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IRREGULARITIES IN THE DISTRIBUTION OF PRIME IDEALS I

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To my father, György E. Révész, on his fiftieth birthday

1. Let us denote, as usual, $\Psi(x)=\sum_{n\leqq x}\Lambda(n)$ where $\Lambda(n)$ is von Mangoldt's function, i.e. $\log p$ for the powers of the prime p and zero for numbers having more than one prime divisor. Then the remainder term in the prime number theorem is $\Delta(x)=\Psi(x)-x$. It is known for a long time, that the order of magnitude of $\Delta(x)$ is closely connected with the non-trivial zeros of $\zeta(s)$, the Riemann zeta function, e.g.

$$\Delta(x)=-\sum_{\varrho}\frac{x^{\varrho}}{\varrho}+O(\log x). \tag{1.1}$$

From this formula it is easy to see an old result of Phragmén, which states, that if θ is the least upper bound for the real parts of the ζ-roots, then

$$\Delta(x)=\Omega(x^{\theta-\varepsilon}). \tag{1.2}$$

However, this is a completely ineffective result. Littlewood [3] pointed out in 1937: "If we give a particular ϱ_0 with $\beta_0>\frac{1}{2}$ then there are no known ways to give an explicit X depending on ϱ_0 and ε, such that

$$|\Delta(x)|>X^{\beta_0-\varepsilon} \tag{1.3}$$

for some x in $[0, X]$." The first effective result in this field is due to P. Turán [11] who could prove the following

THEOREM (P. Turán). *If $\varrho_0=\beta_0+i\gamma_0$, $\beta_0\geqq\frac{1}{2}$ is an arbitrary non-trivial zero of $\zeta(s)$, then for*

$$T>\max\{c_0, c_1(\varrho_0)\} \tag{1.4}$$

one has

$$\max_{1\leqq x\leqq T}|\Delta(x)|>\frac{T^{\beta_0}}{|\varrho_0|^{\frac{10\log T}{\log\log T}}}\exp\left(-c_2\frac{\log T\log\log\log T}{\log\log T}\right) \tag{1.5}$$

with effectively calculable c_i—s.

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But (1.1) suggests that the oscillation of $\Delta(x)$ is of order $\frac{x^{\beta_0}}{|\varrho_0|}$, which is larger than the right side of (1.5). The complete answer in this respect was given by J. Pintz [5], who proved the undermentioned

THEOREM (J. Pintz). *If* $0<\varepsilon<\frac{1}{50}$ *and* $\zeta(\varrho_0)=0$ *for* $\varrho_0=\beta_0+i\gamma_0$, $\beta_0=\frac{1}{2}+\delta>\frac{1}{2}$ *then for* $\gamma_0>\frac{c_3}{\varepsilon^2}$ *and*

(1.6) $$H^{\varepsilon^2/4} > \max(\gamma_0, c_4)$$

we have in the interval

(1.7) $$[H, H^{100\log\gamma_0}]$$

an x *and an* x', *for which*

(1.8) $$\Delta(x) > (1-\varepsilon)\frac{x^{\beta_0}}{|\varrho_0|}, \qquad \Delta(x') < -(1-\varepsilon)\frac{(x')^{\beta_0}}{|\varrho_0|}.$$

Here the c_j*-s are effective constants, too.*

2. As the situation seems to be very similar in the theory of the remainder term in the prime-ideal theorem, the question for similar results is natural.

Denote by $K=K(\vartheta)$ an algebraic number field with degree n and discriminant Δ over $\mathbf{Q}$. P denotes always a prime ideal, $\mathrm{N}P$ the norm of P in K. As usual, let

$$\Psi_K(x) = \sum_{m\leqq x} G(m), \qquad G(m) = \sum_{\substack{P,k \\ \mathrm{N}P^k=m}} \log \mathrm{N}P$$

and let $\Delta_K(x)=\Psi_K(x)-x$ be the remainder term in the prime-ideal theorem. Let $\zeta_K(s)$ be the Dedekind zeta function of K, and so in $\sigma>1$

(2.1) $$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{m=1}^{\infty} \frac{G(m)}{m^s}.$$

By $\varrho=\beta+i\gamma$ we shall denote always non-trivial roots of $\zeta_K(s)$.

It is easy to obtain results like (1.2), but these will be again ineffective. So the natural aim would be to get effective results, in which the constants depends also effectively on the parameters of K. Such a result was first attained by W. Staś [8], who proved

THEOREM (W. Staś). *If* $0<\eta\leqq 2$, $\zeta_K(\varrho_0)=0$, $\varrho_0=\beta_0+i\gamma_0$ *and* $\beta_0\geqq\frac{1}{2}$, *then for*

(2.2) $$T > \max(c_5, c_\eta, c_K, c_{\varrho_0})$$

we have

(2.3) $$\max_{1\leqq x\leqq T} |\Delta_K(x)| > T^{\beta_0}\exp\left\{-(\eta+1)\frac{\log T \log\log\log T}{\log\log T}\right\}.$$

Here c_5, c_6 are explicitly calculable effective absolute constants, and

$$c_\eta = \exp\exp\exp\left(\frac{128}{\eta}\right), \tag{2.4}$$

$$c_K = \max\left\{\exp\left(c_6^n|\Delta|^{3/2}\right),\ \exp\exp\left(\frac{3}{2}n+2\right)^8\right\}, \tag{2.5}$$

$$c_{\varrho_0} = \exp\exp\left(|\varrho_0|^{1/\eta}+|\varrho_0|^{28/\eta}\right). \tag{2.6}$$

3. In the present paper we show, that the expectable oscillation can be proved. With the above notations, and explicit absolute constants c, C we assert:

THEOREM. *If* $\varrho_0=\beta_0+i\gamma_0$, $\zeta_K(\varrho_0)=0$, $\beta_0\geqq\frac{1}{2}$ *and* $\gamma_0\geqq 0$ *then for every*

$$Y > C\max\{(\gamma_0+2),\ n,\ \log|\Delta|\} \tag{3.1}$$

there exists an

$$x\in[Y,\ Y^{c(n\log(\gamma_0+10)+\log|\Delta|)}] \tag{3.2}$$

for which

$$|\Delta_K(x)| > \frac{x^{\beta_0}}{\sqrt{\gamma_0^2+50}}. \tag{3.3}$$

We remark, that (3.1) is much nicer, than the condition (2.2) with the constants (2.4), (2.5) and (2.6), and the oscillation is the expected one. The constant 50 in (3.3) is not important. In the examination of $\max_{1\leqq x\leqq T}|\Delta_K(x)|$, our result is not as good as (2.3), in view of the localization (3.2). In the classical case $K=\mathbf{Q}$, when $\zeta_K(s)$ is the Riemann zeta function and $\Delta_K(x)$ is the remainder term of the prime number formula the above version of Pintz's theorem was proved by himself in a lecture held in 1980 at the Eötvös Loránd University (Budapest).

In the special case, when $\beta_0+i\gamma_0$ is a zero of ζ_K having minimal distance from the real axis, we get the classical Landau-type estimate with a uniform localization. To obtain this, we remark, that by the symmetry of the ζ_K-roots to $\sigma=\frac{1}{2}$, we can take $\beta_0\geqq\frac{1}{2}$, and that if $N_K(T)$ denotes the number of ζ_K-roots having imaginary part not exceeding $T>2$ in absolute value, by classical methods it can be easily seen that[1]

$$N_K(T) = \frac{n}{\pi}T\log T+\frac{\log|\Delta|-n-n\log 2\pi}{\pi}T+O((n+\log|\Delta|)\log T). \tag{3.4}$$

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[1] It is well-known that the remainder term is $\frac{1}{\pi}\operatorname{Im}\int_2^{\frac{1}{2}+iT}\frac{\zeta_K'}{\zeta_K}(s)\,ds$, from which $\int_2^{2+iT}$ is $O(n)$ by (2.1) and the estimate of $G(m)$ in the proof of Lemma 1. An application of Littlewood's lemma (see e.g. [10] § 9.9) gives that for some T' with $|T-T'|\leqq 1$ the horizontal integral is $O((n+\log|\Delta|)\log T)$. Taking into account $N_K(T)-N_K(T')=O((n+\log|\Delta|)\log T)$ (c.f. Lemma 4) we have (3.4).

Note added in proof. Professor J. Kaczorowsky called my attention that an alternative proof can be found in [7].

From the fact, that for a sufficiently large constant the main term exceeds the remainder, we conclude, that all the ζ_K functions have at least one zero $\beta_0+i\gamma_0$ for which $|\gamma_0|$ does not exceed an absolute constant and for which $\beta_0\geqq\frac{1}{2}$. Applying our Theorem to this zero, we get

COROLLARY. *There exist positive, absolute, effective constants* c_0, c', c'' *that for every algebraic number field K and any*

$$Y > c' \max\{n, \log|\Delta|\}, \tag{3.5}$$

we can find an

$$x\in[Y, Y^{c''(n+\log|\Delta|)}] \tag{3.6}$$

for which

$$|\Delta_K(x)| > c_0\sqrt{x}. \tag{3.7}$$

I am deeply indebted to J. Pintz for drawing my attention to the field, and for many helpful remarks during the time of my work.

4. In the course of proof $c_1, c_2, \ldots$ denote always positive, absolute, effectively calculable constants.

LEMMA 1.

$$|\Delta_K(x)| \leqq \frac{n}{\log 2}\, x \log^2 x. \tag{4.1}$$

PROOF. While $0\leqq G(m)\leqq\frac{n}{\log 2}\log^2 m$ ([8], Lemma 2) this is trivial.

LEMMA 2. *For* $-\infty<t<\infty$

$$\left|\frac{1}{\zeta_K\left(\frac{3}{2}+it\right)}\right| < \left(\zeta\left(\frac{3}{2}\right)\right)^n. \tag{4.2}$$

PROOF. This is Corollary 2 and Corollary 3 in p. 295 of [4] with $d=\frac{1}{2}$.

LEMMA 3. *For* $-1\leqq\sigma\leqq 4$, $-\infty<t<\infty$ *one has*

$$|(s-1)\zeta_K(s)| \leqq c_1^n |\Delta|^{3/2}(|t|+1)^{(3/2)n+2}. \tag{4.3}$$

PROOF. This is Lemma 3 in [9].

LEMMA 4. *If* $T\geqq 0$, *then*

$$\sum_{|\gamma-T|\leqq 1} 1 \leqq c_2 n \log(T+2)+c_3\log|\Delta|. \tag{4.4}$$

PROOF. We use Jensen's inequality for $r=\sqrt{2}$, $R=\frac{5}{2}$ around $s_0=\frac{3}{2}+iT$.

(4.5)
$$\sum_{|\gamma-T|\leq 1} 1 \leqq 2 \sum_{\substack{\varrho \\ \beta\geqq 1/2 \\ |\gamma-T|\leqq 1}} 1 \leqq 2 \sum_{\substack{\varrho \\ |\varrho-s_0|\leqq\sqrt{2}}} 1 \leqq$$
$$\leqq \frac{2}{\log\frac{R}{r}} \log \max_{|s-s_0|\leqq R} \left|\frac{\zeta_K(s)(s-1)}{\zeta_K(s_0)(s_0-1)}\right| \leqq c_4 \log\left(\frac{M}{\zeta_K\left(\frac{3}{2}+iT\right)\cdot\frac{1}{2}}\right),$$

where

(4.6)
$$M = \max_{\substack{|\sigma-(3/2)|\leqq(5/2) \\ |t|\leqq T+(5/2)}} |\zeta_K(s)(s-1)| < c_5^n |\Delta|^{3/2}(T+2)^{(3/2)n+2},$$

in view of Lemma 3, and so by Lemma 2 we get

(4.7)
$$\sum_{|\gamma-T|\leqq 1} 1 < c_4 \log\left(c_6^n |\Delta|^{3/2}(T+2)^{(3/2)n+2}\right) < c_2\, n \log(T+2) + c_3 \log|\Delta|.$$

Lemma 4 trivially implies

LEMMA 5. *Let l be a natural number. Then*

(4.8)
$$\sum_{|\gamma-T|\leqq l} 1 \leqq c_7\, l(n \log(|T|+l+2)+\log|\Delta|).$$

LEMMA 6. *When* $-\infty<t<\infty$, *we have for* $s=-\frac{1}{2}+it$

(4.9)
$$\left|\frac{1}{\zeta_K\left(-\frac{1}{2}+it\right)}\right| \leqq c_8^n \frac{1}{|\Delta|\,|s|^n},$$

where

(4.10)
$$c_8 = 2\pi\zeta\left(\frac{3}{2}\right).$$

PROOF. If F denotes the canonical polynomial of ϑ over $\mathbf{Q}$, we denote the number of its real zeros by r_1, and the number of the complex conjugate pairs of roots by r_2, so $r_1+2r_2=n$. Following Landau, we denote the constant in the functional equation (see e.g. [2], Satz 154, Satz 155) by A, i.e.

(4.11)
$$A = \sqrt{|\Delta|}\, 2^{-r_2}\pi^{-n/2}.$$

For the proof of the Lemma we write

$$\frac{1}{\zeta_K(s)} = \frac{\zeta_K(1-s)}{\zeta_K(s)} \frac{1}{\zeta_K(1-s)}.$$

Now let $s=-\frac{1}{2}+it$, then

$$\frac{1}{|\zeta_K(1-s)|} < \left(\zeta\left(\frac{3}{2}\right)\right)^n,$$

and for $f(s)=\dfrac{\zeta_K(1-s)}{\zeta_K(s)}$ we have by the functional equation ([2], Satz 155, 2)

$$f(s) = A^{2s-1}\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right)^{r_1}\left(\frac{\Gamma(s)}{\Gamma(1-s)}\right)^{r_2} =$$

(4.12)

$$= A^{2s-1}\left(\frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right)^{r_1}\frac{1}{\left(\frac{s}{2}\right)^{r_1}}\left(\frac{\Gamma(s+2)}{\Gamma(1-s)}\right)^{r_2}\frac{1}{(s(s+1))^{r_2}},$$

$$(4.13)\quad |f(s)| = A^{-2}\left|\frac{\Gamma\left(\frac{3}{4}+i\frac{t}{2}\right)}{\Gamma\left(\frac{3}{4}-i\frac{t}{2}\right)}\right|^{r_1}\left|\frac{\Gamma\left(\frac{3}{2}+it\right)}{\Gamma\left(\frac{3}{2}-it\right)}\right|^{r_2}\frac{2^{r_1}}{|s|^{r_1+r_2}|s+1|^{r_2}} = A^{-2}2^{r_1}\frac{1}{|s|^{r_1+2r_2}}.$$

In view of (4.11) and $r_1+2r_2=n$, we get from this the lemma.

LEMMA 7. *For* $-\infty<t_0<\infty$, $s_0=-\dfrac{1}{2}+it_0$

$$(4.14)\quad \left|\frac{\zeta_K'}{\zeta_K}(s_0)+\frac{1}{s_0-1}\right| < c_9\, n\log(|t_0|+2)+3\log|\Delta|.$$

PROOF. Let $g(s)=\zeta_K(s)(s-1)$, $r=\dfrac{1}{3}$. Then

$$(4.15)\quad \frac{1}{|g(s_0)|} \leqq \frac{1}{|\zeta_K(s_0)|} \leqq \frac{1}{|\Delta|}c_8^n\frac{1}{|s_0|^n}$$

by Lemma 6, and

$$(4.16)\quad \max_{|s-s_0|\leqq r}|g(s)| \leqq c_1^n|\Delta|^{3/2}\left(|t_0|+\frac{4}{3}\right)^{(3/2)n+2}$$

by Lemma 3.

Since $g(s)$ is regular and nonvanishing in the whole circle $|s-s_0|\leqq r$, we can apply Satz 4.3, Anhang [6]

$$(4.17)\quad \left|\frac{g'}{g}(s_0)\right| \leqq \frac{2}{r}\log\max_{|s-s_0|\leqq r}\left|\frac{g(s)}{g(s_0)}\right|,$$

and so considering our estimates (4.15) and (4.16)

$$(4.18)\quad \begin{aligned}\left|\frac{\zeta_K'}{\zeta_K}(s_0)+\frac{1}{s_0-1}\right| &\leqq 6\log\left\{(c_1c_8)^n\sqrt{|\Delta|}\left(\frac{|t_0|+\frac{4}{3}}{|s_0|}\right)^n\left(|t_0|+\frac{4}{3}\right)^{(1/2)n+2}\right\} \leqq \\ &\leqq c_9\, n\log(|t_0|+2)+3\log|\Delta|.\end{aligned}$$

The main tool in the proof of the theorem will be Turán's powersum method. Here we shall use Cassels' powersum theorem [1] which we formulate as

LEMMA 8. *For arbitrary complex numbers* $z_1, z_2, \dots, z_N$ *we have*

$$\max_{1\leqq\nu\leqq 2N-1} \left| \frac{\sum_{j=1}^{N} z_j^\nu}{z_1^\nu} \right| \geqq 1. \tag{4.19}$$

Substituting here $z_j = e^{\alpha_j a}$ we get from (4.19)

LEMMA 9. *For arbitrary complex numbers* $\alpha_1, \alpha_2, \dots, \alpha_N$ *and for any* $a>0$

$$\max_{a\leqq t\leqq(2N-1)a} \left| \frac{\sum_{j=1}^{N} e^{\alpha_j t}}{e^{\alpha_1 t}} \right| \geqq 1. \tag{4.20}$$

5. PROOF of the theorem. We use the "kernel-function" e^{ks^2+Hs} with real $k>1$, for which the well-known integral formula

$$\frac{1}{2\pi i}\int_{(b)} e^{ks^2+Hs}\,ds = \frac{1}{2\sqrt{\pi k}}\exp\left(-\frac{H^2}{4k}\right) \tag{5.1}$$

holds for every real k and complex H. If we define for $\sigma>1$ the convergent integral

$$\begin{aligned} D(s) &= \int_1^\infty \Delta_K(x)\frac{d}{dx}(x^{-s})\,dx = -s\int_1^\infty \frac{\Psi_K(x)-x}{x^{s+1}}\,dx = \\ &= \frac{\zeta_K'}{\zeta_K}(s)+\frac{s}{s-1} = \frac{\zeta_K'}{\zeta_K}(s)+\frac{1}{s-1}+1, \end{aligned} \tag{5.2}$$

then we can see from the latter form that D is a meromorf function.

Now we define

$$U = \frac{1}{2\pi i}\int_{(2)} D(s+i\gamma_0)\,e^{ks^2+\mu s}\,ds. \tag{5.3}$$

We can evaluate this integral in two ways; by passing the path of integration to $\sigma = -\frac{1}{2}$ and estimating the obtained sum of residues by the powersum method, or by using the definition of D and interchanging the order of integration. We will see, that our kernel function in this way relates the behaviour of $\Delta_K(x)$ around e^μ and the residues of the $\frac{\zeta_K'}{\zeta_K}$ function near to ϱ_0. Let $k=\frac{\mu}{16}$ where μ will be chosen later with the property $\mu\geqq a = 8\log Y \geqq 40$.

$$\begin{aligned} U &= \frac{1}{2\pi i}\int_{(2)}\left(\frac{\zeta_K'}{\zeta_K}(s+i\gamma_0)+\frac{s+i\gamma_0}{s+i\gamma_0-1}\right)e^{ks^2+\mu s}\,ds = \\ &= \frac{1}{2\pi i}\int_{(-1/2)}\left(\frac{\zeta_K'}{\zeta_K}(s+i\gamma_0)+\frac{s+i\gamma_0}{s+i\gamma_0-1}\right)e^{ks^2+\mu s}\,ds + \sum_{\varrho} e^{k(\varrho-i\gamma_0)^2+\mu(\varrho-i\gamma_0)}. \end{aligned} \tag{5.4}$$

Denoting the sum of residues by $W(\mu, \varrho_0)=W$, we have by $\mu>k>1$ and by Lemma 7

$$|W-U| \leqq \frac{1}{2\pi} \int_{-\infty}^{\infty} \{c_9 n \log(|t+\gamma_0|+2)+3\log|\Delta|+1\} e^{k\left(\frac{1}{4}-t^2\right)-\frac{\mu}{2}} dt \leqq$$

$$\leqq e^{\frac{k}{4}-\frac{\mu}{2}} \left\{ n \log(\gamma_0+2)\, c_{10} \int_{-\infty}^{\infty} (1+\log(|t|+2))\, e^{-t^2} dt + 3\log|\Delta| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} dt \right\} <$$

$$< e^{-\frac{\mu}{3}} \{c_{11} n \log(\gamma_0+2)+3\log|\Delta|\} < 1, \tag{5.5}$$

if

$$\mu > 3\log[c_{11} n \log(\gamma_0+2)+3\log|\Delta|]. \tag{5.6}$$

Let

$$V \stackrel{\text{def}}{=} \sum_{|\gamma-\gamma_0|<8} e^{\left(\frac{1}{16}(\varrho-i\gamma_0)^2+(\varrho-i\gamma_0)\right)\mu}. \tag{5.7}$$

The number of summands in V is less than $c_{12}(n\log(\gamma_0+10)+\log|\Delta|)$, according to Lemma 5. Using Lemma 4 we have

$$|W-V| \leqq \sum_{|\gamma-\gamma_0|\geqq 8} e^{k(\beta^2-(\gamma-\gamma_0)^2)+\mu\beta} < e^{\mu+k} \sum_{\nu=8}^{\infty} e^{-k\nu^2} \sum_{\nu\leqq|\gamma-\gamma_0|<\nu+1} 1 <$$

$$< e^{\mu+k} \sum_{\nu=8}^{\infty} 2(c_2 n \log(\nu+\gamma_0+2)+c_3\log|\Delta|) e^{-k\nu^2} < \tag{5.8}$$

$$< e^{\mu+k-64k} c_{13}(n\log(\gamma_0+2)+\log|\Delta|) < 1,$$

if

$$\mu > \log[c_{13}(n\log(\gamma_0+2)+\log|\Delta|)]. \tag{5.9}$$

Now we use Lemma 9 for V $\left(a>0,\ \alpha_j=\frac{1}{16}(\varrho-i\gamma_0)^2+(\varrho-i\gamma_0),\ 1\leqq N<c_{12}(n\log(\gamma_0+\right.$ $\left.+10)+\log|\Delta|),\ \alpha_1=\frac{1}{16}\beta_0^2+\beta_0 \text{ (from the summand for } \varrho_0)\right)$.

We get the existence of a

$$\mu\in[a,\ 2c_{12}(n\log(\gamma_0+10)+\log|\Delta|)a], \tag{5.10}$$

for which

$$|V| \geqq e^{k\beta_0^2+\mu\beta_0}. \tag{5.11}$$

On the other hand, interchanging the order of integrations and derivation we have

$$U = \frac{1}{2\pi i} \int_{(2)} \left(\int_1^{\infty} \Delta_K(x) \frac{d}{dx}(x^{-s-i\gamma_0})\, dx \right) e^{ks^2+\mu s} ds =$$

$$= \int_1^{\infty} \Delta_K(x) \frac{d}{dx}\left[x^{-i\gamma_0} \frac{1}{2\pi i} \int_{(2)} e^{ks^2+(\mu-\log x)s} ds \right] dx =$$

(5.12)

$$= \int_1^\infty \Delta_K(x) \frac{d}{dx}\left[x^{-i\gamma_0} \frac{1}{2\sqrt{\pi k}} \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\right] dx =$$

$$= \frac{1}{2\sqrt{\pi k}} \int_1^\infty \frac{\Delta_K(x)}{x} x^{-i\gamma_0} \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\left(-i\gamma_0 - \frac{2(\log x-\mu)}{4k}\right) dx.$$

Now let $U = U_1 + U_2 + U_3$ where

$$(5.13)\quad U_1 = \int_1^{e^{\mu-14k}} = \int_1^{e^{(1/8)\mu}}, \quad U_2 = \int_{e^{\mu-14k}}^{e^{\mu+14k}} = \int_{e^{(1/8)\mu}}^{e^{(15/8)\mu}}, \quad U_3 = \int_{e^{\mu+14k}}^{\infty} = \int_{e^{(15/8)\mu}}^{\infty}.$$

Using Lemma 1 and $\mu \geqq 40$, we gain for U_1 and U_3

$$|U_1| < \int_1^{e^{\mu-14k}} \frac{n}{\log 2} \log^2 x \exp(-49k)\left(\gamma_0 + \frac{\mu}{2k}\right) dx <$$

(5.14)

$$< \frac{n}{\log 2}(\gamma_0+8)\frac{\mu^2}{64}\exp\left(-\frac{49}{16}\mu\right) e^{(1/8)\mu} < 1,$$

if

(5.15)
$$\mu > \log(n(\gamma_0+2)),$$

and

$$|U_3| < \frac{1}{2\sqrt{\pi k}} \int_{e^{(15/8)\mu}}^\infty \frac{n}{\log 2} \log^2 x \exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\left(\gamma_0 + \frac{\log x - \mu}{2k}\right) dx =$$

$$= \frac{1}{2\sqrt{\pi k}} \frac{n}{\log 2} \int_{(7/8)\mu}^\infty (\mu+y)^2 \left(\gamma_0 + \frac{y}{2k}\right) \exp\left(-\frac{y^2}{4k}\right) e^{\mu+y}\, dy <$$

$$< \frac{e^{\mu+k}}{2\sqrt{\pi k}} \frac{n}{\log 2} \int_{(7/8)\mu}^\infty \left(\frac{15}{7} y\right)^2 (\gamma_0+2) \frac{y}{2k} \exp\left(-\frac{y^2}{4k} + y - k\right) dy =$$

(5.16)
$$= \frac{e^{\mu+k} n(\gamma_0+2)}{2\log 2 \sqrt{\pi k}} \left(\frac{15}{7}\right)^2 \frac{1}{2k} \int_{14k}^\infty y^3 \exp\left(-\left(\frac{y}{2\sqrt{k}} - \sqrt{k}\right)^2\right) dy <$$

$$< \frac{e^{\mu+k} n(\gamma_0+2)}{2\log 2 \sqrt{\pi k}} \left(\frac{15}{7}\right)^2 \frac{1}{2k} \int_{14k}^\infty \left[2\sqrt{k}\, \frac{7}{6}\left(\frac{y}{2\sqrt{k}} - \sqrt{k}\right)\right]^3 \exp\left(-\left(\frac{y}{2\sqrt{k}} - \sqrt{k}\right)^2\right) dy =$$

$$= e^{\mu+k} \frac{n(\gamma_0+2)}{\sqrt{\pi}} \frac{175}{12\log 2} \int_{6\sqrt{k}}^\infty t^3 \exp(-t^2)\, 2\sqrt{k}\, dt =$$

$$= e^{17k} n(\gamma_0+2)\sqrt{\mu} \frac{175}{48\sqrt{\pi}\log 2} \left\{ [t^2(-e^{-t^2})]_{6\sqrt{k}}^\infty - \int_{6\sqrt{k}}^\infty 2t(-e^{-t^2})\, dt \right\} <$$

$$< e^{17k} n(\gamma_0+2)\, 3\sqrt{\mu}\, (36k+1) e^{-36k} < n(\gamma_0+2)\, 7\mu^{3/2} e^{-(19/16)\mu} < 1,$$

if

$$\mu > \log(n(\gamma_0+2)). \tag{5.17}$$

Now from (5.5), (5.8), (5.11), (5.13), (5.14) and (5.16) we have

$$|U_2| \geqq e^{k\beta_0^2+\mu\beta_0}-4 \tag{5.18}$$

with a suitable μ satisfying (5.10) and the further conditions (5.6), (5.9), (5.15) and (5.17), for which it is enough if

$$a > 3\log\{c_{14}(n(\gamma_0+2)+\log|\Delta|)\}. \tag{5.19}$$

Let us suppose, that with some A

$$|\Delta_K(x)| < Ax^{\beta_0} \tag{5.20}$$

for all x in $[e^{\mu/8}, e^{(15/8)\mu}]$. Then for this A we get

$$|U_2| \leqq A\frac{1}{2\sqrt{\pi k}}\int_{e^{\mu-14k}}^{e^{\mu+14k}} x^{\beta_0}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)|7+i\gamma_0|\frac{dx}{x} =$$

$$= A\frac{1}{2\sqrt{\pi k}}\sqrt{49+\gamma_0^2}\int_{-14k}^{14k}\exp\left(-\frac{y^2}{4k}\right)e^{\beta_0(y+\mu)}\,dy < \tag{5.21}$$

$$< A\sqrt{49+\gamma_0^2}\,e^{\beta_0\mu+\beta_0^2k}\frac{1}{2\sqrt{\pi k}}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{y}{2\sqrt{k}}-\beta_0\sqrt{k}\right)^2\right]dy =$$

$$= Ae^{\beta_0\mu+\beta_0^2k}\sqrt{49+\gamma_0^2}.$$

By (5.18) and (5.21) we have for any A satisfying (5.20)

$$A > \frac{1}{\sqrt{49+\gamma_0^2}}\left(1-\frac{4}{e^{\mu\beta_0+k\beta_0^2}}\right). \tag{5.22}$$

Now, as $\sqrt{1-x}<1-\frac{x}{3}$ for $0\leqq x\leqq 0.1$, and as $\mu>a$ and $\beta_0\geqq\frac{1}{2}$, we have with suitable c_{15} and c_{16} for any

$$a > 4\log(\gamma_0+2)+c_{15} \tag{5.23}$$

for $\gamma_0>c_{16}$

$$\sqrt{\frac{49+\gamma_0^2}{50+\gamma_0^2}} < 1-\frac{1}{3(50+\gamma_0^2)} < 1-\frac{4}{(\gamma_0+2)^{33/16}} < 1-\frac{4}{e^{\mu(33/64)}} < 1-\frac{4}{e^{\mu\beta_0+k\beta_0^2}},$$

for $\gamma_0<c_{16}$

$$\sqrt{\frac{49+\gamma_0^2}{50+\gamma_0^2}} < 1-\frac{1}{3(50+c_{16}^2)} < 1-\frac{4}{e^{(1/2)c_{15}}} < 1-\frac{4}{e^{\mu\beta_0+k\beta_0^2}},$$

that is, by (5.22)

$$A > \frac{1}{\sqrt{50+\gamma_0^2}}.$$

So we have an $x\in[e^{(1/8)\mu}, e^{(15/8)\mu}]$ with the chosen μ for which (3.3) holds. Finally, considering (5.10) and the restrictions (5.19) and (5.23) for a, for every

$$Y = e^{a/8} > \max\{e^{3/8\log(c_{14}(n(\gamma_0+2)+\log|\Delta|))}, e^{(1/2)\log(\gamma_0+2)+(1/8)c_{15}}\},$$

so for every

$$Y > C\max\{\log|\Delta|, n, (\gamma_0+2)\}$$

we get the existence of an x,

$$x\in[e^{\mu/8}e^{(15/8)\mu}]\subset[Y, Y^{c(n\log(\gamma_0+10)+\log|\Delta|)}]$$

for which (3.3) holds. Q.e.d.

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