Acta Math. Hung.
54 (3—4) (1989), 329—341.

PERIODIC DECOMPOSITIONS OF CONTINUOUS FUNCTIONS

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1. Introduction. Let $f\colon \mathbf{R}\to\mathbf{R}$ be a real function and let $\alpha_1, \dots, \alpha_n$ be given real numbers. We say that $f=f_1+\dots+f_n$ is an $(\alpha_1, \dots, \alpha_n)$-decomposition of f if f_i is periodic mod α_i for every $i=1, \dots, n$. If $\mathcal{F}$ is a class of real functions and each f_i belongs to $\mathcal{F}$ then we say that $f=f_1+\dots+f_n$ is an $(\alpha_1, \dots, \alpha_n)$-decomposition in $\mathcal{F}$.

Let Δ_α denote the difference operator, that is let

$$\Delta_\alpha f(x) = f(x+\alpha)-f(x) \quad (x\in\mathbf{R};\, f\colon \mathbf{R}\to\mathbf{R}).$$

If $f=f_1+\dots+f_n$ is an $(\alpha_1, \dots, \alpha_n)$-decomposition then $\Delta_{\alpha_i} f_i=0$ for every i and, as the operators Δ_{α_i} commute, we obtain

$$\Delta_{\alpha_1}\dots\Delta_{\alpha_n} f = 0. \tag{1}$$

A class $\mathcal{F}$ of real functions is said to have the decomposition property if, for every $f\in\mathcal{F}$ and $\alpha_1, \dots, \alpha_n\in\mathbf{R}$, (1) implies that f has an $(\alpha_1, \dots, \alpha_n)$-decomposition in $\mathcal{F}$. Neither the class $\mathbf{R}^{\mathbf{R}}$ of all functions $f\colon \mathbf{R}\to\mathbf{R}$ nor $C(\mathbf{R})$, the class of all continuous functions defined on $\mathbf{R}$, has the decomposition property. Indeed, if f is the identity function $f(x)=x$ then $\Delta_\alpha\Delta_\beta f=0$ for every $\alpha, \beta\in\mathbf{R}$. On the other hand, if, say, $\alpha=\beta$, then f does not have an (α, β)-decomposition since f is not periodic. Also, for any α and β, f does not have an (α, β)-decomposition in $C(\mathbf{R})$, since otherwise it would be bounded.

Our main result states that $BC(\mathbf{R})$, the class of all bounded continuous functions has the decomposition property. The first result in this direction was proved by M. Wierdl in [7]. He showed that if $f\in BC(\mathbf{R})$ and $\Delta_{\alpha_1}\Delta_{\alpha_2} f=0$, where α_1 and α_2 are incommensurable, then f has a continuous (α_1, α_2)-decomposition. Our theorem extends this result for every system of numbers $\alpha_1, \dots, \alpha_n$. The proof is given in the next section.

We denote by $\|f\|$ the sup norm of f, that is we put

$$\|f\| = \sup\{|f(x)|: x\in\mathbf{R}\} \quad (f\colon \mathbf{R}\to\mathbf{R}).$$

Let C_n denote the smallest number with the following property: whenever a function $f\in BC(\mathbf{R})$ satisfies (1) then f has an $(\alpha_1, \dots, \alpha_n)$-decomposition $f=f_1+\dots+f_n$ in $C(\mathbf{R})$ such that $\|f_i\|\leqq C_n\|f\|$ $(i=1, \dots, n)$. In Section 3 we show that $C_n\leqq 2^{n-2}$ for every $n\geqq 2$; in particular we have $C_2=1$. We also prove that if $\alpha_1, \dots, \alpha_n$ are pairwise incommensurable then (1) implies the existence of an $(\alpha_1, \dots, \alpha_n)$-decomposition with $\|f_i\|\leqq\left(2-\dfrac{1}{n}\right)\|f\|$. Probably neither of the bounds 2^{n-2} (for $n>2$)

and $2-\frac{1}{n}$ is sharp; the determination of the best constants in these theorems proves to be surprisingly difficult.

Among the unbounded continuous functions satisfying (1) are the polynomials of degree less than n. This raises the following question: which functions can be written in the form $f=p+f_1+\ldots+f_n$, where p is a polynomial of degree $<n$, f_i are continuous and $\Delta_{\alpha_i} f_i=0$ $(i=1, \ldots, n)$. Again, (1) is a necessary condition. However, (1) is not sufficient, as it was shown by I. Z. Ruzsa and M. Szegedy. In Section 4 we discuss this phenomenon and give a necessary and sufficient condition for the existence of such a decomposition.

In Section 5 we conclude with some remarks and problems.

2. The decomposition property of $BC(\mathbf{R})$. In this section we prove the following theorem.

2.1. THEOREM. *Let $\alpha_1, \ldots, \alpha_n$ be real numbers and let $f \in BC(\mathbf{R})$. Then f has an $(\alpha_1, \ldots, \alpha_n)$-decomposition in $C(\mathbf{R})$ if and only if* (1) *holds.*

In the following series of lemmas first we consider the case when the α_i's are pairwise incommensurable, next when they are (pairwise) commensurable, and finally we turn to the mixed case. We shall use the notation

$$M(f)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f\,dx$$

whenever $f\colon \mathbf{R}\to\mathbf{R}$ is locally summable and the limit exists. We remark that for every $f\in L^\infty(\mathbf{R})$ and $\alpha\in\mathbf{R}$ we have $M(\Delta_\alpha f)=0$, since

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \Delta_\alpha f\,dx=\lim_{T\to\infty}\frac{1}{T}\left(\int_T^{T+\alpha} f\,dx-\int_0^\alpha f\,dx\right)=0.$$

2.2. LEMMA. *Let β, γ be incommensurable reals and let $f\in C(\mathbf{R})$ be periodic* mod γ. *Then*

$$f_k(x)=\frac{1}{k}\sum_{j=0}^{k-1} f(x+j\beta) \quad (x\in\mathbf{R},\ k\in\mathbf{N})$$

converges uniformly on $\mathbf{R}$ to the constant $M(f)$ as $k\to\infty$.

PROOF. Since $M(f)=\frac{1}{\gamma}\int_0^\gamma f\,dx$, the statement follows from the well-known fact that the sequence $\left\{k\frac{\beta}{\gamma}\right\}_{k=1}^\infty$ is uniformly distributed in [0, 1] (see [2], p. 390).

2.3. LEMMA. *Suppose that f has an $(\alpha_1, \ldots, \alpha_n)$-decomposition in $C(\mathbf{R})$, where $\alpha_1, \ldots, \alpha_n$ are pairwise incommensurable. Then*

(i) *for every $i=1, \ldots, n$ the functions*

$$f_{i,k}(x)=\frac{1}{k}\sum_{j=0}^{k-1} f(x+j\alpha_i) \quad (x\in\mathbf{R},\ k\in\mathbf{N})$$

converge uniformly to a function f_i as $k\to\infty$;

(ii) $\|f_i\| \leqq \|f\|$ $(i=1, \dots, n)$;

(iii) *whenever* $f=g_1+\dots+g_n$ *is an* $(\alpha_1, \dots, \alpha_n)$*-decomposition of* f *in* $C(\mathbf{R})$ *then* g_i-f_i *is constant for every* i,

(iv) *if* $M(f)=0$ *then* $f=f_1+\dots+f_n$ *is the only* $(\alpha_1, \dots, \alpha_n)$*-decomposition of* f *in* $C(\mathbf{R})$ *with* $M(f_i)=0$ $(i=1, \dots, n)$.

PROOF. Suppose that $f=g_1+\dots+g_n$ is an $(\alpha_1, \dots, \alpha_n)$-decomposition of f in $C(\mathbf{R})$. The previous lemma implies that for every i, $f_{i,k}$ converges uniformly to $g_i+\sum_{j\neq i} M(g_j)$. This proves (i) and (iii) simultaneously, while (ii) is obvious from (i). Since $M(f_i)=\lim_{k\to\infty} M(f_{i,k})=M(f)$ for every i, (iv) follows.

2.4. LEMMA. *Let* $f\in C(\mathbf{R})$, *let* β, γ *be incommensurable reals and suppose* $\Delta_\gamma f=0$. *Then the following are equivalent.*

(i) *There exists* $K>0$ *such that* $\left|\sum_{j=0}^{k-1} f(x+j\beta)\right| \leqq K$ *for every* $x\in\mathbf{R}$ *and* $k\in\mathbf{N}$;

(ii) *there is* $g\in C(\mathbf{R})$ *such that* $\Delta_\gamma g=0$ *and* $\Delta_\beta g=f$.

PROOF. This is a special case of a well-known result in ergodic theory. Indeed, putting Y for the torus $\mathbf{R}/\gamma\mathbf{Z}$, $\Theta(x)=x+\beta$ is a homeomorphism of Y and Y is a minimal orbit-closure since $\{\Theta^n(x)\colon n\in\mathbf{N}\}$ is dense in Y for every x. With this notation Theorem 14.11 of [3] (p. 135) implies our Lemma.

In order to make this paper self-contained, we offer a simple proof of (i)⇒(ii) (the other implication being obvious). In the following proof we use the ideas of [7].

Let $E=\{n\beta+k\gamma\colon n, k\in\mathbf{Z}, n>0\}$, then E is everywhere dense in $\mathbf{R}$, as β, γ are incommensurable. For $x=n\beta+k\gamma\in E$ we define

$$g(x) = \sum_{j=0}^{n-1} f(j\beta).$$

Then g is bounded on E by (i) and obviously $g(x+\gamma)=g(x)$ holds for every $x\in E$. We put

$$\bar{g}(y) = \limsup_{\substack{x\to y\\ x\in E}} g(x), \quad \underline{g}(y) = \liminf_{\substack{x\to y\\ x\in E}} g(x)$$

for every $y\in\mathbf{R}$. Since $g(x+\beta)-g(x)=f(x)$ holds for every $x\in E$ and f is continuous, it is easy to check that for every $y\in\mathbf{R}$

$$\bar{g}(y+\beta)-\bar{g}(y)=f(y) \quad\text{and}\quad \underline{g}(y+\beta)-\underline{g}(y)=f(y). \tag{2}$$

Putting $h=\bar{g}-\underline{g}$, this implies $h(y+\beta)=h(y)$ for every y. Now $\bar{g}$, $\underline{g}$ are periodic mod γ and hence so is h. Therefore

$$h(y+n\beta+k\gamma) = h(y)$$

holds for every $y\in\mathbf{R}$ and $n, k\in\mathbf{Z}$, $n>0$, that is $h(y+x)=h(y)$ for every $y\in\mathbf{R}$ and $x\in E$. Since h is the difference of an upper semicontinuous and a lower semicontinuous function, h itself is upper semicontinuous. As E is everywhere dense, this implies, by the previous equality, that $h(y+z)\geqq h(y)$ for every $y, z\in\mathbf{R}$. Therefore h is constant. Then $\bar{g}=g+h$ shows that $\bar{g}$ is upper and lower semicontinuous simultaneously, and thus $\bar{g}$ is continuous. Finally, (2) completes the proof.

2.5. Lemma. *Let $\alpha_1, \ldots, \alpha_n$ be pairwise incommensurable reals. Let $f \in BC(\mathbf{R})$ and suppose that* (1) *holds. Then f has an $(\alpha_1, \ldots, \alpha_n)$-decomposition in $C(\mathbf{R})$.*

Proof. We argue by induction on n. For $n=1$ the statement is obvious. Let $n>1$ and suppose that the assertion is true for $n-1$. Let $g=\Delta_{\alpha_n} f$, then $g \in BC(\mathbf{R})$, $M(g)=0$ and $\Delta_{\alpha_1} \ldots \Delta_{\alpha_{n-1}} g=0$. Thus, by the induction hypothesis and Lemma 2.3 (iv), we have a decomposition

$$\Delta_{\alpha_n} f = \sum_{i=1}^{n-1} g_i, \tag{3}$$

where $g_i \in C(\mathbf{R})$, $\Delta_{\alpha_i} g_i=0$ and $M(g_i)=0$ for every $i=1, \ldots, n-1$. Applying (3) at $x+j\alpha_n$ and summing up for $j=0, \ldots, N-1$, we get

$$\Delta_{N\alpha_n} f = \sum_{i=1}^{n-1} G_{i,N}, \tag{4}$$

where

$$G_{i,N}(x) = \sum_{j=0}^{N-1} g_i(x+j\alpha_n) \quad (x \in \mathbf{R},\ 1 \leqq i \leqq n-1).$$

Now, for every fixed N, (4) is an $(\alpha_1, \ldots, \alpha_{n-1})$-decomposition of $\Delta_{N\alpha_n} f$ in $C(\mathbf{R})$ with $M(G_{i,N})=N \cdot M(g_i)=0$ $(i=1, \ldots, n-1)$. Hence, by Lemma 2.3 (i), (ii) and (iv),

$$\|G_{i,N}\| \leqq \|\Delta_{N\alpha_n} f\| \leqq 2\|f\| \quad (i = 1, \ldots, n-1).$$

This shows that $\left|\sum_{j=0}^{N-1} g_i(x+j\alpha_n)\right| \leqq 2\|f\|$ for every $x \in \mathbf{R}$, $N \in \mathbf{N}$ and $i=1, \ldots, n-1$. Hence, by Lemma 2.4, we obtain $g_i=\Delta_{\alpha_n} f_i$, where $f_i \in C(\mathbf{R})$ and $\Delta_{\alpha_i} f_i=0$ $(i=1, \ldots, n-1)$. We define $f_n=f-\sum_{i=1}^{n-1} f_i$. Then we have

$$\Delta_{\alpha_n} f_n = \Delta_{\alpha_n} f - \sum_{i=1}^{n-1} \Delta_{\alpha_n} f_i = g - \sum_{i=1}^{n-1} g_i = 0$$

which completes the proof.

2.6. Lemma. *Let $\alpha_1, \ldots, \alpha_n$ $(n \geqq 2)$ be pairwise incommensurable reals and $f \in C(\mathbf{R})$. If $\Delta_{\alpha_1} \ldots \Delta_{\alpha_{n-1}} f=\Delta_{\alpha_n} f=0$ then f is constant.*

Proof. By the previous lemma, f has an $(\alpha_1, \ldots, \alpha_{n-1})$-decomposition $f=f_1+\ldots+f_{n-1}$ in $C(\mathbf{R})$. Then $f_1+\ldots+f_{n-1}-f$ is an $(\alpha_1, \ldots, \alpha_n)$-decomposition of the zero function. By Lemma (2.3) (iii), in this decomposition each term must be constant. In particular, f is constant.

Next we turn to the case when the numbers $\alpha_1, \ldots, \alpha_n$ are (pairwise) commensurable.

2.7. Lemma. *Let f: $\mathbf{R} \to \mathbf{R}$ be a bounded function, let $\alpha \in \mathbf{R}$, $n \in \mathbf{N}$ and suppose $\Delta_\alpha^n f=0$. Then $\Delta_\alpha f=0$.*

PROOF. We prove the lemma for $n=2$; the general case then follows by induction. Let $\Delta_\alpha f=g$, then $\Delta_\alpha g=0$ by assumption. For every $x\in\mathbf{R}$ and $N\in\mathbf{N}$ we have

$$f(x+N\alpha)=f(x)+\sum_{i=0}^{N-1}\Delta_\alpha f(x+i\alpha)=f(x)+Ng(x)$$

and hence $\|g\|\leqq\frac{2}{N}\|f\|$. Letting $N\to\infty$ we obtain $\Delta_\alpha f=g=0$.

In the sequel we shall use the following notation. If $\alpha_1, \dots, \alpha_n$ are non-zero and (pairwise) commensurable then we define the least common multiple of $\alpha_1, \dots, \alpha_n$ by

$$\alpha=[\alpha_1, \dots, \alpha_n]=\min\{\beta>0\colon \beta/\alpha_i\in\mathbf{Z}\quad (i=1, \dots, n)\}.$$

We shall denote $m_i=\alpha/\alpha_i$ so that

$$m_1\alpha_1=\dots=m_n\alpha_n=\alpha.$$

Also, we define the greatest common divisor of $\alpha_1, \dots, \alpha_n$ by

$$(\alpha_1, \dots, \alpha_n)=\max\{\beta>0\colon \alpha_i/\beta\in\mathbf{Z}\quad (i=1, \dots, n)\}.$$

2.8. LEMMA. *Let $\alpha_1, \dots, \alpha_n$ be positive and commensurable, and let $f\in BC(\mathbf{R})$. If* (1) *holds then f has an $(\alpha_1, \dots, \alpha_n)$-decomposition in $C(\mathbf{R})$.*

PROOF. From (1) it follows that

$$\Delta_{k_1\alpha_1}\cdots\Delta_{k_n\alpha_n}f=0 \tag{5}$$

for every $k_1, \dots, k_n\in\mathbf{Z}$. In particular, with $k_i=m_i$ we obtain $\Delta_\alpha^n f=0$ and hence, by Lemma 2.7, $\Delta_\alpha f=0$. We may write (5) in the form

$$\sum_{\substack{e\in\{0,1\}^n\\ e=(e_1, \dots, e_n)}}(-1)^{e_1+\dots e_n}f(x+e_1k_1\alpha_1+\dots+e_nk_n\alpha_n)=0\quad (x\in\mathbf{R}).$$

Summing up for $0\leqq k_i\leqq m_i-1$ we obtain with $M=m_1\dots m_n$

$$M\cdot f(x)=-\sum_{e\neq 0}(-1)^{e_1+\dots+e_n}\sum_{0\leqq k_i\leqq m_i-1}f(x+e_1k_1\alpha_1+\dots+e_nk_n\alpha_n).$$

Grouping the terms according to the last non-zero e_j, we get the representation $f=\sum_{j=1}^n f_j$, where

$$f_j(x)=\sum_{\substack{e=\{0,1\}^n\\ e_j=1\\ e_{j+1}=\dots=e_n=0}}(-1)^{e_1+\dots+e_{j-1}}\frac{1}{M}\sum_{0\leqq k_i\leqq m_i-1}f(x+e_1k_1\alpha_1+\dots+e_jk_j\alpha_j). \tag{6}$$

It is easy to check, using $\Delta_\alpha f=\Delta_{m_j\alpha_j}f=0$ that $\Delta_{\alpha_j}f_j=0$ for each $j=1, \dots, n$.

Now we turn to the proof of Theorem 2.1. Suppose that f, $\alpha_1, \dots, \alpha_n$ satisfy the conditions of the theorem, where $\alpha_1, \dots, \alpha_n$ are arbitrary reals. If $\alpha_i=0$ for some i then $f_i=f$, $f_j=0$ $(j\neq i)$ will provide an $(\alpha_1, \dots, \alpha_n)$-decomposition of f in $C(\mathbf{R})$. Therefore we assume $\alpha_i\neq 0$ $(i=1, \dots, n)$. We shall write the system $\alpha_1, \dots, \alpha_n$ as

$$\alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_b, \dots, \varrho_1, \dots, \varrho_r\quad (a+b+\dots+r=n), \tag{7}$$

where the grouping is made according to commensurability, i.e. the quotient of two numbers in the system is rational if and only if they are denoted by the same Greek letter. We put

$$\alpha = [\alpha_1, \ldots, \alpha_a], \ldots, \ \varrho = [\varrho_1, \ldots, \varrho_r].$$

Now (1) implies, as in the proof of Lemma 2.8, that $\Delta_\alpha^a \ldots \Delta_\varrho^r f = 0$ and hence, by Lemma 2.7,

$$\Delta_\alpha \ldots \Delta_\varrho f = 0. \tag{8}$$

Since the numbers $\alpha, \ldots, \varrho$ are pairwise incommensurable, Lemma 2.5 provides an $(\alpha, \ldots, \varrho)$-decomposition

$$f = f_\alpha + f_\beta + \ldots + f_\varrho \tag{9}$$

in $C(\mathbf{R})$. Let $g = \Delta_{\alpha_1} \ldots \Delta_{\alpha_a} f_\alpha$. Since $\Delta_\beta^b \ldots \Delta_\varrho^r \Delta_{\alpha_1} \ldots \Delta_{\alpha_a} f = 0$, another application of Lemma 2.7 gives $\Delta_\beta \ldots \Delta_\varrho \Delta_{\alpha_1} \ldots \Delta_{\alpha_a} f = 0$. Hence

$$\Delta_\beta \ldots \Delta_\varrho g = \Delta_\beta \ldots \Delta_\varrho \Delta_{\alpha_1} \ldots \Delta_{\alpha_a} (f - f_\beta - \ldots - f_\varrho) = 0.$$

We also have $\Delta_\alpha g = 0$ and, as $\alpha, \ldots, \varrho$ are pairwise incommensurable, g is constant by Lemma 2.6. Since g is the difference of a bounded function, we have $M(g) = 0$ and hence $g = 0$. That is,

$$\Delta_{\alpha_1} \ldots \Delta_{\alpha_a} f_\alpha = 0 \tag{10}$$

and then an application of Lemma 2.8 yields a continuous $(\alpha_1, \ldots, \alpha_a)$-decomposition of f_α. Similarly, we obtain a continuous $(\beta_1, \ldots, \beta_b)$-decomposition of f_β, etc. Replacing $f_\alpha, \ldots, f_\varrho$ by these decompositions, we get a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition of f. This completes the proof of Theorem 2.1.

We conclude this section with the following immediate corollary of Theorem 2.1.

2.9. Theorem. *If $f \in BC(\mathbf{R})$ has an $(\alpha_1, \ldots, \alpha_n)$-decomposition then f has an $(\alpha_1, \ldots, \alpha_n)$-decomposition in $C(\mathbf{R})$.*

3. Norms of the decompositions. By the *norm* of the decomposition $f = f_1 + \ldots + f_n$ we mean $\max_{1 \leq i \leq n} \|f_i\|$. For every $n \geq 1$ we shall denote by C_n the greatest lower bound of all positive numbers C with the following property: whenever $f \in BC(\mathbf{R})$ satisfies (1) then f has a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition of norm not exceeding $C\|f\|$. In this section we show that $C_n \leq 2^{n-2}$ holds for every $n > 1$.

Our following results show that the more "independent" the numbers $\alpha_1, \ldots, \alpha_n$ are, the sharper estimates on the norm of the decomposition we can prove.

3.1. Lemma. *Suppose that $f \in BC(\mathbf{R})$ satisfies* (1), *where $\alpha_1, \ldots, \alpha_n$ are pairwise incommensurable. Then f has a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition with norm not exceeding $\left(2 - \frac{1}{n}\right)\|f\|$.*

Proof. Let $f = g_1 + \ldots + g_n$ be a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition of f. We define $f_{i,k}$ and f_i as in Lemma 2.3 (i). As we proved in Lemma 2.3, $f_i = g_i + \sum_{j \neq i} M(g_j)$ holds for every i. Summing up these equalities for $i = 1, \ldots, n$ we obtain

$$\sum_{i=1}^n f_i = f + (n-1) \sum_{i=1}^n M(g_j) = f + (n-1) M(f).$$

Now we put $h_i=f_i-\left(1-\frac{1}{n}\right)M(f)$. Then $h_1, \ldots, h_n$ is an $(\alpha_1, \ldots, \alpha_n)$-decomposition of f and $\|h_i\| \leqq \|f_i\|+\left(1-\frac{1}{n}\right)|M(f)| \leqq \left(2-\frac{1}{n}\right)\|f\|$ for every i.

3.2. LEMMA. *Suppose that* $\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}$ *are linearly independent over the rationals. If* $f \in BC(\mathbf{R})$ *satisfies* (1) *then*

(i) $\sup f=\sum_{i=1}^{n} \max f_i$, $\inf f=\sum_{i=1}^{n} \min f_i$ *hold for every continuous* $(\alpha_1, \ldots, \alpha_n)$-*decomposition of* f;

(ii) *there is a continuous* $(\alpha_1, \ldots, \alpha_n)$-*decomposition* $f=f_1+\ldots+f_n$ *such that* $\|f\|=\sum_{i=1}^{n}\|f_i\|$.

PROOF. Let $f=f_1+\ldots+f_n$ be a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition. Let f_i attain its maximum at x_i $(i=1, \ldots, n)$, and let $\varepsilon>0$ be arbitary. By the uniform continuity of f_i, there is $\delta>0$ such that $|f_i(x)-f_i(y)|<\varepsilon/n$ hold whenever $|x-y|<\delta$ and $i=1, \ldots, n$. Since $\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}$ are linearly independent, we may apply Kronecker's theorem and find $x \in \mathbf{R}$ and $k_1, \ldots, k_n \in \mathbf{Z}$ such that

$$\left|\frac{x}{\alpha_i}-k_i-\frac{x_i}{\alpha_i}\right|<\frac{\delta}{|\alpha_i|} \quad (i=1, \ldots, n)$$

(see [2], p. 382). Then $|x-k_i\alpha_i-x_i|<\delta$ for every i and hence

$$\sup f \geqq f(x)=\sum_{i=1}^{n} f_i(x)=\sum_{i=1}^{n} f_i(x-k_i\alpha_i) \geqq$$

$$\geqq \sum_{i=1}^{n} f_i(x_i)-\sum_{i=1}^{n}|f_i(x-k_i\alpha_i)-f_i(x_i)|>\sum_{i=1}^{n} \max f_i-\varepsilon .$$

Letting $\varepsilon \to 0$ we obtain $\sup f \geqq \sum_{i=1}^{n} \max f_i$. As the reverse inequality is obvious, this proves the first equality in (i), while the second follows by considering $-f$.

Now let $f=g_1+\ldots+g_n$ be an arbitrary $(\alpha_1, \ldots, \alpha_n)$-decomposition of f in $C(\mathbf{R})$. We put

$$f_i=g_i-\frac{1}{2}(\max g_i+\min g_i) \quad (i=1, \ldots, n-1)$$

and $f_n=f-\sum_{i=1}^{n} f_i$. Then $f=f_1+\ldots+f_n$ is an $(\alpha_1, \ldots, \alpha_n)$-decomposition; we show that $\|f\|=\sum_{i=1}^{n}\|f_i\|$ holds. By (i) we have

$$\sup f=\sum_{i=1}^{n} \max f_i=\sum_{i=1}^{n-1}\|f_i\|+\max f_n$$

and

$$\inf f = \sum_{i=1}^{n} \min f_i = -\sum_{i=1}^{n-1} \|f_i\| + \min f_n.$$

Now we have either $\|f_n\| = \max f_n$ or $\|f_n\| = -\min f_n$ and hence

$$\sum_{i=1}^{n} \|f_i\| \leqq \max(\sup f, -\inf f) \leqq \|f\|.$$

Since $\|f\| \leqq \sum_{i=1}^{n} \|f_i\|$ is obvious, (ii) is proved.

3.3. Corollary. *If α_1, α_2 are incommensurable and $f \in BC(\mathbf{R})$ satisfies $\Delta_{\alpha_1}\Delta_{\alpha_2} f = 0$ then there is a continuous (α_1, α_2)-decomposition $f = f_1 + f_2$ with $\|f\| = \|f_1\| + \|f_2\|$.*

Proof. If α_1, α_2 are incommensurable then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}$ are linearly independent over the rationals and the previous lemma applies.

This result shows that the bound $2 - \frac{1}{n}$ in Lemma 3.1 is not sharp for $n=2$. The next result proves the same for $n=3$.

3.4. Theorem. *If $f \in BC(\mathbf{R})$ satisfies $\Delta_{\alpha_1}\Delta_{\alpha_2}\Delta_{\alpha_3} f = 0$, where $\alpha_1, \alpha_2, \alpha_3$ are pairwise incommensurable then there is a continuous $(\alpha_1, \alpha_2, \alpha_3)$-decomposition of f with norm not exceeding $\|f\|$.*

Proof. Let λ denote the infimum of the norms of the terms of all possible continuous $(\alpha_1, \alpha_2, \alpha_3)$-decompositions of f. Since all these decompositions differ from each other by additive constants, it is easy to see that λ is, in fact, a minimum. Suppose that λ is attained in the decomposition $f = g_1 + g_2 + g_3$ and let, say, $\|g_1\| = \lambda$. Then $f - g_1$ has an (α_2, α_3)-decomposition and, applying Corollary 3.3, we obtain a continuous (α_2, α_3)-decomposition $f - g_1 = f_2 + f_3$ such that

$$\|f_2\| + \|f_3\| = \|f - g_1\| \leqq \|f\| + \lambda.$$

Since $f = g_1 + f_2 + f_3$, the minimality of λ implies $\lambda \leqq \|f_2\|$ and $\lambda \leqq \|f_3\|$. Hence $2\lambda \leqq \|f_2\| + \|f_3\| \leqq \|f\| + \lambda$, and thus $\lambda \leqq \|f\|$. Also, $\lambda + \|f_3\| \leqq \|f\| + \lambda$ showing $\|f_3\| \leqq \|f\|$. We obtain similarly $\|f_2\| \leqq \|f\|$ and hence $f = g_1 + f_2 + f_3$ is the desired decomposition.

Now we turn to the proof of $C_n \leqq 2^{n-2}$. First we establish the result for $n=2$.

3.5. Theorem. $C_2 = 1$.

Proof. Suppose that $f \in BC(\mathbf{R})$ satisfies $\Delta_{\alpha_1}\Delta_{\alpha_2} f = 0$. If α_1, α_2 are incommensurable then Corollary 3.3 implies that there is a continuous (α_1, α_2)-decomposition of f with norm $\leqq \|f\|$. Thus we may assume that α_1, α_2 are commensurable. We may even suppose that α_1, α_2 are relatively prime integers. Let $f = g_1 + g_2$ be a continuous (α_1, α_2)-decomposition of f. We put

$$M(x) = \max_{i \in \mathbf{Z}} g_1(x+i), \quad m(x) = \min_{i \in \mathbf{Z}} g_1(x+i) \quad (x \in \mathbf{R}),$$

then M and m are continuous and periodic mod 1. We define

$$f_1=g_1-\frac{1}{2}(M+m),\quad f_2=g_2+\frac{1}{2}(M+m).$$

Then $f=f_1+f_2$ is a continuous (α_1,α_2)-decomposition of f; we show that $\|f_i\|\leqq\|f\|$ $(i=1,2)$. Let

$$N(x)=\max_{i\in\mathbf{Z}}f_2(x+i),\quad n(x)=\min_{i\in\mathbf{Z}}f_2(x+i)\quad(x\in\mathbf{R}).$$

Let $x\in\mathbf{R}$ be fixed. There are integers j,k such that $g_1(x+j)=M(x)$ and $f_2(x+k)=N(x)$. Since α_1,α_2 are relatively prime, the system of congruences

$$s\equiv j\ (\mathrm{mod}\ \alpha_1),\quad s\equiv k\ (\mathrm{mod}\ \alpha_2)$$

is solvable; let s be a solution. Then

$$f_1(x+s)=f_1(x+j)=M(x)-\frac{1}{2}\big(M(x)+m(x)\big)=\frac{1}{2}\big(M(x)-m(x)\big)$$

and $f_2(x+s)=f_2(x+k)=N(x)$. Therefore

(11) $$\|f\|\geqq f(x+s)=f_1(x+s)+f_2(x+s)=\frac{1}{2}\big(M(x)-m(x)\big)+N(x).$$

A similar argument proves

$$-\|f\|\leqq-\frac{1}{2}\big(M(x)-m(x)\big)+n(x),$$

that is

(12) $$\|f\|\geqq\frac{1}{2}\big(M(x)-m(x)\big)-n(x).$$

For every x we have either $N(x)\geqq 0$ or $n(x)\leqq 0$. Therefore (11) and (12) imply $\frac{1}{2}\big(M(x)-m(x)\big)\leqq\|f\|$. Since $\|f_1\|\leqq\max_{x\in\mathbf{R}}\frac{1}{2}\big(M(x)-m(x)\big)$ this proves $\|f_1\|\leqq\|f\|$. Also, (11) and (12) imply $N(x)\leqq\|f\|$ and $-n(x)\leqq\|f\|$. Since $\|f_2\|\leqq\max_{x\in\mathbf{R}}\max\big(N(x),-n(x)\big)$, $\|f_2\|\leqq\|f\|$ follows. This completes the proof.

3.6. THEOREM. *For every* $n\geqq 2$, $C_n\leqq 2^{n-2}$.

PROOF. Suppose that $f\in BC(\mathbf{R})$ satisfies (1). If $\alpha_1,\dots,\alpha_n$ are commensurable then, as we proved in Lemma 2.8, f has a decomposition $f=f_1+\dots+f_n$, where f_j is defined by (6). The sum defining f_j consists of 2^{j-1} terms of norm not exceeding $\|f\|$. Hence $\|f_j\|\leqq 2^{j-1}\|f\|$ holds for every $j=1,\dots,n$. Let $g=f-\sum_{j=1}^{n-2}f_j$, then g has an (α_{n-1},α_n)-decomposition and

$$\|g\|\leqq\Big(1+\sum_{j=1}^{n-2}2^{j-1}\Big)\|f\|=2^{n-2}\|f\|.$$

By Theorem 3.5, there is a continuous (α_{n-1}, α_n)-decomposition $g=g_{n-1}+g_n$ such that $\|g_i\| \leqq 2^{n-2}\|f\|$ $(i=n-1, n)$, and hence $f=\sum_{i=1}^{n-2} f_i+g_{n-1}+g_n$ is an $(\alpha_1, \ldots, \alpha_n)$-decomposition with norm $\leqq 2^{n-2}\|f\|$.

Next suppose that $\alpha_1, \ldots, \alpha_n$ are not commensurable. Then (7) contains at least two groups. Now (8) and Lemma 3.1 imply that the $(\alpha, \ldots, \varrho)$-decomposition (9) can be chosen with $\|f_\alpha\| \leqq 2\|f\|, \ldots, \|f_\varrho\| \leqq 2\|f\|$. Applying (10) and taking into consideration that $\alpha_1, \ldots, \alpha_a$ are commensurable, we obtain an $(\alpha_1, \ldots, \alpha_a)$-decomposition of f_α with norm not exceeding

$$2^{a-2}\|f_\alpha\| \leqq 2^{n-3}\cdot 2\|f\| = 2^{n-2}\|f\|.$$

The same argument applies for $f_\beta, \ldots, f_\varrho$ and hence we can find an $(\alpha_1, \ldots, \alpha_n)$-decomposition of f with norm $\leqq 2^{n-2}\|f\|$.

4. Quasi-decompositions. We say that $f=p+f_1+\ldots+f_n$ is $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition of the function f, if p is a polynomial of degree $<n$ and $\Delta_{\alpha_i} f_i=0$ $(i=1, \ldots, n)$. If f has an $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition then, as we remarked in the introduction, (1) must hold. On the other hand, the existence of an $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition in $C(\mathbf{R})$ imposes a stronger condition on f than (1). In order to see this let us confine ourselves to the case of $n=2$.

If $f=p+g+h$ is a continuous (α, β)-quasi-decomposition then p is linear, say, $p(x)=cx+d$. Hence

$$\frac{1}{k} f(k\alpha) = c\alpha+\frac{1}{k} d+\frac{1}{k} g(k\alpha)+\frac{1}{k} h(k\alpha) = c\alpha+O\left(\frac{1}{k}\right) \tag{13}$$

since g and h are bounded. Let $F=\Delta_\alpha f$, then $F=c\alpha+\Delta_\alpha h$ and thus $M(F)=c\alpha$. Since F is periodic mod β, this gives $c\alpha=\frac{1}{\beta}\int_0^\beta F\,dx$. Also,

$$f(k\alpha) = \sum_{i=0}^{k-1} F(i\alpha)+f(0),$$

and hence (13) can be written in the form

$$\frac{1}{k}\sum_{i=0}^{k-1} F(i\alpha)-\frac{1}{\beta}\int_0^\beta F\,dx = O\left(\frac{1}{k}\right) \quad (k\to\infty). \tag{14}$$

Now, if α and β are incommensurable, then the sequence $\{i\alpha\}_{i=0}^\infty$ is uniformly distributed mod β and hence

$$\frac{1}{k}\sum_{i=0}^{k-1} F(i\alpha) \to \frac{1}{\beta}\int_0^\beta F\,dx$$

holds for *every* function F continuous and periodic mod β. However, the convergence is not necessarily as fast as $O\left(\frac{1}{k}\right)$. Let F be a continuous function, periodic mod β, for which (14) does not hold. Then we can construct a continuous f with $\Delta_\alpha f = F$ (we even can prescribe f on $[0, \alpha]$ supposing $f(\alpha) - f(0) = F(0)$). Then f will satisfy $\Delta_\alpha \Delta_\beta f = 0$ but will not have an (α, β)-quasi-decomposition in $C(\mathbf{R})$. We remark that one can construct continuous functions violating (14) by elementary constructions (this is what M. Szegedy did) as well as using trigonometric series.

In our next theorem we shall give a necessary and sufficient condition for the existence of a continuous $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition by making use of the following generalized modulus of continuity

$$\delta_n(f) = \sup_{h\in\mathbf{R}} \|\Delta_h^n f\| = \sup\left\{\left|\sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x+jh)\right| : x,\ h\in\mathbf{R}\right\} \quad (f\in C(\mathbf{R})).$$

Whitney proved in [6] that there is a constant K_n such that

$$\inf\{\|f-p\| : p \text{ is a polynomial of degree } < n\} \leqq K_n \delta_n(f)$$

holds for every $f \in C(\mathbf{R})$.

4.1. Theorem. *A function $f\in C(\mathbf{R})$ has an $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition in $C(\mathbf{R})$ if an only if (1) and $\delta_n(f) < \infty$ hold simultaneously.*

Proof. If $f = p + f_1 + \ldots + f_n$ is a continuous $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition then

$$\delta_n(f) = \sup_{h\in\mathbf{R}} \|\Delta_h^n f\| = \sup_{h\in\mathbf{R}} \|\Delta_h^n (f-p)\| \leqq 2^n \|f-p\| \leqq 2^n (\|f_1\| + \ldots + \|f_n\|) < \infty.$$

In the other direction, if (1) and $\delta_n(f) < \infty$ hold then, by Whitney's theorem, there is a polynomial p of degree $< n$ such that $f - p \in BC(\mathbf{R})$. Since $\Delta_{\alpha_1} \ldots \Delta_{\alpha_n}(f-p) = 0$, we may apply Theorem 2.1 and obtain a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition of $f-p$.

Our next result is a simple application of this condition.

4.2. Theorem. *A function f has an $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition in $C(\mathbf{R})$ with a linear p if and only if (1) holds and f is uniformly continuous.*

Proof. The necessity is obvious. Suppose that f is uniformly continuous and satisfies (1). We may assume $\alpha_i > 0$ $(i = 1, \ldots, n)$. Let $h \in \mathbf{R}$ be fixed. We can choose integers k_i such that $h_i \overset{\text{def}}{=} h - k_i \alpha_i \in [0, \alpha_i)$ $(i = 1, \ldots, n)$. Then we have

$$\Delta_h^n f(x) = \Delta_{h_1 + k_1\alpha_1} \ldots \Delta_{h_n + k_n\alpha_n} f(x) =$$

$$= \sum_{\mathbf{e}\in\{0,1\}^n} (-1)^e f\big(x + e_1(h_1 + k_1\alpha_1) + \ldots + e_n(h_n + k_n\alpha_n)\big) =$$

$$= \sum_{\mathbf{e}} (-1)^e (f + \Delta_{e_1 h_1 + \ldots + e_n h_n} f)(x + e_1 k_1 \alpha_1 + \ldots + e_n k_n \alpha_n) =$$

$$= \Delta_{k_1\alpha_1} \ldots \Delta_{k_n\alpha_n} f(x) + \sum_{\mathbf{e}} (-1)^e \Delta_{(\mathbf{e},\mathbf{h})} f\big(x(\underline{e})\big),$$

where we denoted $e=e_1+\ldots+e_n$, $(\mathbf{e},\mathbf{h})=e_1h_1+\ldots+e_nh_n$ and $x(\mathbf{e})=x+e_1k_1\alpha_1+\ldots$ $\ldots+e_nk_n\alpha_n$. Since $|(\mathbf{e},\mathbf{h})|\leqq n\cdot\max_i(\alpha_i)\overset{\text{def}}{=\!=}\alpha$, we obtain

$$|\Delta_h^n f(x)| \leqq 0+2^n\sup\{|f(x+t)-f(x)|:\ x\in\mathbf{R},\ |t|\leqq\alpha\}<\infty$$

by the uniform continuity of f. This upper bound of $|\Delta_h^n f(x)|$ does not depend on h, therefore $\delta_n(f)<\infty$. By the previous theorem this implies that f has a continuous $(\alpha_1,\ldots,\alpha_n)$-quasi-decomposition $f=p+f_1+\ldots+f_n$. Here p must be uniformly continuous, and hence p is linear, which completes the proof.

5. Remarks and problems. 5.1. The following theorem is a special case of [5] Theorem 2.1. Let $\mathcal{F}$ be a translation-invariant linear space of bounded functions defined on $\mathbf{R}$. Suppose that there is a vector topology $\mathcal{T}$ on $\mathcal{F}$ such that $\mathcal{T}$ is translation-invariant, $f_n\in\mathcal{F}$, $\|f_n\|\to 0$ implies $f_n\to 0$ in $\mathcal{T}$, and $\{f\in\mathcal{F}:\ \|f\|\leqq 1\}$ is compact in $\mathcal{T}$. Then $\mathcal{F}$ has the decomposition property. While this condition establishes the decomposition property of several function classes (e.g. the classes of all bounded functions, all bounded Lipschitz functions or $L^\infty(\mathbf{R})$; see [5]), it cannot be applied for $BC(\mathbf{R})$. It was shown by V. Totik that there is no topology $\mathcal{T}$ on $BC(\mathbf{R})$ satisfying the conditions above. That is, the existence of such a topology is not necessary for the decomposition property of the class $\mathcal{F}$.

5.2. If a function f has an $(\alpha_1,\ldots,\alpha_n)$-decomposition in $C(\mathbf{R})$ then f is obviously uniformly almost periodic (u.a.p.; see [1]). On the other hand, if f is u.a.p. then condition (1) easily implies that each term of the Fourier series of f is periodic mod one of the numbers $\alpha_1,\ldots,\alpha_n$. That is, the $(\alpha_1,\ldots,\alpha_n)$-decomposition of f can be obtained by grouping the terms of the Fourier series of f according to the periods (cf. Bohr's theorem [1], p. 44). This observation suggests that a simple proof of Theorem 2.1 would be possible by showing that each bounded and continuous solution of the difference equation (1) must be u.a. p. This is an obvious corollary of our theorem; however, an independent proof would be desirable. We may ask the following, more general question. Is every bounded, continuous solution of a difference equation

$$\sum_{i=1}^{n} A_i f(x+a_i) = 0$$

necessarily u.a.p.? (We remark that, according to a theorem of J.-P. Kahane [4], p. 43, a bounded solution is u.a.p. if and only if it is uniformly continuous.)

5.3. Our results on the norms of the decompositions raise several questions. What is the best constant in place of $2-\frac{1}{n}$ in Lemma 3.1? What is the exact value of C_n? In particular, is $C_n=1$ true for every n?

5.4. Now we pose some problems concerning decompositions $f=f_1+\ldots+f_n$ with measurable f_i. Suppose that f is continuous and has an $(\alpha_1,\ldots,\alpha_n)$-decomposition with measurable f_i. Does f have an $(\alpha_1,\ldots,\alpha_n)$-decomposition in $C(\mathbf{R})$? (This amounts to ask whether f is necessarily bounded.) If not, does f have an $(\alpha_1,\ldots,\alpha_n)$-quasi-decomposition in $C(\mathbf{R})$? We remark that no non-constant polynomial can have an $(\alpha_1,\ldots,\alpha_n)$-decomposition with measurable f_i. (More generally, no function f with $\lim_{x\to\infty}|f(x)|=\infty$ can be the sum of finitely many periodic, measurable functions.) We also remark that x^m *does* have an $(\alpha_1, \dots, \alpha_n)$-decomposition (with non-measurable terms) supposing that $m<n$ and $\alpha_1, \dots, \alpha_n$ are linearly independent over the rationals. This was proved by M. Wierdl in [7].

5.5. The following result of Wierdl is also closely related to our topic. If $f\in C(\mathbf{R})$ satisfies (1) then the limit

$$\lim_{x\to\infty} \frac{f(x)}{x^{n-1}}$$

exists and is finite.

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(Received April 17, 1987)

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