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IRREGULARITIES IN THE DISTRIBUTION OF PRIME IDEALS II

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1. Denote by K an algebraic number field, by n and Δ the degree and the discriminant of it. Let $F(m)$ be the number of ideals in K with norm m, and

$$G(m) = \sum_{\substack{P,k \\ NP^k=m}} \frac{\log m}{k}, \tag{1.1}$$

where P runs over the prime ideals of K, NI is the norm of the ideal I, and k runs over the positive integers. Denote by $\zeta_K(s)$ the Dedekind zeta function of K. We have the absolutely convergent expansions

$$\zeta_K(s) = \sum_{m=1}^{\infty} \frac{F(m)}{m^s} \qquad (\sigma > 1), \tag{1.2}$$

and

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{m=1}^{\infty} \frac{G(m)}{m^s} \qquad (\sigma > 1). \tag{1.3}$$

Let us denote by $\Delta_K(x)$ the remainder term in the prime ideal theorem:

$$\Delta_K(x) := \Psi_K(x) - x = \sum_{m \le x} G(m) - x. \tag{1.4}$$

The aim of this paper is to investigate the connection between the domain in which $\zeta_K(s)$ does not vanish and the oscillation of $\Delta_K(x)$. In this connection W. Staś and K. Wiertelak [18] proved the following theorems:

THEOREM (W. Staś—K. Wiertelak). *Suppose that* $\zeta_K(s) \ne 0$ *in the domain*

$$\sigma > 1 - c_K \eta(|t|) \qquad c_K \le 1 \tag{1.5}$$

where c_K *is a constant*[1] *depending on* K, *and* $\eta(t)$ *is for* $t \ge 0$ *a decreasing function,*

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[1] c always denote an explicitly calculable effective constant, which is absolute if it has only an integer index, and depends on some object only when this object is denoted in the index of c.

having a continuous derivative $\eta'(t)$ *and satisfying the following conditions:*

(1.6) $$0 < \eta(t) \leq \frac{1}{2},$$

(1.7) $$\eta'(t) \to 0 \quad as \quad t \to \infty,$$

(1.8) $$\frac{1}{\eta(t)} = O(\log t) \quad as \quad t \to \infty.$$

Let α *be a fixed number in* $0<\alpha<1$, *and*

(1.9) $$\omega(x) = \min_{t \geq 1} (\eta(t) \log x + \log t).$$

Then for $x \to \infty$

(1.10) $$|\Delta_k(x)| < c_{\alpha,\eta} \left\{ \frac{n^2 \log^2(|\Delta|+1)}{c_K^2} x \exp\left(-\frac{c_K \alpha}{2} \omega(x)\right)\right\}.$$

THEOREM (W. Staś—K. Wiertelak). *Let* $\eta(t)$ *be a function which satisfies the conditions of the preceding theorem and also the condition*

(1.11) $$\eta(t) \leq c_1 \quad for \quad |t| > c_2$$

where c_1 *is a sufficiently small constant and* $\omega(x)$ *is defined as in* (1.9). *Suppose further that* (1.10) *holds. Then* $\zeta_K(s) \neq 0$ *in the domain*

(1.12) $$\sigma > 1 - \frac{\log t}{400 \log \left[\omega^{-1}\left(\frac{4}{\alpha c_0} \log t\right)\right]}$$

$$t > \max\left\{c_3, \left(\frac{n}{c_K} \log(|\Delta|+1)\right)^{10}, |\Delta|+1, \eta^{-1}(e^{-n^2})\right\},$$

where ω^{-1} *and* η^{-1} *denotes the inverse functions for* ω *and* η.

2. We have to remark, that in the special case $K=\mathbf{Q}$, where $\Delta(x)$ is the remainder of the prime number theorem and $\zeta(s)$ is the Riemann zeta function, a long development was done from the time of the first proof of the prime number theorem in 1896. A general theorem of Ingham ([4], Theorem 22), which uses first the functions η and ω with the definitions and properties listed above, handle the problem of obtaining an estimate for $\Delta(x)$ from a general zero-free domain of $\zeta(s)$. In the other direction E. Schmidt [14] and J. E. Littlewood [9] began investigations, and a decisive step was done by P. Turán (see [19], p. 150), whose power-sum method played an essential role in the further improvement reached by Staś [17]. Finally, J. Pintz [12] could strengthen Ingham's result to such an extent, that it was optimal (apart from constants and ε in the exponent), and he also could prove its optimality by giving a converse of it, which is naturally also optimal and so essentially settles the question in this important special case. His theorems sounds as follows:

THEOREM (J. Pintz). *Suppose that* $\zeta(s)\neq 0$ *in the domain*

$$\sigma > 1-\eta(|t|) \quad \left(0 < \eta(t) \leqq \frac{1}{2}\right), \tag{2.1}$$

where $\eta(t)$ *is a continuous decreasing function for* $t \geqq 0$. *Let* $0<\varepsilon<1$ *be fixed and* $\omega(x)$ *as in* (1.9). *Then we have*

$$\Delta(x) = O\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right). \tag{2.2}$$

THEOREM (J. Pintz). *Suppose that* $\zeta(s)$ *has an infinity of zeros in the domain* (2.1) *where* $\eta(t)$ *is a continuous decreasing function, but we make the further assumption that for* $g(u):=\eta(e^u)$

$$g'(u) \nearrow 0 \quad \text{for} \quad u\to\infty, \tag{2.3}$$

by which we now mean that $g'(u)$ *tends to* 0 *monotonically increasingly for* $u>c_4$ *and if* $\lim_{u\to\infty} g(u)=0$ *then* $g'(u)$ *tends to* 0 *strictly monotonically increasingly for* $u>c_4$. *Let* ε *be a fixed real number with* $0<\varepsilon<1$ *and let* ω *be the function defined in* (1.9) *which has now the form*

$$\omega(x) = \min_{u\geqq 0}\left(g(u)\log x+u\right). \tag{2.4}$$

Then we have

$$\Delta(x) = \Omega_{\pm}\left(\frac{x}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{2.5}$$

3. In Pintz's theorems well-known deep results were combined with a new method, which was built also on the powersum method. Following Pintz's method, we shall prove the undermentioned results, which corresponds to those of Pintz, and so are also optimal (apart from the constants implied by the 0-symbol and ε in the exponent).

THEOREM 1. *Suppose that* $\zeta_K(s)\neq 0$ *in the domain*

$$\sigma > 1-\eta(|t|), \tag{3.1}$$

where $\eta(t)$ *is for* $t\geqq 0$ *a continuous nonincreasing function and* $0<\eta(t)\leqq 1/2$.
Let $0<\varepsilon<1$ *be fixed, further let*

$$\omega(x) = \min_{t\geqq 1}\left(\eta(t)\log x+\log t\right). \tag{3.2}$$

Then we have

$$\Delta_K(x) = O_{K,\varepsilon,\eta}\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right) \tag{3.3}$$

where the O-constant can be explicitely determined in dependence of K, ε *and the function* $\eta(t)$.

Roughly speaking, this theorem improves the corresponding result of Staś—Wiertelak by a factor 2 in the exponent and at the same time we suppose less about the function $\eta(t)$. For the proof of the result in the other direction, described in Theorem 3, we also need Theorem 2. So we state

THEOREM 2. *Let* $0<\varepsilon<0.1$ *and let us assume the existence of a zero* $\varrho_0=\beta_0+i\gamma_0$ *of* $\zeta_K(s)$ *with the condition, that with the constant*[2]

$$G_K := \max\left\{(n+\log|\Delta|)^2, \frac{1}{1-E_K}\right\}, \tag{3.4}$$

where

$$E_K := \max\{\beta_0 \colon \zeta_K(\beta_0)=0\}, \tag{3.5}$$

and for a sufficiently large absolute constant c_5

$$\log\gamma_0 > \left(\frac{c_5 n}{\varepsilon}\right)^{3/2} \max\left\{\frac{G_K}{n}, \exp\left(\left(\frac{c_5 n}{\varepsilon}\right)^2\right)\right\}. \tag{3.6}$$

Then for every Y *for which*

$$\log Y > \left(\frac{c_5 n}{\varepsilon}\right)^5 \log\gamma_0 \tag{3.7}$$

we have in the interval

$$I=[Y, Y^{1+\varepsilon}] \tag{3.8}$$

an x *for which*

$$|\Delta_K(x)| > \frac{x^{\beta_0}}{\gamma_0^{1+\varepsilon}}. \tag{3.9}$$

THEOREM 3. *Suppose that* $\zeta_K(s)$ *has an infinity of zeros in the domain*

$$\sigma \geqq 1-\eta(|t|) \quad \left(0<\eta(|t|)\leqq\frac{1}{2}\right), \tag{3.10}$$

where $\eta(t)$ *is for* $t\geqq 0$ *a continuous nonincreasing function, for which with the notation*

$$g(u) := \eta(e^u) \quad u \geqq 0 \tag{3.11}$$

we have

$$g'(u) \nearrow 0 \quad \textit{for} \quad u\to\infty \tag{3.12}$$

with the meaning described after (2.3).

Let $0<\varepsilon<0.1$ *be fixed and* $\omega(x)$ *be the function*

$$\omega(x) = \min_{t\geqq 1}(\eta(t)\log x+\log t) = \min_{u\geqq 0}(g(u)\log x+u). \tag{3.13}$$

[2] We remark that by Lemma 13 $G_K \ll \max(n, |\Delta|^{c_9})$.

Then we have for an infinite sequence of x_k values tending to infinity

(3.14) $$|\Delta_K(x_k)| > \frac{x_k}{e^{(1+\varepsilon)\omega(x_k)}}.$$

I am deeply indebted to J. Pintz for drawing my attention to the problem and giving significant help by his comments during my work.

4. Before we begin the proofs, let us fix some notations. We take an element ϑ from K, which generates K, and denote its canonical polynomial by f, so n is the degree of f, and f has r_1 real and $2r_2$ complex roots. Denote $r=r_1+r_2-1<r_1+2r_2=n$, then r is the number of elements of any system of fundamental unit elements of K (see [7], Satz 137). M denotes the absolute value of the determinant of any such system which depends only on K (see [7], Satz 138, 139), and w denote the (by [7], Satz 135 finite) number of unity roots in K, and $d_{r+1}=1$ if $r_2=0$ and 2 if $r_2>0$. h will stand for the (by [7], Satz 125 finite) class number of K. Following Landau, we introduce the constants

(4.1) $$\lambda = \frac{2^n \pi^{r_2} M}{w d_{r+1}\sqrt{|\Delta|}},$$

and

(4.2) $$A = 2^{-r_2}\pi^{-n/2}\sqrt{|\Delta|}.$$

By $\varrho=\beta+i\gamma$ we denote a non-trivial zero of $\zeta_K(s)$, $\sum_\varrho$ means a sum extended over all of them, and for restricted sums we shall subscribe the special restrictions. We define

(4.3) $$\Theta_K = \sup\{\mathrm{Re}\,\varrho:\ \zeta_K(\varrho)=0\} = \sup_\varrho \beta,$$

so $1/2\leqq\Theta_K\leqq 1$ and the Riemann hypothesis for K means $\Theta_K=1/2$. We will use the facts written in [7] Satz 155 about the pole and the trivial zeros of $\zeta_K(s)$ without repeated citations, and write

(4.4) $$\frac{\zeta_K'}{\zeta_K}(s) = \frac{r}{s} + \sum_{j=0}^{\infty} a_j s^j,$$

(4.5) $$\zeta_K(s) = \frac{h\lambda}{s-1} + \sum_{j=0}^{\infty} b_j (s-1)^j,$$

(4.6) $$\frac{\zeta_K'}{\zeta_K}(s) = \frac{-1}{s-1} + \sum_{j=0}^{\infty} e_j (s-1)^j.$$

We denote the distance of a real σ and the nearest integer by $\|\sigma\|$:

(4.7) $$\|\sigma\| := \min_{n\in\mathbf{Z}} |n-\sigma|.$$

The constants implied by the O and $\ll$ symbols are effective constants, and absolute ones except when the dependence of some parameters is explicitly stated.

5. First we state some lemmas, most of them being well known. For the sake of completeness and to obtain explicit dependence on the parameters of the field (where it is needed) we give their proofs (or exact references for them).

LEMMA 1.

$$G(m) \leqq \frac{n}{\log 2} \log^2 m. \tag{5.1}$$

LEMMA 2.

$$|\Delta_K(x)| < \frac{n}{\log 2} x \log^2 x. \tag{5.2}$$

PROOF. Lemma 1 is Lemma 2 of [13], and Lemma 2 is a trivial consequence.

LEMMA 3. *For any* $\sigma > 1$ *and* $-\infty < t < \infty$ *we have*

$$|\zeta_K(\sigma+it)| \leqq \zeta(\sigma)^n, \tag{5.3}$$

$$\left|\frac{1}{\zeta_K(\sigma+it)}\right| \leqq \zeta_K(\sigma) \leqq \zeta(\sigma)^n. \tag{5.4}$$

PROOF. The second inequality of (5.4) is a special case of (5.3) which is Corollary 3 on p. 295 of [10], while the first inequality of (5.4) is Corollary 2 on p. 295 of [10].

LEMMA 4. *Let* $0 < B \leqq 1/2$ *be a nonnegative parameter. Then in the domain*

$$\mathscr{D}_B = \{s = \sigma + it \colon \sigma \leqq -B \text{ and if } |t| \leqq B \text{ then } \|\sigma\| > B\} \tag{5.5}$$

we have

$$\left|\frac{\zeta'_K}{\zeta_K}(s)\right| < C_B n \log(|s|+1) + \log|\Delta|. \tag{5.6}$$

PROOF. By [7], p. 112, formula (184) we have from the functional equation

$$\frac{\zeta'_K}{\zeta_K}(s) = -\frac{\zeta'_K}{\zeta_K}(1-s) + n \log 2\pi - \log|\Delta| + (r_1+r_2)\frac{\pi}{2} \operatorname{ctg} \frac{\pi s}{2} -$$

$$-r_2 \frac{\pi}{2} \operatorname{tg} \frac{\pi s}{2} - n \frac{\Gamma'}{\Gamma}(1-s). \tag{5.7}$$

As $\sigma < -B$, $1-\sigma > 1+B > 1$ and by Lemma 1

$$\left|\frac{\zeta'_K}{\zeta_K}(1-s)\right| \leqq \sum_{m=1}^{\infty} \frac{G(m)}{m^{1+B}} < \frac{n}{\log 2} \sum_{m=1}^{\infty} \frac{\log^2 m}{m^{1+B}} = n c_1(B). \tag{5.8}$$

By the periodicity of $\operatorname{ctg}(\pi s/2)$ and $\operatorname{tg}(\pi s/2)$ by 2 it suffices to consider them only in $\{s = \sigma + it \colon -2-B \leqq \sigma < -B\} \cap \mathscr{D}_B$. But there are no poles of these meromorf functions in this domain and for $-2-B \leqq \sigma \leqq -B$, $|t| > 1$, we have the trivial estimates

$$\left|\operatorname{tg} \frac{\pi s}{2}\right| < \frac{e^{\pi/2}+e^{-\pi/2}}{e^{\pi/2}-e^{-\pi/2}}, \quad \left|\operatorname{ctg} \frac{\pi s}{2}\right| < \frac{e^{\pi/2}+e^{-\pi/2}}{e^{\pi/2}-e^{-\pi/2}}, \tag{5.9}$$

so these functions have an absolute bound depending only on B in the whole domain.

As for $\Gamma'/\Gamma(z)$, $z=x+iy$, $x\geqq 1+B$, we can use the identity

(5.10)
$$\frac{\Gamma'}{\Gamma}(z)=\frac{\Gamma'}{\Gamma}(z-1)+\frac{1}{z-1}$$

$[x]-1$ times, and obtain $\left(\text{as in } 1\leqq\sigma<2 \ \left|\frac{\Gamma'}{\Gamma}(s)\right|=O(\log(|t|+2))\right)$

$$\left|\frac{\Gamma'}{\Gamma}(z)\right|\leqq\left|\sum_{k=1}^{[x]-1}\frac{1}{x-k}\right|+\left|\frac{\Gamma'}{\Gamma}(z-[x]+1)\right|<\log x+O(\log(y+2))\ll\log(|z|+2).$$

If $1-z\in\mathcal{D}_B$, $\|x-[x]+1\|\geqq B$ or $|y|\geqq B$, and so for $s\in\mathcal{D}_B$

(5.11)
$$\left|\frac{\Gamma'}{\Gamma}(1-s)\right|\ll\log(|1-s|+2)<c_2(B)\log(|s|+1).$$

Writing (5.11), (5.9) and (5.8) in (5.7), by $r_1+r_2\leqq n$ we get (5.6).

LEMMA 5. *If* $T\in\mathbf{R}$ *and* $L\in\mathbf{N}$, *we have*

(5.12)
$$\sum_{\substack{\varrho\\|\gamma-T|\leqq 1}}1\ll n\log(|T|+2)+\log|\Delta|,$$

(5.13)
$$\sum_{\substack{\varrho\\|\gamma-T|\leqq L}}1\ll nL\log(|T|+L+2)+L\log|\Delta|,$$

(5.14)
$$\sum_{\substack{\varrho\\0\leqq\gamma\leqq T}}1\ll n|T|\log(|T|+2)+(|T|+2)\log|\Delta|.$$

Further, if $|T|\geqq 2$, *then all these estimates hold with* $|T|$ *instead of* $|T|+2$.

PROOF. The assertion of (5.12) is Lemma 4 of [13], while the others are trivial mplications of it.

LEMMA 6. *For* $-1/2\leqq\sigma\leqq 4$ *we have*

(5.15)
$$\frac{\zeta_K'}{\zeta_K}(s)=\sum_{\substack{\varrho\\|\gamma-t|\leqq 1}}\frac{1}{s-\varrho}-\frac{1}{s-1}+\frac{r}{s}+O((n+\log|\Delta|)\log^2(|t|+2)).$$

Consequently, if $|t|\geqq 2$ *we have*

(5.16)
$$\frac{\zeta_K'}{\zeta_K}(s)=\sum_{\substack{\varrho\\|\gamma-t|\leqq 1}}\frac{1}{s-\varrho}+O((n+\log|\Delta|)^2\log^2|t|).$$

PROOF. Since the number b appearing in Satz 179 in [7] is

$$b=-\sum_{\varrho}{}^{*}\frac{1}{\varrho}-\log A$$

(where the $*$ means, that the $\varrho'\neq 1/2$ numbers of the sum are taken together with their pairs ϱ'' for which $\varrho'+\varrho''=1$ for the sake of convergence). We get from it

$$\frac{\zeta_K'}{\zeta_K}(s) = b-\frac{1}{s-1}-\frac{1}{s}-\frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)-r_2\frac{\Gamma'}{\Gamma}(s)+\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right). \tag{5.17}$$

Since the sum is absolute convergent, we can take together members as in $\sum^*$, and after it the $\sum^*$ for b and the $\sum_{\varrho}\frac{1}{\varrho}$ part of the together taken sum cancel each other. Further in $-1/2\leqq\sigma\leqq 4$ we can use

$$\frac{\Gamma'}{\Gamma}(s)=-\frac{1}{s}+O(\log(|t|+2)). \tag{5.18}$$

Finally, if we write separately those finite many members of the transformed sum for which $|\gamma|\leqq 1$, we get

$$\begin{aligned}\frac{\zeta_K'}{\zeta_K}(s) &= \frac{r}{s}-\frac{1}{s-1}+\sum_{\substack{\varrho\\ |\gamma|\leqq 1}}\frac{1}{s-\varrho}+\\ &+\sum_{\substack{\varrho\\ \gamma>1}}{}'\left(\frac{1}{s-\varrho}+\frac{1}{s-(1-\varrho)}\right)+O((n+\log|\Delta|)\log(|t|+2)).\end{aligned} \tag{5.19}$$

We can suppose $t\geqq 0$. Making use of (5.12) we obtain:

(5.20)

$$\begin{aligned}&\left|\sum_{\substack{\varrho\\ \gamma>t+1}}\left(\frac{1}{s-\varrho}+\frac{1}{s-(1-\varrho)}\right)\right|=\left|\sum_{\substack{\varrho\\ \gamma>t+1}}\frac{2s-1}{(s-\varrho)(s-1+\varrho)}\right|\ll\sum_{\substack{\varrho\\ \gamma>t+1}}\frac{t+1}{(\gamma-t)(\gamma+t)}\ll\\ &\ll(t+1)\sum_{j=1}^{\infty}\frac{n\log(t+j+2)+\log|\Delta|}{j(2t+j)}\ll(t+1)(n+\log|\Delta|)\int_{t+1}^{\infty}\frac{1+\log x}{(x-t)(x+t)}dx<\\ &<(n+\log|\Delta|)(t+1)\left\{\frac{1+\log(2t+2)}{t+1}\int_{t+1}^{2t+2}\frac{1}{x-t}dx+\int_{2t+2}^{\infty}\frac{1+\log x}{\left(\frac{x^2}{2}\right)}dx\right\}\ll\\ &\ll(n+\log|\Delta|)\log^2(t+2),\end{aligned}$$

(5.21)

$$\begin{aligned}&\left|\sum_{\substack{\varrho\\ 1<\gamma<t-1}}\left(\frac{1}{s-\varrho}+\frac{1}{s-(1-\varrho)}\right)\right|\ll(t+1)\sum_{\substack{\varrho\\ 1<\gamma<t-1}}\frac{1}{(t-\gamma)(t+\gamma)}<\sum_{\substack{\varrho\\ 1<\gamma<t-1}}\frac{1}{t-\gamma}\ll\\ &\ll(n+\log|\Delta|)\sum_{j=1}^{[t]}\frac{\log(t+2)}{j}<(n+\log|\Delta|)\log^2(t+2),\end{aligned}$$

$$\begin{aligned}&\left|\sum_{\substack{\varrho\\ |\gamma|\leqq 1\\ |t-\gamma|>1}}\frac{1}{s-\varrho}\right|+\left|\sum_{\substack{\varrho\\ \gamma>1\\ |t-\gamma|\leqq 1}}\frac{1}{s-(1-\varrho)}\right|<\sum_{\substack{\varrho\\ |\gamma|\leqq 1}}1+\sum_{\substack{\varrho\\ |t-\gamma|\leqq 1}}1\ll\\ &\ll(n+\log|\Delta|\log(t+2).\end{aligned} \tag{5.22}$$

Now (5.19)—(5.22) imply the Lemma.

LEMMA 7. *For any t there exist some T, $|T-t|\leqq 1$ for which*

$$\left|\frac{\zeta_K'}{\zeta_K}(\sigma+iT)\right| < c_6(n+\log|\Delta|)^2\log^2(|t|+2) \quad \textit{for all } \sigma \textit{ in } -\frac{1}{2}\leqq\sigma\leqq 4. \tag{5.23}$$

PROOF. Let $t\geqq 0$ and consider the poles of ζ_K'/ζ_K in $[0,1]\times[t-1,t+1]$. By Lemma 5 there are not more than $c_7(n+\log|\Delta|)\log(|t|+2)$, and so we have a horizontal line $y=T$ which avoid all of them at least $R=\dfrac{1}{c_7(n+\log|\Delta|)\log(|t|+2)+1}$ far, and $|t-T|\leqq 1-R$. By Lemma 6 and Lemma 5

$$\left|\frac{\zeta_K'}{\zeta_K}(\sigma+iT)\right| \leqq \sum_{\substack{\varrho\\|\gamma-T|\leqq 1}}\left|\frac{1}{\sigma+iT-\varrho}\right|+O((n+\log|\Delta|)\log^2(|t|+2)) <$$

$$< \sum_{\substack{\varrho\\|\gamma-T|\leqq 1}}(c_7(n+\log|\Delta|)\log(|t|+2)+1)+O((n+\log|\Delta|)\log^2(|t|+2)) < \tag{5.24}$$

$$< c_6(n+\log|\Delta|)^2\log^2(|t|+2).$$

LEMMA 8. *Let*

$$\delta(y)=\begin{cases}0 & \textit{if}\ \ 0<y<1\\ \dfrac{1}{2} & \textit{if}\ \ y=1\\ 1 & \textit{if}\ \ y>1.\end{cases}$$

For arbitrary $y>0$, $H>0$, $T>0$ we have

$$\left|\frac{1}{2\pi i}\int_{H-iT}^{H+iT}\frac{y^s}{s}\,ds-\delta(y)\right|<\begin{cases}\dfrac{H}{T} & \textit{if}\ \ y=1\\ y^H\min\left(1,\dfrac{1}{T|\log y|}\right) & \textit{if}\ \ y\neq 1.\end{cases} \tag{5.25}$$

PROOF. This is the Lemma of Chapter 17 in [2], p. 105.

LEMMA 9. *For any $Q>1$, $T>0$ and $x=[x]+1/2>1$ we have*

$$\left|\Psi_K(x)-\frac{1}{2\pi i}\int_{Q-iT}^{Q+iT}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds\right| \ll n\frac{x^Q}{T}\left(\log^3 x+\frac{1}{(Q-1)^3}\right). \tag{5.26}$$

PROOF. Let $k=[x]$, so $x=k+1/2$. If $(3/4)x\leqq m\leqq k$ then

$$\left|\log\frac{x}{m}\right| = -\log\left(1-\frac{x-m}{x}\right)\geqq\frac{x-m}{x}\geqq\frac{3}{4}\frac{x-m}{m},$$

and if $m\geqq k+1$, then

$$\left|\log\frac{x}{m}\right| = -\log\left(1-\frac{m-x}{m}\right)\geqq\frac{m-x}{m}.$$

Applying the above inequalities and Lemma 8 we get by Lemma 1

$$\left|\Psi_K(x)-\frac{1}{2\pi i}\int_{Q-iT}^{Q+iT}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds\right| < \sum_{m=1}^{\infty} G(m)\left(\frac{x}{m}\right)^Q \frac{1}{T\left|\log\frac{x}{m}\right|} \ll$$

$$\ll \frac{x^Q}{T} n\left\{\sum_{m\le(3/4)x}\frac{\log^2 m}{m}+\sum_{(3/4)x<m<(5/4)x}\frac{\log^2 m}{m\left|\frac{m-k-1/2}{m}\right|}+\sum_{m\ge(5/4)x}\frac{\log^2 m}{m^Q}\right\}\ll$$

$$\ll \frac{x^Q n}{T}\left\{\log^3 x+\log^2 x\sum_{l=1}^{x}\frac{1}{1/2l}+\int_x^\infty\frac{\log^2 t}{t^Q}\right\}\ll$$

$$\ll \frac{nx^Q}{T}\left(\log^3 x+\frac{1}{(Q-1)^3}\right).$$

LEMMA 10. *If* $x=[x]+1/2>1$, $T>3$ *and* a_0 *is the constant defined by* (4.4) *then we have*

$$\Delta_K(x)=-(r\log x+a_0)-\frac{r_1}{2}\log\left(1-\frac{1}{x^2}\right)-r_2\log\left(1-\frac{1}{x}\right)-\sum_{\varrho}\frac{x^\varrho}{\varrho} \tag{5.27}$$

and

$$\Delta_K(x)=-(r\log x+a_0)-\frac{r_1}{2}\log\left(1-\frac{1}{x^2}\right)-r_2\log\left(1-\frac{1}{x}\right)- \\ -\sum_{\substack{\varrho\\ |\gamma|<T}}\frac{x^\varrho}{\varrho}+O\left(\frac{(n+\log|\Delta|)^2 x}{T}\left(\log^3 x+\frac{\log^2 T}{\log x}\right)\right). \tag{5.28}$$

PROOF. It suffices to show (5.28) for those values of T, which satisfy (5.23) (with T in place of $|t|+2$) and $T>2$, since for other $T'>3$ Lemma 7 guarantees the existence of a T in the interval $[T'-1, T'+1]$ with this property, and by Lemma 5

$$\left|\sum_{\min(T,T')\le\gamma<\max(T,T')}\frac{x^\varrho}{\varrho}\right|<\sum_{\substack{\varrho\\ |\gamma-T'|\le 1}}\frac{x}{T'-1}\ll$$

$$\ll(n+\log|\Delta|)\log T\frac{x}{T}<\frac{(n+\log|\Delta|)x}{T}\max\left(\log^3 x,\ \frac{\log^2 T}{\log x}\right).$$

Let $Q=1+1/\log x$ and $R=[R]+1/2>0$. We define the broken line

$$L=L_1\cup L_2\cup L_3, \tag{5.29}$$

where

$$L_1=[Q-iT,-R-iT],\quad L_2=[-R-iT,-R+iT], \\ L_3=[-R+iT,Q+iT]. \tag{5.30}$$

By the residuum principle

$$\frac{1}{2\pi i}\int_{Q-iT}^{Q+iT}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds = \tag{5.31}$$
$$= \frac{1}{2\pi i}\int_L\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds + \sum_{\substack{z_j\\ -R<\operatorname{Re} z_j<Q\\ |\operatorname{Im} z_j|<T}} \operatorname{Res}\left[\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s};\ z_j\right].$$

As ζ_K has one simple pole at $s=1$, there ζ_K'/ζ_K has residuum -1, while at the nontrivial zeros from the region $|\operatorname{Im} z_j|<T$ we get the sum over the ϱ-s in (5.28). At $s=0$ with the notation (4.4)

(5.32)
$$\operatorname{Res}\left[\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s};\ 0\right] = \operatorname{Res}\left[-\left(\frac{r}{s}+a_0+\ldots\right)(1+s\log x+\ldots)\frac{1}{s};\ 0\right] = -r\log x - a_0,$$

while from the multiplicity of the negative zeros of ζ_K follows that at the negative integers we obtain, as the sum over the corresponding residues

$$r_2\sum_{l=-1}^{-[R]}\frac{x^l}{l} + r_1\sum_{l=-1}^{-[R/2]}\frac{x^{2l}}{2l}. \tag{5.33}$$

Collecting the residues and applying Lemma 9, since $x^Q = ex$, we get from (5.31)

$$\Bigg|\Psi_K(x) - \Bigg\{\frac{1}{2\pi i}\int_L\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds + x - \tag{5.34}$$
$$-\sum_{\substack{\varrho\\ |\gamma|<T}}\frac{x^\varrho}{\varrho} - r\log x - a_0 - r_2\sum_{l=1}^{[R]}\frac{x^{-l}}{l} - \frac{r_1}{2}\sum_{l=1}^{[R/2]}\frac{x^{-2l}}{l}\Bigg\}\Bigg| \ll n\frac{x}{T}\log^3 x.$$

We can estimate the integrals over L_1 and L_3, using in $-1/2\leqq\sigma\leqq Q$ that T satisfies (5.23) and in $\sigma\leqq -1/2$ Lemma 4 (e.g. with $B=1/3$), and get independently of R for $j=1$ and $j=3$

$$\left|\frac{1}{2\pi i}\int_{L_j}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds\right| \ll$$
$$\ll (n+\log|\Delta|)^2\left(\max_{\sigma\leqq -1/2}\frac{\log^2(|\sigma+iT|+1)}{|\sigma+iT|} + \frac{\log^2 T}{T}\right)\int_{-\infty}^{Q}x^\sigma\,d\sigma \ll \tag{5.35}$$
$$\ll \frac{(n+\log|\Delta|)^2 x\log^2 T}{T\log x},$$

and

$$\left|\frac{1}{2\pi i}\int_{L_2}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds\right| \ll \frac{(n+\log|\Delta|)\log(T+R+1)x^{-R}}{R}T, \tag{5.36}$$

so letting $R\to\infty$ and using in (5.34) that by $x>1$ the series over l are convergent Taylor expansions, we get from (5.34) by (5.35) and (5.36)

$$\Psi_K(x)-x-\left\{-r\log x-a_0-r_2\log\left(1-\frac{1}{x}\right)-\frac{r_1}{2}\log\left(1-\frac{1}{x^2}\right)-\sum_{\substack{\varrho\\|\gamma|<T}}\frac{x^\varrho}{\varrho}\right\}=$$

(5.37)

$$=O\left((n+\log|\Delta|)^2\frac{x}{T}\left(\log^3 x+\frac{\log^2 T}{\log x}\right)\right)$$

which gives (5.28). (5.27) follows by letting $T\to\infty$.

LEMMA 11. *We have for any* $0<\varepsilon<(1-F_K)/2$

$$\left|\sum_{\substack{\varrho\\|\gamma|\le 1}}\frac{x^\varrho}{\varrho}\right|\ll\frac{(n+\log|\Delta|)^2}{\varepsilon^2}x^{F_K+\varepsilon} \tag{5.38}$$

where

$$F_K:=\max\left\{\frac{1}{2},\ \max\{\beta:\ \zeta_K(\beta+i\gamma)=0,\ |\gamma|\le 4\}\right\}. \tag{5.39}$$

PROOF. By Lemma 7 we can find to $t=2$ a T in $1\le T\le 3$ for which

$$\left|\frac{\zeta_K'}{\zeta_K}(\sigma+iT)\right|\ll(n+\log|\Delta|)^2\quad\text{for all}\quad\sigma\quad\text{in}\quad-\frac{1}{2}\le\sigma\le 4. \tag{5.40}$$

By symmetry it holds also for $\sigma-iT$ values if $-1/2\le\sigma\le 4$. On the segments

$$L_1=[F_K+\varepsilon-iT,\ F_K+\varepsilon+iT],\quad L_3=[1-F_K-\varepsilon-iT,\ 1-F_K-\varepsilon+iT]$$

by the definition of F_K and $\varepsilon<(1-F_K)/2$ we avoid the poles of ζ_K'/ζ_K at least with ε, and so by Lemma 6 and Lemma 5

(5.41)

$$\left|\frac{\zeta_K'}{\zeta_K}(s)\right|\ll\frac{n}{\varepsilon}+\sum_{\substack{\varrho\\|\gamma|<T+1}}\frac{1}{\varepsilon}+O(n+\log|\Delta|)\ll\frac{n+\log|\Delta|}{\varepsilon}\quad\text{for}\quad s\in L_1\quad\text{and}\quad s\in L_3.$$

Now we apply the residuum theorem on the rectangle R with vertical sides L_1 and L_3, and get (as by the symmetry of the zeros of ζ_K to $\sigma=1/2$ we have all the nontrivial zeros of ζ_K with $|\gamma|\le T$ (<4) in R)

$$-\sum_{\substack{\varrho\\|\gamma|<T}}\frac{x^\varrho}{\varrho}=\frac{1}{2\pi i}\int_{\partial R}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds. \tag{5.42}$$

By (5.39) and (5.41), since $\varepsilon<1-F_K-\varepsilon$,

$$\left|\frac{1}{2\pi i}\int_{\partial R}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x}{s}\,ds\right|\ll \tag{5.43}$$

$$\ll(n+\log|\Delta|)^2\frac{x^{F_K+\varepsilon}}{T}+\frac{n+\log|\Delta|}{\varepsilon}\left(\frac{x^{F_K+\varepsilon}}{F_K+\varepsilon}+\frac{x^{1-F_K-\varepsilon}}{1-F_K-\varepsilon}\right)\ll\frac{(n+\log|\Delta|)^2x^{F_K+\varepsilon}}{\varepsilon^2}.$$

Further by Lemma 5, and $1<T<4$

$$\left|\sum_{\substack{\varrho \\ 1<|\gamma|<T}} \frac{x^{\varrho}}{\varrho}\right| < \sum_{\substack{\varrho \\ 1<|\gamma|<T}} \frac{x^{F_K}}{1} \ll (n+\log|\Delta|)x^{F_K}. \tag{5.44}$$

Now (5.42), (5.43) and (5.44) give the Lemma.

LEMMA 12. *Let us denote by $N_K(\sigma, T)$ the number of zeros of ζ_K in the parallelogram $\beta>\sigma$, $|\gamma|<T$. Then we have*

$$N_K(\sigma, T) \ll_K T^{(n+2)(1-\sigma)} (\log^2 T)^{2n^2+4+1-\sigma}. \tag{5.45}$$

PROOF. This is the Corollary of Sokolovskii's paper [15].

LEMMA 13. *$\zeta_K(s)\neq 0$ in the domain*

$$\sigma \geqq 1-\frac{c_8}{n\log(|t|+2)+\log|\Delta|}, \tag{5.46}$$

except for at most one real and simple zero β_0 for which

$$\beta_0 \leqq 1-|\Delta|^{-c_9}. \tag{5.47}$$

PROOF. This is Lemma 2.3 of [6].

LEMMA 14. *For the coefficients a_0 and e_0 in (4.4) and (4.6) we have*

$$|a_0| \ll \frac{n+\log|\Delta|}{1-F_K}, \tag{5.48}$$

and

$$|e_0| \ll \frac{n+\log|\Delta|}{1-F_K}. \tag{5.49}$$

Consequently, by Lemma 13 and the definition of F_K in (5.39) we have

$$|a_0| < \max\left(c_{10}(n+\log|\Delta|)^2,\ c_{11}(n+\log|\Delta|)|\Delta|^{c_9}\right) \ll n^2|\Delta|^{c_9+2}, \tag{5.50}$$

and

$$|e_0| < \max\left(c_{12}(n+\log|\Delta|)^2,\ c_{13}(n+\log|\Delta|)|\Delta|^{c_9}\right) \ll n^2|\Delta|^{c_9+2}. \tag{5.51}$$

PROOF. By the functional equation (for this form see, e.g. [7] formula (184) on p. 112)

(5.52)

$$\frac{\zeta_K'}{\zeta_K}(s) = -\frac{\zeta_K'}{\zeta_K}(1-s)+n\log 2\pi-\log|\Delta|+(r_1+r_2)\frac{\pi}{2}\operatorname{ctg}\frac{\pi s}{2}-r_2\frac{\pi}{2}\operatorname{tg}\frac{\pi s}{2}-n\frac{\Gamma'}{\Gamma}(1-s).$$

Using the Laurent expansions of $\frac{\pi}{2}\operatorname{ctg}\frac{\pi s}{2}$, $\frac{\pi}{2}\operatorname{tg}\frac{\pi s}{2}$, $\frac{\Gamma'}{\Gamma}(1-s)$ and of $\frac{\zeta_K'}{\zeta_K}(s)$ (see

(4.4), (4.6)) in (5.2) we get

$$\frac{\zeta_K'}{\zeta_K}(s) = \frac{r}{s}+a_0+\sum_{j=1}^{\infty} a_j s^j = \frac{-1}{s}-e_0-\sum_{j=1}^{\infty} e_j(-s)^j+n\log 2\pi-\log|\Delta|+$$

$$+(r_1+r_2)\left\{\frac{1}{s}+\sum_{j=0}^{\infty} d_{2j+1}s^{2j+1}\right\}+r_2\sum_{j=0}^{\infty} f_{2j+1}s^{2j+1}-n\frac{\Gamma'}{\Gamma}(1)+n\sum_{j=1}^{\infty} g_j s^j.$$

Equating coefficients in (5.53) we find

$$a_0 = n\left(\log 2\pi-\frac{\Gamma'}{\Gamma}(1)\right)-\log|\Delta|-e_0, \tag{5.54}$$

so it suffices to prove (5.49). By the maximum principle it follows, that for any $0<R<1-F_K$

$$|e_0| \leq \max_{|s-1|=R}\left|\frac{\zeta_K'}{\zeta_K}(s)+\frac{1}{s-1}\right|. \tag{5.55}$$

From Lemmas 5 and 6 and from $r<n$ we get

$$\max_{|s-1|=\frac{1-F_K}{2}}\left|\frac{\zeta_K'}{\zeta_K}(s)+\frac{1}{s-1}\right| \ll \frac{n+\log|\Delta|}{\frac{1-F_K}{2}} \ll \frac{n+\log|\Delta|}{1-F_K}. \tag{5.56}$$

For the proof of Theorem 2 Turán's powersum method is essential. We shall apply it in the following continuous form:

LEMMA 15. *Let $\alpha_j\in\mathbf{C}$ for $j=1, \dots, N$ and $d>0$ be arbitrary. Then we have*

$$\max_{a\leq t\leq a+d}\frac{\left|\sum_{j=1}^{N} e^{\alpha_j t}\right|}{\max_{1\leq j\leq N}|e^{\alpha_j t}|} \geq \left(\frac{1}{4e\left(\frac{a}{d}+1\right)}\right)^N. \tag{5.57}$$

PROOF. For $z_j\in\mathbf{C}$ $(j=1, \dots, N)$ and $m>0$ we have by the second main theorem of the powersum method (see [5])

$$\max_{m\leq\nu\leq m+N}\frac{\left|\sum_{j=1}^{N} z_j^\nu\right|}{\max_{1\leq j\leq N}|z_j^\nu|} \geq \left(\frac{N}{4e(m+N)}\right)^N.$$

Choosing $m=Na/d$, $z_j=e^{\alpha_j(a/m)}=e^{\alpha_j(d/N)}$ $(j=1, \dots, N)$, we get the continuous form.

Our last lemma is a somewhat modified form of a result proved in the Appendix of [20].

LEMMA 16. *Let $T>30$, $\delta>q=\dfrac{\log\log\log T}{\log\log T}$ and denote the number of roots of ζ_K in the rectangle $|t-T|\leq\delta/2$, $a-\delta\leq\sigma<a$ by $M_K(a, T, \delta)$. Then if ζ_K has no*

zeros in the domain $\sigma \geqq a$, $|t-T| \leqq 2$, *then we have*

$$M_K(a, T, \delta) \ll \delta(G_K + n \log T) \tag{5.58}$$

where G_K *is defined by* (3.4).

PROOF. For $\delta > 1/20$ Lemma 5 can be used, so we can suppose $q < \delta < 1/20$. For any $\varrho = \beta + i\gamma$ and $s = \sigma + iT$

$$\operatorname{Re}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) = \frac{\sigma-\beta}{(\sigma-\beta)^2 + (T-\gamma)^2} + \frac{\beta}{\beta^2+\gamma^2},$$

hence for any $s = \sigma + iT$ with $\sigma > a$ by the condition

$$\left|\sum_{\varrho}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right)\right| \geqq \operatorname{Re} \sum_{\substack{\varrho \\ |\gamma-T| \leqq 2}} \left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) - \left|\sum_{\substack{\varrho \\ |\gamma-T|>2}}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right)\right| \geqq$$

$$\geqq \sum_{\substack{\varrho \\ \beta \geqq a-\delta \\ |\gamma-T| \leqq \delta/2}} \frac{\sigma-a}{(\sigma+\delta-a)^2 + \frac{\delta^2}{4}} - \left|\sum_{\substack{\varrho \\ |\gamma-T|>2}}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right)\right| \geqq \tag{5.59}$$

$$\geqq \frac{\sigma-a}{2(\sigma-a+\delta)^2} M_K(a, T, \delta) - \left|\sum_{\substack{\varrho \\ |\gamma-T|>2}}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right)\right|.$$

Let $z = 2 + iT$, then by Lemma 5 and $T > 30$ (>2)

$$\left|\sum_{\substack{\varrho \\ |\gamma-T|>2}}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) - \sum_{\substack{\varrho \\ |\gamma-T|>2}}\left(\frac{1}{z-\varrho} + \frac{1}{\varrho}\right)\right| = \left|\sum_{\substack{\varrho \\ |\gamma-T|>2}}\left(\frac{1}{s-\varrho} - \frac{1}{z-\varrho}\right)\right| = \tag{5.60}$$

$$= \left|\sum_{\substack{\varrho \\ |\gamma-T|>2}} \frac{s-z}{(s-\varrho)(z-\varrho)}\right| \ll |s-z| \sum_{m=2}^{\infty} \frac{n \log(T+m) + \log|\Delta|}{m^2} \ll n \log T + \log|\Delta|.$$

Also by Lemma 5, since $|z-\varrho| \geqq 2-\beta > 1$ and $T > 30$

$$\left|\sum_{\substack{\varrho \\ |\gamma-T| \leqq 2}}\left(\frac{1}{z-\varrho} + \frac{1}{\varrho}\right)\right| \ll n \log T + \log|\Delta|. \tag{5.61}$$

Now collecting (5.59), (5.60) and (5.61) we see

(5.62)

$$M_K(a, T, \delta) \ll \frac{(\sigma-a+\delta)^2}{\sigma-a}\left\{\left|\sum_{\varrho}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right)\right| + \left|\sum_{\varrho}\left(\frac{1}{z-\varrho} + \frac{1}{\varrho}\right)\right| + n \log T + \log|\Delta|\right\}.$$

We shall estimate $\sum_{\varrho}\left(\frac{1}{w-\varrho} + \frac{1}{\varrho}\right)$ where $w = u + iv$ $u \geqq 1/2$, $v \geqq 2$. We use $\Gamma'/\Gamma(w) = O(\log v)$ for $1/4 \leqq u \leqq 2$ and (with the notation described there) formula (5.17), and get by Lemmas 5 and 13

$$(5.63)\quad \left|\sum_{\varrho}\left(\frac{1}{w-\varrho}+\frac{1}{\varrho}\right)\right| \leqq \left|\frac{\zeta_K'}{\zeta_K}(w)\right|+|\log A|+\left|\sum_{\varrho}{}^{*}\frac{1}{\varrho}\right|+O(1)+O(n\log v) \ll$$

$$\ll \left|\frac{\zeta_K'}{\zeta_K}(w)\right|+n+\log|\Delta|+n\log v+\sum_{j=1}^{\infty}\frac{(n+\log|\Delta|)\log(j+2)}{j^2}+(n+\log|\Delta|)^2\min_{\varrho}\left|\frac{1}{\varrho}\right|.$$

By Lemma 11 (1.3), and the definition of the constants G_K, E_K in (3.4)—(3.5), we get for z from (5.63)

$$(5.64)\quad \left|\sum_{\varrho}\left(\frac{1}{z-\varrho}+\frac{1}{\varrho}\right)\right| \ll G_K+n\log T.$$

Similarly for $s=\sigma+iT$, since $T>4$

$$(5.65)\quad \left|\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)\right| \ll G_K+n\log T+\left|\frac{\zeta_K'}{\zeta_K}(s)\right|.$$

In $u>a$, $|v-T|\leqq 2$ $\zeta_K(w)\neq 0$, and so the $\sigma\geqq 1$ halfplane with this rectangle is a domain, where $\log\zeta_K(w)$ is an analytic function. Let us define the following disks around $z=2+iT$

$$(5.66)\quad \begin{aligned} \mathscr{D}_1 &= \{w\colon\ |w-z|\leqq R=2-a\} \\ \mathscr{D}_2 &= \left\{w\colon\ |w-z|\leqq r=2-a-\frac{1}{2}q\right\} \\ \mathscr{D}_3 &= \{w\colon\ |w-z|\leqq r_3=2-a-6q\} \\ \mathscr{D}_4 &= \{w\colon\ |w-z|\leqq r_4=2-a-7q\} \\ \mathscr{D}_5 &= \left\{w\colon\ |w-z|\leqq r_5=\frac{1}{2}\right\}. \end{aligned}$$

Since in $u>-1$ and $v>2$ (see, e.g. [16], Lemma 7) $|\zeta_K(w)|<|\Delta|^{c_{14}}v^{c_{15}n}$, Lemma 1 leads at once to

$$(5.67)\quad \begin{aligned} &\operatorname{Re}\log\frac{\zeta_K(w)}{\zeta_K(z)} = \log\left|\frac{\zeta_K(w)}{\zeta_K(z)}\right| \ll \\ &\ll n\log v+\log|\Delta|+\sum_{m=2}^{\infty}\frac{n\log m}{m^2} \ll n\log v+\log|\Delta|;\quad w\in\mathscr{D}_1. \end{aligned}$$

Using the Borel—Carathéodory theorem (see, e.g. [3], p. 53)

$$\max_{|w-z|\leqq r}|f(w)-f(z)| \leqq \frac{2r}{R-r}\left\{\max_{|w-z|\leqq R}\operatorname{Re}f(w)-\operatorname{Re}f(z)\right\}$$

for the analytic function $f(w)=\log\dfrac{\zeta_K(w)}{\zeta_K(z)}$, $R=2-a$, $r=2-a-1/2\,q$, we get from (5.67)

$$(5.68)\qquad |\log \zeta_K(w)| \ll \frac{1}{q}\{n \log T+\log |\Delta|\} \quad \text{for} \quad w\in \mathcal{D}_2.$$

Now we apply the three-circle theorem to $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_5$ (where $|\log \zeta_K(w)|=O(n)$ for $w\in \mathcal{D}_5$ by Lemma 1 and (1.1)—(1.3)), and get

$$(5.69)\qquad \begin{aligned} |\log \zeta_K(w)| &\ll \left(\frac{1}{q}\left(n+\frac{\log |\Delta|}{\log T}\right)\log T\right)^{\alpha} n^{1-\alpha} \ll \\ &\ll \left(n+\frac{\log |\Delta|}{\log T}\right)\left(\frac{\log T}{q}\right)^{\alpha} \quad \text{for} \quad w\in \mathcal{D}_3, \end{aligned}$$

where

$$(5.70)\qquad \begin{aligned} \alpha &= \frac{\log r_3-\log r_5}{\log r-\log r_5} = \frac{\log (4-2a-12q)}{\log (4-2a-q)} = 1+\frac{\log\left(1-\frac{11q}{4-2a-q}\right)}{\log (4-2a-q)} < \\ &< 1-\frac{11q}{(4-2a-q)\log (4-2a-q)} < 1-\frac{11q}{3\log 3} < 1-3q \end{aligned}$$

since $q<\delta<1/20$ and $a\geqq 1/2$ can be supposed because if $a<1/2$ then by the symmetry of the ζ_K-roots to $\sigma=1/2$ the condition gives $M_K(a, T, \delta)=0$. Now from (5.69) and (5.70)

$$(5.71)\qquad \begin{aligned} &|\log \zeta_K(w)| \ll \\ &\ll \left(n+\frac{\log |\Delta|}{\log T}\right)\frac{\log T}{q}(\log T)^{-2\frac{\log\log\log T}{\log\log T}} \ll \left(n+\frac{\log |\Delta|}{\log T}\right) q\log T \quad \text{for} \quad w\in \mathcal{D}_3. \end{aligned}$$

We apply Cauchy's coefficient estimate in $\mathcal{D}_4$, and get

$$(5.72)\qquad \left|\frac{\zeta_K'}{\zeta_K}(w)\right| \ll \left(n+\frac{\log |\Delta|}{\log T}\right)\log T = n\log T+\log |\Delta| \quad \text{for} \quad w\in \mathcal{D}_4.$$

Let $\sigma=a+6q+\delta$, so $s=\sigma+iT\in \mathcal{D}_4$, and (5.72) applies to s. From (5.72), (5.62), (5.64) and (5.65) we obtain

$$(5.73)\qquad M_K(a, T, \delta) \ll \frac{(6q+2\delta)^2}{6q+\delta}\{n\log T+\log |\Delta|+G_K\} \ll \delta(n\log T+G_K).$$

6. Proof of Theorem 1. We shall handle the case $\lim_{t\to\infty} \eta(t)=H>0$ and the case $\lim_{t\to\infty} \eta(t)=0$ separately. First let

$$(6.1)\qquad \lim_{t\to\infty} \eta(t) = H > 0,$$

i.e. we suppose the quasi-Riemann hypothesis (if $H=1/2$, the Riemann hypothesis). It follows, that

$$(6.2)\qquad F_K \leqq \Theta_K \leqq 1-H.$$

From Lemma 11 for all $\delta<H/2$ we have

$$\left|\sum_{\substack{\varrho\\|\gamma|\le 1}}\frac{x^\varrho}{\varrho}\right|\ll (n+\log|\Delta|)^2\frac{1}{\delta^2}x^{F_K+\delta}. \tag{6.3}$$

Now we apply the explicit formula described in Lemma 10, and get (choosing $T=x$ in (5.28)) for any $x=[x]+1/2>3$

$$\Delta_K(x)=-a_0-\sum_{\substack{\varrho\\|\gamma|<x}}\frac{x^\varrho}{\varrho}+O((n+\log|\Delta|)^2\log^3 x). \tag{6.4}$$

The contribution of the zeros with $|\gamma|\le 1$ is estimated in (6.3), while Lemma 5 can be used to the remaining part. If we choose $\delta=1/\log x$, we get

$$\begin{aligned}\left|\sum_{\substack{\varrho\\|\gamma|<x}}\frac{x^\varrho}{\varrho}\right| &\ll (n+\log|\Delta|)^2\frac{1}{\delta^2}x^{F_K+\delta}+\sum_{m\le x}\frac{x^{1-H}(n+\log|\Delta|)\log(m+2)}{m}<\\ &<(n+\log|\Delta|)^2x^{1-H}\left(\log^2 x+\sum_{m\le x}\frac{\log(m+2)}{m}\right)\ll\\ &\ll_{H,\varepsilon'}(n+\log|\Delta|)^2x^{1-H+H\varepsilon'}.\end{aligned} \tag{6.5}$$

A short reflection to Lemma 14 and (6.2) turns (6.4) by (6.5) to

$$|\Delta_K(x)|\ll_{H,\varepsilon'}(n+\log|\Delta|)^2x^{1-H+H\varepsilon'} \tag{6.6}$$

which is now valid for any $x>3$, since $\Delta_K(x)$ changes in the interval $[[x],[x]+1)$ not more than $2n\log^2 x$ by Lemma 1. Now if $t>t_0(\varepsilon,\eta)$ then $\eta(t)<H(1+\varepsilon/3)$, and so for

$$x>x_0:=t_0^{\frac{3}{H\varepsilon}}(\varepsilon,\eta) \tag{6.7}$$

we have

$$\begin{aligned}\omega(x)&\le\eta(x^{\frac{H\varepsilon}{3}})\log x+\log x^{\frac{H\varepsilon}{3}}<H\left(1+\frac{\varepsilon}{3}\right)\log x+\frac{H\varepsilon}{3}\log x=\\ &=H\left(1+\frac{2}{3}\varepsilon\right)\log x<\frac{1}{1-\varepsilon}H\left(1-\frac{\varepsilon}{3}\right)\log x,\end{aligned} \tag{6.8}$$

so if $x>x_0(\varepsilon,\eta)$ and $\varepsilon'=\varepsilon/3$,

$$(1-\varepsilon)\omega(x)<H(1-\varepsilon')\log x,$$

and thus by (6.6)

$$|\Delta_K(x)|\ll_{\varepsilon,\eta}(n+\log|\Delta|)^2\frac{x}{e^{(1-\varepsilon)\omega(x)}}, \tag{6.9}$$

proving our theorem in this case.

The case $\lim_{t\to\infty}\eta(t)=0$ is much deeper. Here we shall use the analogue of Carlson's density theorem, described in Lemma 12. For any $0<\varepsilon<1/2$ this lemma gives (as $n\geqq 2$)

$$N_K(1-\varepsilon, T) \ll_K T^{(n+2)\varepsilon}(\log^2 T)^{7n^2} \ll_{K,\varepsilon} T^{9n\varepsilon}. \tag{6.10}$$

We apply the explicit formula (5.28) of Lemma 10 to obtain (6.4), and for $\delta\leqq\eta(4)/2$ we have (6.3), too, by Lemma 11. By Lemma 14 we can estimate a_0, and as $F_K\leqq\leqq 1-\eta(4)$, we get for $\delta=\eta(4)/2$

$$\Delta_K(x) = -\sum_{\substack{\varrho\\ 1<|\gamma|<x}} \frac{x^\varrho}{\varrho} + O\left((n+\log|\Delta|)^2\log^3 x + \frac{(n+\log|\Delta|)^2}{\eta(4)^2} x^{1-\frac{\eta(4)}{2}}\right). \tag{6.11}$$

Now we estimate the weights of the x-powers by Lemma 5 and get

$$\sum_{\substack{\varrho\\ 1<|\gamma|<x}} \frac{1}{|\varrho|} \ll (n+\log|\Delta|) \sum_{m\leqq x} \frac{\log(m+2)}{m} \ll (n+\log|\Delta|)\log^2 x. \tag{6.12}$$

Let φ be choosen later, then the contribution of zeros with $\beta\leqq 1-\varphi$ in (6.11) is by (6.12)

$$\left|\sum_{\substack{\varrho\\ 1<|\gamma|<x\\ \beta\leqq 1-\varphi}} \frac{x^\varrho}{\varrho}\right| \ll (n+\log|\Delta|)x^{1-\varphi}\log^2 x. \tag{6.13}$$

Finally, we have to consider the contribution of zeros near $\sigma=1$. By (6.10) for any $\varepsilon>9n\varphi$ we obtain

$$\begin{aligned}\left|\sum_{\substack{\varrho\\ 1<|\gamma|<x\\ \beta>1-\varphi}} \frac{x^\varrho}{\varrho}\right| &\ll_{K,\varphi} \sum_{k=1}^{[\log x]+1} \frac{x^{1-\eta(e^k)}}{e^{k-1}}(e^k)^{9n\varphi} \ll x \sum_{k=1}^{[\log x]+1} \frac{e^{-(\varepsilon-9n\varphi)k}}{e^{\eta(e^k)\log x + k(1-\varepsilon)}} < \\ &< \frac{x}{e^{(1-\varepsilon)\omega(x)}} \sum_{k=1}^{\infty} e^{-(\varepsilon-9n\varphi)k} \ll \frac{x}{e^{(1-\varepsilon)\omega(x)}} \frac{1}{\varepsilon-9n\varphi}.\end{aligned} \tag{6.14}$$

Setting $\varphi=\varepsilon/10n$ and collecting our estimates (6.11), (6.13) and (6.14) we arrive to

$$|\Delta_K(x)| \ll_{K,\varepsilon} \frac{x}{e^{(1-\varepsilon)\omega(x)}} + \frac{1}{\eta(4)^2} x^{1-\frac{\eta(4)}{2}} + x^{1-\frac{\varepsilon}{10n}}\log^2 x. \tag{6.15}$$

Since $\log x = O_\delta(x^\delta)$ and $e^{\omega(x)} = O_{\eta,\varepsilon'}(x^{\varepsilon'})$, from (6.15) follows

$$|\Delta_K(x)| \ll_{K,\varepsilon,\eta} \frac{x}{e^{(1-\varepsilon)\omega(x)}}.$$

7. Proof of Theorem 2. In the following we will use always that c_5 is chosen sufficiently large. We introduce the notations

$$\varepsilon' = \frac{\varepsilon}{c_5 n} = \frac{1}{\lambda}, \quad \mu = k\lambda^2 \tag{7.1}$$

where k is a real number to be chosen later in the range

$$\varepsilon'^2(1+4\varepsilon')\log Y \leqq k \leqq \varepsilon'^2(1+5\varepsilon')\log Y, \tag{7.2}$$

or, equivalently

$$(1+4\varepsilon')\log Y \leqq \mu \leqq (1+5\varepsilon')\log Y. \tag{7.3}$$

First we find a convenient zero of ζ_K. Let $\varrho_1=\beta_1+i\gamma_1$ be a ζ_K-zero with the maximal real part β_1 among the zeros for which

$$0 \leqq \gamma \leqq \gamma_0$$

and $\varrho_{j+1}=\beta_{j+1}+i\gamma_{j+1}$ a zero with maximal real part among those satisfying

$$\gamma_j \leqq \gamma \leqq \gamma_j+2\lambda, \quad \beta > \beta_j+\frac{1}{\log Y}. \tag{7.4}$$

After not more than [1/2 log Y] steps this process cannot be continued, and we get a zero $\varrho_M=\beta_M+i\gamma_M$ for which

$$\gamma_M \leqq \gamma_1+(2M-2)\lambda \leqq \gamma_0+\log^2 Y \tag{7.5}$$

and for which the regions

$$0 \leqq t \leqq \gamma_M, \quad \sigma > \beta_M \tag{7.6}$$

and

$$\gamma_M \leqq t \leqq \gamma_M+2\lambda, \quad \sigma > \beta_M+\frac{1}{\log Y} \tag{7.7}$$

are zerofree.

If $\gamma_j \leqq \gamma_0$, by $\beta_j \geqq \beta_0$ trivially

$$\frac{x^{\beta_j}}{\gamma_j^{1+\varepsilon}} \geqq \frac{x^{\beta_0}}{\gamma_0^{1+\varepsilon}}, \tag{7.8}$$

and if $\gamma_j \geqq \gamma_0$, it means that $\beta_j > \beta_{j-1}+1/\log Y$ and further $\gamma_j < 2\gamma_{j-1}$, which results for every $x \geqq Y$ the inequality

$$\frac{x^{\beta_j}}{\gamma_j^{1+\varepsilon}} \geqq \frac{x^{\beta_{j-1}} e^{\frac{\log x}{\log Y}}}{(2\gamma_{j-1})^{1+\varepsilon}} > \frac{x^{\beta_{j-1}}}{\gamma_{j-1}^{1+\varepsilon}}. \tag{7.9}$$

Hence for any $x \geqq Y$ we can prove from (7.8) and (7.9) by induction (7.8) for every j without the condition $\gamma_j > \gamma_0$. This, using (7.8) for $j=M$, gives that it is enough to prove (3.9) with ϱ_M instead of ϱ_0. Now we distinguish two cases, according to

$$\gamma_M > \gamma_0^{\varepsilon'^{3/2}} \quad \text{(Case I)}, \quad \text{or} \qquad \gamma_M \leqq \gamma_0^{\varepsilon'^{3/2}} \quad \text{(Case II)}. \tag{7.10}$$

Let us define

$$\gamma' = \begin{cases} \gamma_M & \text{in Case I} \\ \gamma_0 & \text{in Case II.} \end{cases} \tag{7.11}$$

If we prove (3.9) for $\beta_M+i\gamma'$ (which, of course, in Case II need not to be a zero of ζ_K) in place of ϱ_0, it implies (3.9), since in Case I we work with ϱ_M instead of ϱ_0, which was justified just before, and in Case II we can simply apply $\beta_M \geqq \beta_0$. So our aim will be to prove, that the indirect assumption

$$|\Delta_K(x)| \leqq \frac{x^{\beta_M}}{\gamma'^{1+\varepsilon}} \quad \text{for} \quad x \in I \tag{7.12}$$

leads to a contradiction: this contradiction will prove the theorem. As the trivial consequences of our notations and conditions (3.6), (3.7), (7.1), (7.2), (7.3), (7.5), (7.10), (7.11), we will use the estimates

$$\gamma' \geqq \gamma_0^{\varepsilon'^{3/2}} > e^{\lambda^2} > \lambda^2, \tag{7.13}$$

$$\gamma_M \leqq \gamma' \leqq \gamma_0 + \log^2 Y \leqq Y^{\varepsilon'^3} + \log^2 Y < e^{\varepsilon'^3\mu} + \mu^2. \tag{7.14}$$

For $\sigma > 1$ we define the analytic function

$$H(s) = \int_1^\infty \Delta_K(x) \frac{d}{dx}(x^{-s})\, dx, \tag{7.15}$$

for which in $\sigma > 1$ by partial integration

$$H(s) = \frac{\zeta_K'}{\zeta_K}(s) + \frac{s}{s-1} \tag{7.16}$$

so $H(s)$ can be defined as a meromorf function in the whole complex plane. We shall use the well-known integral formula

$$\frac{1}{2\pi i} \int_{(2)} e^{Vs^2+Zs}\, ds = \frac{1}{2\sqrt{\pi V}} \exp\left(-\frac{Z^2}{4V}\right) \tag{7.17}$$

valid for any $V > 0$ and arbitrary complex Z. Let further

$$U := \frac{1}{2\pi i} \int_{(2)} H(s+i\gamma_M) e^{ks^2+\mu s}\, ds = \frac{1}{2\pi i} \int_{(2)} \int_1^\infty \Delta_K(x) \frac{d}{dx}(x^{-s-i\gamma_M}) e^{ks^2+\mu s}\, dx\, ds =$$

$$\begin{aligned} &= \int_1^\infty \Delta_K(x) \frac{d}{dx}\left(x^{-i\gamma_M} \frac{1}{2\pi i} \int_{(2)} e^{ks^2+(\mu-\log x)s}\, ds\right) dx = \\ &= \int_1^\infty \Delta_K(x) \frac{d}{dx}\left(x^{-i\gamma_M} \frac{1}{2\sqrt{\pi k}} \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\right) dx = \\ &= \frac{1}{2\sqrt{\pi k}} \int_1^\infty \frac{\Delta_K(x)}{x} x^{-i\gamma_M} \left(-i\gamma_M + \frac{\mu-\log x}{2k}\right) \exp\left(-\frac{(\mu-\log x)^2}{4k}\right) dx. \end{aligned} \tag{7.18}$$

The idea is to estimate U from above using (7.12) in the latter expression, and find a contradiction by giving a greater lower estimate using the defining formula of U

and choosing a suitable k satisfying (7.2). We split up U into three parts, namely

(7.19) $$U_1=\int\limits_1^{e^{\mu-3\lambda k}},\quad U_2=\int\limits_{e^{\mu-3\lambda k}}^{e^{\mu+3\lambda k}},\quad U_3=\int\limits_{e^{\mu+3\lambda k}}^{\infty},$$

so by (7.1) and (7.3) we have

(7.20) $$[e^{\mu-3\lambda k}, e^{\mu+3\lambda k}]\subset I.$$

Taking account (7.13), (7.14), (7.12) and (7.20), we obtain

$$|U_2|\leqq\frac{1}{2\sqrt{\pi k}}\int\limits_{e^{\mu-3\lambda k}}^{e^{\mu+3\lambda k}}\left|\frac{\Delta_K(x)}{x}\right|\left(\gamma_M+\frac{|\mu-\log x|}{2k}\right)\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)dx\leqq$$

$$\leqq\frac{1}{2\sqrt{\pi k}}\int\limits_{e^{\mu-3\lambda k}}^{e^{\mu+3\lambda k}}\left(\gamma_M+\frac{3}{2}\lambda\right)\frac{x^{\beta_M-1}}{\gamma'^{1+\varepsilon}}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx\leqq$$

(7.21) $$\leqq\frac{\left(1+\frac{3}{2}\varepsilon'\right)\gamma'}{2\sqrt{\pi k}\,\gamma'^{1+\varepsilon}}\int\limits_{-3\lambda k}^{3\lambda k}e^{\beta_M(\mu+y)-\frac{y^2}{4k}}dy<$$

$$<\frac{e^{\mu\beta_M+k\beta_M^2}\left(1+\frac{3}{2}\varepsilon'\right)}{\gamma'^{\varepsilon}2\sqrt{\pi k}}\int\limits_{-\infty}^{\infty}e^{-\left(\frac{y}{2\sqrt{k}}-\beta_M\sqrt{k}\right)^2}dy=\left(1+\frac{3}{2}\varepsilon'\right)\frac{e^{\mu\beta_M+k\beta_M^2}}{\gamma'^{\varepsilon}}.$$

For U_1 and U_3 Lemma 2 and (7.14) gives

(7.22)
$$|U_1|\leqq\frac{1}{2\sqrt{\pi k}}\int\limits_1^{e^{\mu-3k\lambda}}\frac{|\Delta_K(x)|}{x}\left(\gamma_M+\frac{\lambda^2}{2}\right)\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx\leqq n\mu^2 2\gamma' e^{\mu-\frac{9}{4}\mu}<1,$$

$$|U_3|\leqq\frac{1}{2\sqrt{\pi k}}\int\limits_{e^{\mu+3k\lambda}}^{\infty}\frac{n}{\log 2}\log^2 x\left(\gamma'+\frac{\log x-\mu}{2k}\right)\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx\leqq$$

$$\leqq\frac{2n\gamma'}{\log 2\sqrt{\pi}}\int\limits_{e^{\mu+3\lambda k}}^{\infty}\left(\frac{\log x}{2\sqrt{k}}\right)^2\frac{(\log x-\mu)}{2\sqrt{k}}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx=$$

(7.23) $$=\frac{4n\gamma'\sqrt{k}}{\log 2\sqrt{\pi}}\int\limits_{\frac{3}{2}\lambda\sqrt{k}}^{\infty}\left(y+\frac{\mu}{2\sqrt{k}}\right)^2 ye^{-y^2}e^{2\sqrt{k}y+\mu}dy<$$

$$<4n\gamma'\sqrt{k}\,e^{\mu+k}\int\limits_{\frac{3}{2}\lambda\sqrt{k}}^{\infty}\frac{\left(\frac{3}{2}\lambda\sqrt{k}+\frac{\mu}{2\sqrt{k}}\right)^2\frac{3}{2}\lambda\sqrt{k}}{\left(\frac{3}{2}\lambda\sqrt{k}-\sqrt{k}\right)^3}(y-\sqrt{k})^3e^{-(y-\sqrt{k})^2}dy<$$

$$<n\gamma' e^{\mu+k}\mu\int\limits_{\left(\frac{3}{2}\lambda-1\right)\sqrt{k}}^{\infty}u^3e^{-u^2}du<e^{(3/2)\mu}\int\limits_{2\mu}^{\infty}ve^{-v}dv<1.$$

So we conclude from (7.21), (7.22) and (7.23) that (7.12) implies

$$|U| < 2\frac{e^{\mu\beta_M + k\beta_M^2}}{\gamma'^{\varepsilon}}. \tag{7.24}$$

Now in (7.18) we shall use the first form, and shift the line of the integration to $\sigma = -1/2$, where by $\log(a+b) < \log(a+1) + \log(b+1)$, (7.14), (7.16) and Lemma 4

$$\left|\frac{1}{2\pi i}\int_{(-1/2)} H(s+i\gamma_M)e^{ks^2+\mu s}\,ds\right| \ll \ll \int_{-\infty}^{\infty} (n+\log|\Delta|)^2 \log(|t|+\gamma_M+2)e^{-\frac{\mu}{2}+\frac{1}{4}-t^2}\,dt < 1, \tag{7.25}$$

and so by the residuum principle and $r \leqq n$

$$U = \sum_{\varrho} e^{k(\varrho - i\gamma_M)^2 + \mu(\varrho - i\gamma_M)} + re^{-k\gamma_M^2 - i\mu\gamma_M} + O(1) = = \sum_{\varrho} \{e^{(\varrho - i\gamma_M)^2 + \lambda^2(\varrho - i\gamma_M)}\}^k + O(n). \tag{7.26}$$

The contribution of the zeros with $|\gamma - \gamma_M| > 2\lambda$ is by Lemma 5

$$\ll (n+\log|\Delta|)\sum_{j=2[\lambda]}^{\infty} e^{\mu + k(1-j^2)}\log(j+\gamma_M+2) < 1. \tag{7.27}$$

Now we divide the remaining zeros into two classes, namely

$$\mathscr{C}_1 := \left\{\varrho\colon \zeta_K(\varrho) = 0,\ |\gamma - \gamma_M| < \varepsilon'^{3/2},\ \left|\beta - \beta_M - \frac{1}{\log Y}\right| < \varepsilon'^{3/2}\right\}, \quad \mathscr{C}_2 := \{\varrho\colon \zeta_K(\varrho) = 0,\ |\gamma - \gamma_M| \leqq 2\lambda\}\setminus \mathscr{C}_1. \tag{7.28}$$

The distinction of the two cases in (7.10) was needful for the proof of

$$M_K\left(\beta_M + \frac{1}{\log Y}, \gamma_M, \varepsilon'^{3/2}\right) = \sum_{\varrho \in \mathscr{C}_1} 1 < c_{14}\varepsilon'^{3/2} n \log \gamma'. \tag{7.29}$$

In Case I by (3.6) and (7.10)

$$\varepsilon'^{3/2} > \frac{\log\log\log \gamma_0^{\varepsilon'^{3/2}}}{\log\log \gamma_0^{\varepsilon'^{3/2}}} > \frac{\log\log\log \gamma_M}{\log\log \gamma_M}, \tag{7.30}$$

and so (7.29) follows from the application of Lemma 16 while in Case II we use (7.10) and Lemma 5 to get

$$M_K\left(\beta_M + \frac{1}{\log Y}, \gamma_M, \varepsilon'^{3/2}\right) < \sum_{\substack{\varrho \\ |\gamma - \gamma_M| \leqq 1}} 1 \ll n \log \gamma_M + \log|\Delta| \ll$$

$$\ll n\varepsilon'^{3/2} \log \gamma_0 = n\varepsilon'^{3/2} \log \gamma'.$$

So (7.29) is valid, and by Lemma 15 we can find a k satisfying (7.2) for which by (7.1)

$$\left|\sum_{\varrho\in\mathscr{C}_1}\{e^{(\varrho-i\gamma_M)^2+\lambda^2(\varrho-i\gamma_M)}\}^k\right| \geqq \left(\frac{1}{4e\frac{1+5\varepsilon'}{\varepsilon'}}\right)^{c_{14}\varepsilon'^{3/2}n\log\gamma'} e^{k\beta_M^2+\mu\beta_M} \geqq \tag{7.31}$$

$$\geqq \gamma'^{-c_{14}n\varepsilon'^{3/2}\log\frac{20}{\varepsilon'}} e^{k\beta_M^2+\mu\beta_M} > 4\frac{e^{k\beta_M^2+\mu\beta_M}}{\gamma'^{\varepsilon}}.$$

As for the remaining zeros of $\mathscr{C}_2$, we have $|\gamma-\gamma_M|\leqq 2\lambda$ and so by (7.6) and (7.7)

$$\beta<\beta_M+\frac{1}{\log Y}<\beta_M+\frac{2}{\mu},$$

which gives by Lemma 5 and the definition of $\mathscr{C}_2$ in (7.28), using (3.6)

$$\left|\sum_{\varrho\in\mathscr{C}_2} e^{k(\varrho-i\gamma_M)^2+\mu(\varrho-i\gamma_M)}\right| \ll$$

$$\ll (n+\log|\Delta|)\lambda\log(\gamma_M+2\lambda+2)e^{k\left\{\left(\beta_M+\frac{2}{\mu}\right)^2-\varepsilon'^3\right\}+\mu\left(\beta_M+\frac{2}{\mu}\right)} < \tag{7.32}$$

$$< c_{15}(n+\log|\Delta|)\lambda\log\gamma' e^{\mu\beta_M+k\beta_M^2} e^{-\frac{\mu}{\lambda^5}} <$$

$$< \frac{\gamma'^{\varepsilon}\gamma'^{\varepsilon}e^{\mu\beta_M+k\beta_M^2}}{Y^{\varepsilon'^5}} < \frac{e^{\mu\beta_M+k\beta_M^2}}{\gamma'^{\varepsilon}}.$$

Finally, collecting (7.26), (7.27), (7.31) and (7.32) we get

$$|U|>3\frac{e^{\mu\beta_M+k\beta_M^2}}{\gamma'^{\varepsilon}}+O(n)>2\frac{e^{\mu\beta_M+k\beta_M^2}}{\gamma'^{\varepsilon}},$$

which gives the contradiction to (7.24), whence the theorem.

8. Proof of Theorem 3. First we shall investigate the case $\lim_{u\to\infty} g(u)=0$. We define

$$u:=\log t,\quad r:=\log x,\quad \tilde{\omega}(r):=\omega(x). \tag{8.1}$$

If $r>c_g$, then for $u<c_{16}$

$$g(u)r+u\geqq g(c_{16})r>g(\sqrt{r})r+\sqrt{r}=o(r), \tag{8.2}$$

and so $g(u)r+u$ takes its minimal value for $u\geqq c_{16}$ — and this minimal value is taken only in one place, since by (3.12) at most for one $u\geqq c_{16}$

$$0=\frac{d}{du}(g(u)r+u)=g'(u)r+1=0\Leftrightarrow g'(u)=-\frac{1}{r}. \tag{8.3}$$

Thus we can define the unique solution of (8.3) as the function of r, and denote it by $u_0(r)$, and for $u>c_g'$ we can define $r_0(u)$ as the unique solution of (8.3) in r, since

g' is strictly monotone by (3.12). Further for $\delta>0$ and $u>\delta r$ we have

$$g(u)r+u>\delta r>g(\sqrt{r})r+\sqrt{r}=o(r), \tag{8.4}$$

and thus we get

$$u_0(r)=o(r), \text{ i.e. } \lim_{u\to\infty}\frac{r_0(u)}{u}=\lim_{r\to\infty}\frac{r}{u_0(r)}=\infty. \tag{8.5}$$

Now let us consider a number u_0, and order the given zeros with $\gamma>e^{u_0}$ in (3.10) according to the increasing imaginary parts, i.e. let

$$\varrho_k=\beta_k+i\gamma_k=\beta_k+ie^{u_k}, \quad \beta_k\geqq 1-g(u_k), \quad u_k\geqq u_{k-1}. \tag{8.6}$$

We want to apply Theorem 2 with an $\varepsilon<1/2-g(u_0)$, for which we suppose by $g(u)\to 0$ that we fixed a u_0 for which $g(u_0)<1/2$. Since $\gamma_k\to\infty$, for $k>k_1(\varepsilon, K)$ γ_k satisfies (3.6), and if we define

$$Y_k=e^{r_0(u_k)} \tag{8.7}$$

we see by (8.5) that for $k>k_2(\varepsilon, n, g)$ (3.7) is satisfied, too. Hence for $k>k_0(\varepsilon, K, g)$ $=\max(k_1, k_2)$ we can apply Theorem 2, and get an x_k with

$$r_0(u_k)\leqq \log x_k\leqq(1+\varepsilon)r_0(u_k), \tag{8.8}$$

for which

$$\begin{aligned}|\Delta_K(x)|&>\frac{x_k^{\beta_k}}{\gamma_k^{1+\varepsilon}}=\frac{x_k}{e^{(1-\beta_k)\log x_k+(1+\varepsilon)u_k}}\geqq\\ &\geqq\frac{x_k}{e^{g(u_k)(1+\varepsilon)r_0(u_k)+(1+\varepsilon)u_k}}\geqq\frac{x_k}{e^{(1+\varepsilon)\omega(Y_k)}}\geqq\frac{x_k}{e^{(1+\varepsilon)\omega(x_k)}}\end{aligned} \tag{8.9}$$

since ω is trivially an increasing function of x. In the case $\lim_{u\to\infty} g(u)=H>0$ we have $H>1-\Theta_K$ and we have a zero on $\sigma=\Theta_K$ or a sequence of zeros with $\beta_k\to\Theta_K-0$ by (4.3). Hence with fixed $\varepsilon>0$ we have at least one $\varrho_0=\beta_0+i\gamma_0$ with $1-\beta_0<$ $<(1+\varepsilon/2)H$. Applying the Theorem of [13] we have for a sequence of sufficiently large x-values tending to infinity the estimate

$$|\Delta_K(x)|>\frac{x^{\beta_0}}{\sqrt{\gamma_0^2+50}}>\frac{x}{e^{(1-\beta_0)\log x+c(\gamma_0)}}>\frac{x}{e^{(1+\varepsilon/2)H\log x+\varepsilon/2H\log x}} \tag{8.10}$$

and so for any $u\geqq 0$

$$|\Delta_K(x)|>\frac{x}{e^{(1+\varepsilon)H\log x}}>\frac{x}{e^{(1+\varepsilon)(g(u)\log x+u)}}\geqq\frac{x}{e^{(1+\varepsilon)\omega(x)}}. \tag{8.11}$$

9. Finally, we can collect our knowledge about the oscillation of $\Delta_K(x)$ as follows. Giving a domain $\sigma\geqq 1-\eta(t)$ by a function η (or g) satisfying the conditions, the following cases are possible:

1) If the domain is zerofree, then

$$\Delta_K(x) = O_{K,\varepsilon,\eta}\left(\frac{x}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{9.1}$$

2) If the domain contains finitely many zeros $\varrho_1, \ldots, \varrho_N$ but all of them have real parts not exceeding $\lim_{u\to\infty} 1-g(u)=1-H$, then (taking into account also $\zeta_K(1+it)\neq 0$) we have

$$\Delta_K(x) = O_{K,\eta,\varepsilon,\varrho_1,\ldots,\varrho_N}\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right). \tag{9.2}$$

3) If the domain has finitely many zeros, $\varrho_1, \ldots, \varrho_N$, but some of them have real parts $>1-H$, we can see similarly to Theorem 1

$$\Delta_K(x) = -\sum_{j=1}^{N}\frac{x^{\varrho_j}}{\varrho_j} + O_{K,\varepsilon,\eta}\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right) = \Omega(x^{1-H}) = \Omega\left(\frac{x}{e^{\omega(x)}}\right). \tag{9.3}$$

4) If the domain contains an infinity of zeros, then

$$\Delta_K(x) = \Omega\left(\frac{x}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{9.4}$$

Till this time the only information known about the real situation is that $\sigma=1$ is zerofree, moreover 1) is true with

$$\eta(t) = \frac{1}{cn^{11}|\Delta|^3\log^{2/3}t\,(\log\log t)^{1/3}} \quad \text{for} \quad t \geqq 4, \tag{9.5}$$

as can be seen in [1], and, on the other hand if $H=1/2$ then 4) must hold since there is an infinity of zeros with Re $\varrho \geqq 1/2$, and so on the boundary or in the interior of the given domain. In this latter case we know a little bit more, since we have by a theorem of Landau [8]

$$\Delta_K(x) = \Omega(\sqrt{x}\log\log\log x), \tag{9.6}$$

which is a generalization of Littlewood's result [9].

REFERENCES

[1] BARTZ, K. M., On a theorem of A. V. Sokolovskiĭ, *Acta Arith.* **34** (1977/78), 113—126. *MR* **57** #5923.
[2] DAVENPORT, H., *Multiplicative number theory*, Second edition, Graduate Texts in Mathematics, 74, Springer-Verlag, New York—Berlin, 1980. *MR* **82m**: 10001.
[3] HOLLAND, A. S. B., *Introduction to the theory of entire functions*, Pure and Applied Mathematics, Vol. 56, Academic Press, New York—London, 1973. *MR* **56** #5882. *Zbl* **278.** 30001.
[4] INGHAM, A. E., *The distribution of prime numbers*, Cambridge Tracts in Math. and Math. Phys. Nr. 30, Cambridge University Press, Cambridge, 1932. *Zbl* **6**, 397.
[5] KOLESNIK, G. and STRAUS, E. G., On the sum of powers of complex numbers, *Studies in Pure Mathematics, To the memory of Paul Turán*, ed. by P. Erdős et al., Akadémiai Kiadó, Budapest, 1983, 427—442.

[6] LAGARIAS, J. C., MONTGOMERY, H. L., and ODLYZKO, A. M., A bound for the least prime ideal in the Chebotarev density theorem, *Invent. Math.* **54** (1979), 271—296.
[7] LANDAU, E., *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Teubner, Leipzig—Berlin, 1918.
[8] LANDAU, E., Über Ideale und Primideale in Idealklassen, *Math. Z.* **2** (1918), 52—154.
[9] LITTLEWOOD, J. E., Sur la distribution des nombres premiers, *C. R. Acad. Sci. Paris* **158** (1914), 1869—1872.
[10] NARKIEWICZ, W., *Elementary and analytic theory of algebraic numbers*, Monografie Matematyczne, Tom 57, PWN-Polish Scientific Publishers, Warszawa, 1974. *MR* **50** #268.
[11] PINTZ, J., On the remainder of the prime number formula I. On a problem of Littlewood, *Acta Arith.* **36** (1980), 341—365.
[12] PINTZ, J., On the remainder of the prime number formula II. On a theorem of Ingham, *Acta Arith.* **37** (1980), 209—220.
[13] RÉVÉSZ, SZ. GY., Irregularities in the distribution of prime ideals I, *Studia Sci. Math. Hungar.* **18** (1983), 57—67.
[14] SCHMIDT, E., Über die Anzahl der Primzahlen unter gegebener Grenze, *Math. Ann.* **57** (1903), 195—204.
[15] SOKOLOVSKIĬ, A. V., Density theorems for a class of zeta functions, *Izv. Akad. Nauk. UzSSR Ser. Fiz. Mat. Nauk* **10** (1966), no 3, 33—40 (in Russian). *MR* **33** #7311.
[16] STAŚ, W., Über eine Anwendung der Methode von Turán auf die Theorie des Restgliedes im Primidealsatz, *Acta Arith.* **5** (1959), 179—195. *MR* **21** #6355.
[17] STAŚ, W., Über die Umkehrung eines Satzes von Ingham, *Acta Arith.* **6** (1960/61), 435—446. *MR* **26** #3679.
[18] STAŚ, W. and WIERTELAK, K., Some estimates in the theory of Dedekind zeta-functions, *Acta Arith.* **23** (1973), 127—135. *MR* **49** #2668.
[19] TURÁN, P., *Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953. *MR* **15**—688.
[20] TURÁN, P., On the so-called density-hypothesis in the theory of the zeta-function of Riemann, *Acta Arith.* **4** (1958), 31—56. *MR* **20** #2304.

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