the only periodic function in $L_p(\mathbf{R})$ is the identically zero function, this implies that if $f \in L_p(\mathbf{R})$ for some $1 \leqq p < \infty$, and $\Delta_{a_1} \dots \Delta_{a_n} f = 0$ for some $a_1, \dots, a_n \in \mathbf{R}$, then $f=0$. This is known; see [3], Corollary 2.7.

5.6. QUESTION. Let L_p^1 denote the class of those measurable functions $f\colon \mathbf{R} \to \mathbf{R}$ for which

$$\sup_x \Big(\int_x^{x+1} |f|^p \, dx \Big)^{1/p} < \infty \quad (1 \leqq p < \infty).$$

Does L_p^1 have the d.pr.? We remark that every periodic function in L_p^1 is S^p almost periodic ([1], p. 77). Thus an affirmative answer to this question would imply that if $f \in L_p^1$ $(1 \leqq p < \infty)$ and $\Delta_{a_1} \dots \Delta_{a_n} f = 0$ then f is S^p almost periodic. Is this true? More generally, is it true that every function $f \in L_p^1$ $(1 \leqq p < \infty)$ satisfying a homogeneous difference equation is S^p almost periodic?

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