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NOTE ON A PROBLEM OF Q. I. RAHMAN AND P. TURÁN

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1.

Q. I. Rahman and P. Turán began to investigate the following problem in 1972. Let

(1.1) $$v(z)=\prod_{j=1}^{n}(z-\xi_j)=\sum_{k=0}^{n}B_k z^k \quad (B_k\in\mathbf{C}),$$

a polynomial of degree n, and

(1.2) $$u(z)=\sum_{k=0}^{t}A_k z^k = A_0\prod_{j=1}^{t}\left(1-\frac{z}{\eta_j}\right) \quad (A_k\in\mathbf{C})$$

a polynomial of degree $t<n$. Consider the rational function

(1.3) $$f(z)=\frac{u(z)}{v(z)}$$

of degree n, and suppose the normalizations

(1.4) $$f(0)=1$$

and

(1.5) $$0<|\xi_1|\leqq\ldots\leqq|\xi_{n-1}|\leqq\xi_n=1.$$

Then we are seeking for the best possible lower estimate of the integral mean

(1.6) $$I^p(f,r)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\varphi})|^p\,d\varphi$$

taken on the circle with radius r, assuming always the natural condition

(1.7) $$(0<)r<|\xi_1|.$$

Q. I. Rahman and P. Turán, based on heuristic arguments, conjectured that there is an extremal function which gives the least possible integral mean for all p and r considered, and this extremal function should be $\frac{1}{1-z^n}$. The conjecture is undecided at the moment. Rahman and Turán reached the following result.

THEOREM (Q. I. Rahman and P. Turán [1]). *If* $1<p\leqq 2$ *and the notations and normalizations* (1.1)—(1.7) *are used, then we have with* $q=\frac{p}{p-1}$ (the usual dual of p)

$$I^p(f,r)\geqq\left\{1+\frac{r^{qn}}{\left(\sum_{k=0}^{n-1}\binom{n}{k}^p r^{pk}\right)^{q/p}}\right\}^{p/q}. \tag{1.8}$$

For large r, (1.8) is obviously far from being optimal, but when $r\to 0$, it is asymptotically

$$\{1+r^{qn}(1+o(1))\}^{p-1}=1+(p-1)r^{qn}(1+o(1)). \tag{1.9}$$

On the other hand

$$\left(\frac{1}{1-z^n}\right)^{p/2}=(1+\sum_{l=1}^{\infty}z^{nl})^{p/2}=\sum_{k=0}^{\infty}\binom{p/2}{k}(\sum_{l=1}^{\infty}z^{nl})^k=$$

$$=1+\sum_{m=1}^{\infty}\left\{\sum_{k=1}^{m}\binom{p/2}{k}\binom{m-1}{k-1}\right\}z^{mn}\quad(|z|<2^{-1/n})$$

and so by the Parseval formula (for $r<2^{-\frac{1}{n}}$)

(1.10)

$$I^p\left(\frac{1}{1-z^n},r\right)=\frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{1}{1-r^ne^{in\varphi}}\right)^{p/2}\right|^2d\varphi=\sum_{m=0}^{\infty}\left\{\sum_{k=1}^{m}\binom{p/2}{k}\binom{m-1}{k-1}\right\}^2r^{2mn}.$$

Hence we have the asymptotical expansion

$$I^p\left(\frac{1}{1-z^n},r\right)=1+\binom{p/2}{1}^2r^{2n}+O(r^{4n})=1+\frac{p^2}{4}r^{2n}(1+o(1)), \tag{1.11}$$

which is equal to (1.9) in the central case $p=2$. When $1<p<2$, Rahman and Turán use the Young—Hausdorff inequality, and therefore they do not obtain a sharp asymptotic in this case.

A further reason to believe in the truth of the conjecture is an observation of Q. I. Rahman, namely

THEOREM (Q. I. Rahman [2]). *Supposing* (1.1)—(1.7) *we have*

$$\max_{0\leqq\varphi\leqq 2\pi}|f(re^{i\varphi})|\geqq\max_{|z|=r}\left|\frac{1}{1-z^n}\right|=\frac{1}{1-r^n}. \tag{1.12}$$

In other words, if $p=\infty$, then the conjecture is true.

On the other hand, if $p=1$ or $2<p<\infty$, there are no known non-trivial results similar to (1.8).

The following result shows that the conjecture is true for small values of r, for arbitrary $p\geqq 1$.

THEOREM 1. *For* $1 \leqq p < \infty$ *and for any* r *and* f *satisfying* (1.1)—(1.7), *there exists a radius* $r_0(p, f) > 0$ *such that for all* $r < r_0(p, f)$ *we have*

$$I^p(f, r) \geqq I^p\left(\frac{1}{1-z^n}, r\right). \tag{1.13}$$

If $f(z) \neq \frac{1}{1-z^n}$ *then there exists a radius* $r_1(p, f) > 0$ *such that for all* $r < r_1(p, f)$ *we have*

$$I^p(f, r) > I^p\left(\frac{1}{1-z^n}, r\right). \tag{1.14}$$

Furthermore, if there is a root of the denominator of f *which has absolute value less than* 1, *then we can take*

$$r_1(p, f) = r_1(n, p, |B_0|) \quad (< |\xi_1|). \tag{1.15}$$

COROLLARY. *If for some* p, *there exists an extremal function such that for all* $r < |\xi_1|$ *it has minimal* p^{th} *integral mean, then this function is* $\frac{1}{1-z^n}$.

2.

Rahman and Turán also investigated the important special case when f is a logarithmic derivative of a polynomial of degree n. Their result is stronger than the immeadiate consequence of (1.8) which would depend also on $|f(0)|$.

THEOREM (Q. I. Rahman and P. Turán [1]). *If* $v(z)$ *is a polynomial satisfying* (1.1) *and* (1.5), $1 < p \leqq 2$ *and* r *is chosen according to* (1.7), *then*

$$I^p\left(\frac{v'}{v}, r\right) \geqq \frac{n^p r^{p(n-1)}}{1 + r^p + \sum_{k=2}^{n-1} \binom{n}{k}^p r^{kp}}. \tag{2.1}$$

Here the estimate is asymptotically

$$n^p r^{p(n-1)}(1 + o(1)) \tag{2.2}$$

as $r \to 0$, and for $v(z) = z^n - 1$

$$I^p\left(\frac{nz^{n-1}}{z^n - 1}, r\right) = n^p r^{p(n-1)} \sum_{k=0}^{\infty} \binom{-p/2}{k} r^{2kn} = n^p r^{p(n-1)}(1 + o(1)), \tag{2.3}$$

which shows that (2.1) is asymptotically[1] sharp for any p in the interval $1 < p \leqq 2$.

[1] Here we regard only the main term, whence this is not determined by a normalizatiton, as it was in the previous section.

Besides, in the case $p=\infty$ the expected theorem can be proved.

THEOREM (Q. I. Rahman [2]). *If* $v(z)$ *is a polynomial subject to* (1.1) *and* (1.5) *and* r *satisfies* (1.7), *then*

$$\max_{|z|=r}\left|\frac{v'(z)}{v(z)}\right| \geqq \max_{|z|=r}\left|\frac{nz^{n-1}}{z^n-1}\right| = \frac{nr^{n-1}}{1-r^n}. \tag{2.4}$$

Now, the extremality of z^n-1 in a small neighbourhood of zero can be proved in this problem, too.

THEOREM 2. *For any* $1\leqq p<\infty$ *and any polynomial* $v(z)$ *subject to* (1.1) *and* (1.5), *there exists a radius* $r_2(p,v)>0$ *such that for all* $r<r_2(p,v)$ *we have*

$$I^p\left(\frac{v'}{v}, r\right) \geqq I^p\left(\frac{nz^{n-1}}{z^n-1}, r\right). \tag{2.5}$$

If $v(z)\neq z^n-1$, *then there exists a radius* $r_3(p,v)>0$, *such that for all* $r<r_3(p,v)$

$$I^p\left(\frac{v'}{v}, r\right) > I^p\left(\frac{nz^{n-1}}{z^n-1}, r\right). \tag{2.6}$$

3.

PROOF OF THEOREM 1. If the Taylor expansion of f around $z=0$ is

$$f(z) = \sum_{\nu=0}^{\infty} g(\nu)z^\nu = 1+\sum_{\nu=1}^{\infty} g(\nu)z^\nu, \tag{3.1}$$

then for $r<r(f)$ we have

$$\sum_{\nu=1}^{\infty} |g(\nu)|r^\nu < 1. \tag{3.2}$$

Since

$$f(z)^{p/2} = \Big(1+\sum_{\nu=1}^{\infty} g(\nu)z^\nu\Big)^{p/2} = \sum_{k=0}^{\infty}\binom{p/2}{k}\Big(\sum_{\nu=1}^{\infty} g(\nu)z^\nu\Big)^k =$$

$$= 1+\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\binom{p/2}{k}\sum_{\substack{l_j\geqq 1\\ l_1+\ldots+l_k=m}} g(l_1)\ldots g(l_k)\right) z^m,$$

hence with the notation

$$h(m) = \sum_{k=1}^{m}\binom{p/2}{k}\sum_{\substack{l_j\geqq 1\\ l_1+\ldots+l_k=m}} g(l_1)\ldots g(l_k) \tag{3.4}$$

we have

$$I^p(f, r) = 1+\sum_{m=1}^{\infty} |h(m)|^2 r^{2m}. \tag{3.5}$$

Thus, if there is a coefficient among $h(1), \ldots, h(n-1)$ different from zero, then (1.10) and (3.5) immediately lead to (1.14).

In the opposite case

(3.6) $$h(j) = 0 \quad (j = 1, \ldots, n-1),$$

(3.4) yields

(3.7) $$g(j) = 0 \quad (j = 1, \ldots, n-1).$$

(1.1), (1.2), (1.3) and (3.1) give

$$\Big(\sum_{\nu=0}^{\infty} g(\nu) z^{\nu}\Big)\Big(\sum_{k=0}^{n} B_k z^k\Big) = \sum_{k=0}^{t} A_k z^k$$

which leads to

(3.8) $$\sum_{\nu=l}^{n+l} g(\nu) B_{n+l-\nu} = 0 \quad (l = 0, 1, 2, \ldots)$$

that is

(3.9) $$g(n+l) = \frac{-1}{B_0} \sum_{\nu=l}^{n+l-1} g(\nu) B_{n+l-\nu}.$$

So if (3.6) and (3.7) hold, then by (1.1)

(3.10) $$g(n) = \frac{-1}{B_0} g(0) B_n = \frac{(-1)^{n+1}}{\xi_1 \ldots \xi_n}.$$

If we take $m=n$ in (3.4) then (3.7), (3.10), (1.10) and (3.5) yield (1.14) if

(3.11) $$|\xi_1| < 1.$$

If

(3.12) $$|\xi_1| = 1$$

then (3.5) and (1.10) give (1.14) in every such case when for some j with $n+1 \leqq j \leqq 2n-1$, $h(j) \neq 0$. So we are entitled to assume further on, that besides (3.6), (3.7) and (3.12)

$$h(j) = 0 \quad (j = n+1, \ldots, 2n-1)$$

holds, too. Since from (3.4) we get

$$g(j) = \frac{1}{\binom{p/2}{1}} \Big\{ h(j) - \sum_{k=2}^{j} \binom{p/2}{k} \sum_{\substack{l_j \geqq 1 \\ l_1 + \ldots l_k = j}} g(l_1) \ldots g(l_k) \Big\},$$

we have

(3.13) $$g(j) = 0 \quad (j = n+1, \ldots, 2n-1)$$

too, since in the inner sum all terms contain at least one factor $g(l)$ with $1 \leqq l \leqq n-1$, and this factor is zero according to (3.7). So we have to prove (1.13)—(1.14) only when (3.7), (3.12) and (3.13) are valid. By (3.8), (3.9) and (3.10) here

$$g(2n) = \frac{-1}{B_0} g(n) B_n = g^2(n)$$

and

(3.14) $$B_1 = B_2 = \ldots = B_{n-1} = 0,$$

further

(3.15) $$g(ln) = g^l(n) \quad (l = 1, 2, \ldots)$$

and

(3.16) $$g(j) = 0 \quad (j \neq ln).$$

This means that

(3.17) $$f(z) = 1 + \sum_{l=1}^{\infty} g^l(n) z^{ln} = \frac{1}{1-g(n)z^n},$$

and so by $v(1)=0$ (c.f. (1.5)[2])

(3.18) $$f(z) = \frac{1}{1-z^n}.$$

In order to prove (1.15) we have to give our argumentation a quantitative form. Suppose that

(3.19) $$|\xi_1| < 1,$$

and denote

(3.20) $$\delta = \max \{|g(1)|, \ldots, |g(n-1)|\}.$$

Since from (1.1) and (1.5)

(3.21) $$|B_k| \leqq \binom{n}{k},$$

from (3.9) we obtain with the choice $l=0$

(3.22) $$|g(n)| \geqq \frac{1}{|\prod_{j=1}^{n} \xi_j|}(1-2^{n-1}\delta) = \frac{1-2^{n-1}\delta}{|B_0|}.$$

On the other hand, by (3.4) and (3.20)

(3.23) $$|h(m)| \geqq \left|\binom{p/2}{1} g(m)\right| - \sum_{k=2}^{m} \left|\binom{p/2}{k}\right| \binom{m-1}{k-1} \left(\max_{1 \leqq l < m} |g(l)|\right)^k.$$

So if for some $1 \leqq l < n$

(3.24) $$|g(l)| = \delta,$$

then by (3.20) and (3.23)

(3.25) $$|h(l)| \geqq \frac{p}{2}\delta - \sum_{k=2}^{l} \left|\binom{p/2}{k}\right| \binom{l-1}{k-1} \delta^k.$$

[2] For the sake of absolute uniqueness we used $\ldots \leqq \xi_n = 1$ in (1.5) instead of the normalization $\ldots \leqq |\xi_n| = 1$ used in [1].

Since for any $1 \leqq p < \infty$ and for $k \geqq 2$

$$\left|\binom{p/2}{k}\right| = \left|\prod_{j=1}^{k} \frac{p+2-2j}{2j}\right| \leqq \left(\frac{p}{2}\right)^{p/2+1}, \tag{3.26}$$

(3.23) implies

$$|h(m)| \geqq \frac{p}{2}|g(m)| - \left(\frac{p}{2}\right)^{p/2+1} 2^{m-1} \max_{\substack{1 \leqq l < n \\ k \geqq 2}} |g(l)|^k \tag{3.27}$$

and thus

$$|g(m)| \leqq \frac{2}{p}|h(m)| + \left(\frac{p}{2}\right)^{p/2} 2^{m-1} \max_{\substack{1 \leqq l < m \\ k \geqq 2}} |g(l)|^k. \tag{3.28}$$

Taking in account (3.26) we derive from (1.10) the estimate

$$I^p\left(\frac{1}{1-z^n}, r\right) \leqq 1 + \left(\frac{p}{2}\right)^2 r^{2n} + \sum_{m=2}^{\infty} \left\{\frac{p}{2} + 2^{m-1}\left(\frac{p}{2}\right)^{p/2+1}\right\}^2 r^{2mn} \leqq \tag{3.29}$$

$$\leqq 1 + \frac{p^2}{4} r^{2n} + \left(\frac{p}{2}\right)^{p+2} \left\{\frac{1}{4} \max_{p \geqq 1} \left(\frac{p}{2}\right)^{-p/2} + \frac{1}{2}\right\}^2 \sum_{m=2}^{\infty} (4r^{2n})^m.$$

Now for any

$$r < r_4(|B_0|, n, p) = \min\left\{\frac{1}{4}, \left(\frac{1-|B_0|}{20\left(\frac{p}{2}\right)^p}\right)^{1/2n}\right\} \tag{3.30}$$

this leads to

$$I^p\left(\frac{1}{1-z^n}, r\right) \leqq 1 + \frac{p^2}{4} r^{2n} + \left(\frac{p}{2}\right)^{p+2} \left(\frac{2+\sqrt{2}}{4}\right)^2 \frac{4}{3} 16 r^{4n} < \tag{3.31}$$

$$< 1 + \frac{p^2}{4} r^{2n} + 20\left(\frac{p}{2}\right)^{p+2} r^{4n} \leqq 1 + \frac{p^2}{4}\{1 + (1-|B_0|)\} r^{2n}.$$

Using (3.9), (3.20) and (3.21) we get by induction

$$|g(n+l)| \leqq \frac{1}{|B_0|} 2^n \max_{l \leqq v \leqq n+l-1} g(v)| \leqq \ldots \leqq \left(\frac{2^n}{|B_0|}\right)^{l+1} \max(1, \delta), \tag{3.32}$$

and so (3.2) is valid for every

$$r < \frac{1}{2\max(1, \delta)} \frac{|B_0|}{2^n}. \tag{3.33}$$

Now we distinguish three cases as follows:

$$\begin{cases} \textit{Case I:} & 0 \leqq \delta \leqq \delta_0 = \min\left(\frac{1-|B_0|}{2^{n+1}}, \left(\frac{2}{p}\right)^{p/2}\right). \\ \textit{Case II:} & \delta_0 < \delta \leqq 1. \\ \textit{Case III:} & 1 < \delta. \end{cases} \tag{3.34}$$

In Cases I and II (3.33) means the condition

$$r < r_5(|B_0|, n) = \frac{|B_0|}{2^{n+1}} (<|\xi_1|), \tag{3.35}$$

and so for these values of r (3.3) and so (3.5) holds. In Case I we get from (3.22) and (3.27)

$$|h(n)| \geqq \frac{p}{2}\frac{1-\delta_0 2^n}{|B_0|} - \left(\frac{p}{2}\right)^{p/2+1} 2^{n-1}\delta_0^2 = \tag{3.36}$$

$$= \frac{p}{2}\frac{1}{|B_0|}\left\{1-\delta_0 2^n\left(\left(\frac{p}{2}\right)^{p/2}\delta_0+1\right)\right\} \geqq \frac{p}{2}\left\{1+\frac{1-|B_0|}{2}\right\}.$$

Collecting (3.5), (3.31) and (3.36) we obtain

$$I^p(f, r) \geqq 1+\frac{p^2}{4}\left\{1+\frac{1-|B_0|}{2}\right\}^2 r^{2n} > 1+\frac{p^2}{4}\{1+(1-|B_0|)\}r^{2n} \geqq I^p\left(\frac{1}{1-z^n}, r\right)$$

when $r<r_4$ and $r<r_5$.

In Case II we can find an index $1 \leqq m < n$ for which[3]

$$|g(m)| \geqq \delta_0\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{m+1-n},$$

but

$$|g(v)| < \delta_0\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{v+1-n} \qquad (v = 1, \dots, m-1).$$

For this m (3.27) gives

(3.37)

$$|h(m)| \geqq \frac{p}{2}\delta_0\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{m+1-n} - \left(\frac{p}{2}\right)^{p/2+1} 2^{n-1}\delta_0^2\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{m-n} \geqq$$

$$\geqq \frac{p}{4}\delta_0\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{-n} = pr_6(|B_0|, n, p) = pr_6,$$

say, and so for $r<r_6$ we get by (3.5)

$$I^p(f, r) \geqq 1+(pr_6)^2 r^{2m} > 1+p^2 r^{2m+2}.$$

Considering the condition (3.30) we may use (3.31) to obtain the required inequality

Now we begin with the case $1<\delta$. Regarding the lower estimation of $h(m)$ we are in an easier position as before. But this is compensated by the fact that we can not guarantee the validity of (3.3) and so that of (3.5) within a fixed circle, since in (3.33) δ can be large. This is caused not by a weak method of computation, but the fact that in this case the numerator of $f(z)$ can be zero in the close neighbourhood of zero, and so for general p (3.3) might be divergent.

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[2] Since $n=1$ is a trivial case, we suppose $n \geqq 2$ and so $\left(\frac{p}{2}\right)^{p/2+1} 2^n > 1$ for $p \geqq 1$.

Similarly to Case II, we find an index $1 \leqq m < n$ for which

$$|g(m)| \geqq \left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{m+1-n},$$

but

$$|g(\nu)| < \left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{\nu+1-n} \quad (\nu = 1, \ldots, m-1),$$

and by (3.27) we get

(3.38)

$$|h(m)| \geqq \frac{p}{2}\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{m+1-n} - \left(\frac{p}{2}\right)^{p/2+1} 2^{n-2}\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{2m-2n} >$$

$$> \frac{p}{4}\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{-n}.$$

In the following lines we restrict ourselves to those values of r for which

$$r < r_7(|B_0|, n, p) = \min\left\{r_4(|B_0|, n, p), \frac{1}{3^n}\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{-n}\right\}, \tag{3.39}$$

and so automatically we have (3.31) and its consequence,

$$I^p\left(\frac{1}{1-z^n}, r\right) < 1 + \frac{p^2}{2} r^{2n}. \tag{3.40}$$

Now, if for some r, (3.3) and hence (3.5) is valid, we get from (3.5) and (3.38) the estimate

$$I^p(f, r) \geqq 1 + \frac{p^2}{16} 3^{2n} r_7^2 r^{2n-2}. \tag{3.41}$$

If we order the roots η_j of $u(z)$ according to nondecreasing order of modulus, and introduce the notation

$$\eta = |\eta_1| = \min_{1 \leqq j \leqq t} |\eta_j|,$$

then f has no singularity and even no zeros if $r < |\xi_1|$ and

$$r < \min\{\eta, r_7\}.$$

Thus for any such r we have both (3.41) and (3.40), which gives (1.15). It follows that we are ready if $\eta \geqq r_7$. If $\eta < r_7$, then we have (3.41) for all $r < \eta$, and even for $r = \eta$ by continuity. Taking into account that (1.6) is a monotonically increasing function of r, we have for any r in the interval

$$\eta < r \leqq \min\{3\eta, r_7\}, \tag{3.42}$$

by an application of (3.41) for $r=\eta$ and by (3.40), that

$$I^p(f,r) \geqq I^p(f,\eta) > 1+\frac{p^2}{16}3^{2n}r_7^2\eta^{2n-2} > 1+\frac{p^2}{2}r_7^2(3\eta)^{2n-2} \geqq \tag{3.43}$$

$$\geqq 1+\frac{p^2}{2}r^{2n} > I^p\left(\frac{1}{1-z^n}, r\right).$$

So if $3\eta \geqq r_7$, then we have (1.15) for all r satisfying (3.39), and so we have completed the proof of Case III. Finally, if $3\eta < r_7$, then for any r with $3\eta \leqq r < r_7$ with the notation $f(z)=\left(1-\frac{z}{\eta_1}\right)F(z)$ we have

(3.44)
$$I^p(f,r) = \frac{1}{2\pi}\int_0^{2\pi}\left|\left(1-\frac{re^{i\varphi}}{\eta_1}\right)F(re^{i\varphi})\right|^p d\varphi > \min_{0\leqq\varphi\leqq 2\pi}\left|1-\frac{re^{i\varphi}}{\eta_1}\right|^p |F(0)| > 2^p,$$

since F is analytic at every point where f is, $F(0)=f(0)=1$, and so for $r<r_7$, $I^p(F,r)$ is also an increasing function of r. Now (3.40) and (3.44) give (1.15) again, so we can colect the results of Cases I, II and III to obtain the final expression

$$r_1(|B_0|, n, p) = \min\{r_4, r_5, r_6, r_7\}.$$

4.

Proof of Theorem 2. From (1.1) we get for all r satisfying (1.7) the formula

$$\frac{v'(z)}{v(z)} = \sum_{j=1}^n \frac{1}{z-\xi_j} = \sum_{j=1}^n \frac{-\frac{1}{\xi_j}}{1-\frac{z}{\xi_j}} = -\sum_{\nu=0}^\infty\left\{\sum_{j=1}^n\left(\frac{1}{\xi_j}\right)^{\nu+1}\right\}z^\nu. \tag{4.1}$$

Denoting

$$z_j = \frac{1}{\xi_j} \tag{4.2}$$

we have

$$|z_1| \geqq \ldots \geqq |z_{n-1}| \geqq z_n = 1. \tag{4.3}$$

Let

$$e(\nu) = \sum_{j=1}^n z_j^\nu. \tag{4.4}$$

Then we have

$$\frac{v'(z)}{v(z)} = -\sum_{\nu=0}^\infty e(\nu+1)z^\nu. \tag{4.5}$$

If

$$m = \min\{\nu \geqq 1 \colon e(\nu) \neq 0\}, \tag{4.6}$$

then it is clear again, that for $m \leqq n-1$ and for

$$r < r(v) \tag{4.7}$$

we can estimate by the leading term trivially and get

$$I^p\left(\frac{v'}{v}, r\right) \geqq \frac{1}{2}|e(m)|^p r^{(m-1)p} > 2n^p r^{(n-1)p} > I^p\left(\frac{nz^{n-1}}{z^n-1}, r\right). \tag{4.8}$$

So we can suppose, that

$$e(1) = \ldots = e(n-1) = 0. \tag{4.9}$$

But in this case the function $\frac{v'(z)}{z^{n-1}v(z)}$ is analytic at 0, and thus $\frac{v'(z)}{z^{n-1}}$ is analytic at 0 too. According to (1.1) this means that $v'(z) = nB_n z^{n-1} = nz^{n-1}$, and $B_{n-1} = \ldots = B_1 = 0$, i.e. $v(z) = B_0 + z^n$. Since $\xi_n = 1$ is a root of v, from this we get $v(z) = z^n - 1$. Q. E. D.

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