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ON THE LEAST PRIME IN AN ARITHMETIC PROGRESSION

by

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The aim of this paper is to give a simple elementary analytic proof for the theorem of Dirichlet which also furnishes an effective estimate for the least prime in an arithmetic progression.

The odd part of the proof is that the obtained estimate has about the same accuracy as the first analytic results which were attained by the use of the complex function theory, the distribution of the zeros of Dirichlet's L-functions, and essentially more complicated tools.

CHOWLA raised the question of presenting for arbitrary k and l a bound depending on k only for the appearence of the first prime in the progression. Denoting the first prime $\equiv l \bmod k$ (where $(k, l)=1$) in the progression by $P(k, l)$, Chowla expressed the conjecture that $P(k, l)<c(\varepsilon)k^{1+\varepsilon}$. In [3] Chowla gave an elementary proof for the inequality $P(k, l)<\exp(ck^{3/2}\log^6 k)$ in the special case where k is a prime $\equiv 3 \bmod 4$. In [4], investigating the prime number formula for arithmetic progressions for variable k he showed that the asymptotic equality

$$\pi(x, k, l) \sim \frac{x}{\varphi(k)\log x}$$

holds if $x \geqq \exp\left(c(\varepsilon)k^{\frac{1}{2}+\varepsilon}\right)$. This implies $P(k, l)<\exp\left(c(\varepsilon)k^{\frac{1}{2}+\varepsilon}\right)$. The famous theorem of LINNIK from 1944 provides $P(k, l)<k^c$. The best known value of c, $c=17$ is due to CHEN [2]. The deepness of Chowla's conjecture can be seen from the fact that even the assumption of the general Riemann hypothesis leads only to the estimate $P(k, l)<c(\varepsilon)k^{2+\varepsilon}$ (see Chowla [4]).

Dirichlet's original proof used L-functions for real $s \geqq 1$ only, and did not give any estimate for the least prime. On the other hand, the results of Chowla [4] and Linnik [6], [7] and other authors use the theory of complex functions and many deep facts concerning the distribution of zeros of L-functions in the critical strip.

In this paper we investigate the possibility to give a simple effective proof for Dirichlet's theorem (which gives a not too large upper bound for $P(k, l)$) restricting ourselves for real $s \geqq 1$ values and thus using no complex function theory and nothing from the zeros of L-functions. We shall prove that this is possible with the estimate $P(k, l)<\exp(ck \cdot \log^{11} k)$ where the proof — including the cited results — uses nothing which would need methods not known at Dirichlet's time.

The key of the following proof is the lower estimation of $L(1, \chi)$, due to BOMBIERI [1], in the case when χ is a real non-principal character. We use this in the form

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$L(1,x)>\frac{c}{\sqrt{k}\log k}$. Though Bombieri's result is relatively new, the theorem in his paper has a simple and elementary proof. Denoting by q the maximum of the sum $\sum_{n=1}^{m}-\chi(n)$ over all m, Bombieri's original assertion is $L(1,\chi)\geqq\frac{\pi}{16(q+1)}$. The applied form can be derived actually from the Pólya—Vinogradov inequality which states

$$\left|\sum_{n=1}^{m}\chi(n)\right|\leqq c\sqrt{k}\log k$$

for arbitrary m. This also can be proved elementary. If instead of this inequality we use the trivial estimate $|\sum_{n=1}^{m}\chi(n)|\leqq k$, but we apply the sharper bound $d(k)<c(\varepsilon)k^{\varepsilon}$ (which is also an elementary fact) instead of the trivial $d(k)<2\sqrt{k}$, then we can get similarly $P(k,l)<\exp\left(C(\varepsilon)k^{1+\varepsilon}\right)$. The further two results, used in the proof, are also based on elementary arguments.

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THEOREM.

$$P(k,l)<\exp(ck\log^{11}k),$$

where c is an explicitly calculable constant.

In the course of proof s (and s_0, s_1) will always denote a real number in the interval $(1,2)$: χ is a character mod k $(k\geqq3)$; χ_0 the principal character,

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

the corresponding L-function. The constants $c_1,\ldots,c_{13}$ will denote explicitly calculable positive absolute constants.

In the proof we will make use of the following assertions[1]:

(1) *The number of real characters* mod $k\leqq2\cdot2^{\nu(k)}\leqq2d(k)<4\sqrt{k}$.

This follows from the explicit form of the caracters (see, e.g. [8]).

(2)
$$\frac{L'}{L}(s,\chi_0)=-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}+\sum_{p|k}\sum_{\alpha=1}^{\infty}\frac{\log p}{p^{\alpha s}}=$$
$$=\frac{\zeta'}{\zeta}(s)+\sum_{p|k}\frac{\log p}{p^s(1-p^{-s})}=\frac{-1}{s-1}+O(\log^2 k).$$

(3)
$$\sum_{p}\sum_{\alpha=2}^{\infty}\frac{\Lambda(p^\alpha)}{p^{\alpha s}}<\sum_{n}\frac{\log n}{n^2\left(1-\frac{1}{n}\right)}<c_1.$$

[1] $\nu(n)$ denotes the number of (distinct) prime divisors of the natural number n, while $d(n)$ is the number of all its divisors.

Further by

$$\Theta(t) = \sum_{n \leqq t} \log p < c_2 t,$$

(4) $$\sum_{p \geqq R} \frac{\log p}{p^s} = -\Theta(R) - \int_R^\infty \frac{\Theta(t)(-s)}{t^{s+1}}\, dt < \frac{c_3}{s-1} R^{1-s}.$$

(5) $$L(s, \chi_0) \leqq \sum_{n=1}^\infty \frac{1}{n^s} < 1 + \frac{1}{s-1} < \frac{2}{s-1}.$$

If $\chi \neq \chi_0$, then

(6) $$|L(s, \chi)| < c_4 \log k,$$

(7) $$|L'(s, \chi)| < c_5 \log^2 k.$$

By the trivial estimate $\left|\sum_{n=a}^b \chi(n)\right| < k$ and Abel's inequality[2] we get namely

$$\left|\sum_{n=1}^\infty \frac{\chi(n)}{n^s}\right| \leqq \left|\sum_{n=1}^k\right| + \left|\sum_{n=k+1}^\infty\right| < \sum_{n=1}^k \frac{1}{n} + \frac{k}{(k+1)^s} < c_4 \log k$$

and

$$\left|\sum_{n=1}^\infty - \frac{\chi(n)\log n}{n^s}\right| \leqq \left|\sum_{n=1}^k\right| + \left|\sum_{n=k+1}^\infty\right| < \sum_{n=1}^k \frac{\log n}{n} + \frac{k \log(k+1)}{(k+1)^s} < c_5 \log^2 k.$$

Now we get lower estimates for the L-functions in the righthand neighbourhood of $s=1$.

If χ is a real non-principal character mod k, then there exist absolute constants c_6, c_7 such that

(8) $$|L(s, \chi)| > \frac{c_6}{\sqrt{k}\log k}$$

for

$$s \in \left(1, 1 + \frac{c_7}{\sqrt{k}\log^3 k}\right) \stackrel{\text{def}}{=} I.$$

Using the mentioned result of Bombieri [1],

$$|L(1, \chi)| > \frac{c}{\sqrt{k}\log k},$$

[2] Abel's inequality: if a_n is a sequence satisfying $\left|\sum_{n=1}^M a_n\right| < S$ for all $M \leqq N$ and b_n is a positive monotonically decreasing sequence, then $\left|\sum_{n=1}^N a_n b_n\right| < S b_1$.

we get by (7)

$$|L(s,\chi)| \geqq |L(1,\chi)|-|L(s,\chi)-L(1,\chi)| \geqq \frac{c}{\sqrt{k}\log k}-\left|\int_1^s L'(t,\chi)\,dt\right| >$$

$$> \frac{c}{\sqrt{k}\log^3 k}-(s-1)c_5\log^5 k > \frac{c}{2}\frac{1}{\sqrt{k}\log k}$$

if

$$s < 1+\frac{c}{2c_5\sqrt{k}\log^3 k}.$$

Now let us consider the case when χ is a complex character. From [5], page 92 we know

(9) $$|L(s,\chi_0)|^3|L(s,\chi)|^4|L(s,\chi^2)|^2 \geqq 1.$$

From this, (5) and (6) we obtain for any s_0 in the interval (1, 2)

$$|L(s_0,\chi)| \geqq |L(s_0,\chi_0)|^{-3/4}|L(s_0,\chi^2)|^{-1/2} >$$

$$> \frac{(s_0-1)^{3/4}}{2^{3/4}\sqrt{c_4\log k}} = c_8(s_0-1)^{3/4}\log^{-1/2}k.$$

Now for any $s\in(1;s_0)$ on account of (7)

$$|L(s,\chi)| \geqq |L(s_0,\chi)|-|L(s_0,\chi)-L(s,\chi)| \geqq c_8(s_0-1)^{3/4}\log^{-1/2}k-$$

$$-\left|\int_s^{s_0} L'(t,\chi)\,dt\right| > c(s_0-1)^{3/4}\log^{-1/2}k-c_5(s_0-1)\log^2 k.$$

If we choose here

$$s_0 = 1+\left(\frac{c_8}{2c_5}\right)^4\log^{-10}k = 1+c_9\log^{-10}k,$$

then for $s\in(1,s_0)$ we get from this

(10) $$|L(s,\chi)| > c_{10}\log^{-8}k.$$

So if $s\in(1,1+c_9\log^{-10}k)\overset{\text{def}}{=}J$ and χ is a complex character, we obtain from this and again from (7) that

(11) $$\left|\frac{L'}{L}(s,\chi)\right| < \frac{c_5\log^2 k}{c_{10}\log^{-10}k} = c_{11}\log^{10}k.$$

Making use of the orthogonality relations of the characters we have

$$\sum_{n\equiv l(k)}\frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty}\frac{1}{\varphi(k)}\sum_{\chi\bmod k}\bar\chi(l)\chi(n)\frac{\Lambda(n)}{n^s} = \frac{1}{\varphi(k)}\sum_{\chi\bmod k}\bar\chi(l)\left(-\frac{L'}{L}(s,\chi)\right) =$$

(12) $$= \frac{1}{\varphi(k)}\left(-\frac{L'}{L}(s,\chi_0)\right)+\frac{1}{\varphi(k)}\sum_{\chi\text{ real}\bmod k}\bar\chi(l)\left(-\frac{L'}{L}(s,\chi)\right)+$$

$$+\frac{1}{\varphi(k)}\sum_{\chi\text{ complex}\bmod k}\bar\chi(l)\left(-\frac{L'}{L}(s,\chi)\right)$$

and applying (1), (2), (3), (7), (8) and (11) in it we get for $s\in I\cap J$

$$\sum_{p\equiv l(k)}\frac{\log p}{p^s}>$$

$$(13)\qquad >\frac{1}{\varphi(k)}\left(\frac{1}{s-1}-c_0\log^2 k\right)-\frac{1}{\varphi(k)}4\sqrt{k}\,\frac{c_5\log^2 k}{\dfrac{c_6}{\sqrt{k}\log k}}-\frac{\varphi(k)}{\varphi(k)}c_{11}\log^{10}k-c_1>$$

$$>\frac{1}{\varphi(k)}\left(\frac{1}{s-1}-c_{12}k\log^{10}k\right)\geqq\frac{1}{2\varphi(k)(s-1)},$$

if we require

$$(14)\qquad 1<s\leqq s_1=1+\frac{1}{2c_{12}k\log^{10}k}$$

(and so $s\in I$, $s\in J$).

If here $s\to 1+0$, we get Dirichlet's original theorem. Moreover, we have from (4) and (13)

$$(15)\qquad \sum_{p<R,\,p\equiv l(k)}\frac{\log p}{p^{s_1}}>\frac{1}{s_1-1}\left(\frac{1}{2\varphi(k)}-c_3R^{1-s_1}\right)=0$$

if

$$(16)\qquad R=\left(2c_3\varphi(k)\right)^{\frac{1}{s_1-1}}<\exp\left(c_{13}k\log^{11}k\right)$$

which proves the theorem.

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